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TRAPEZOID TYPE INEQUALITIES FOR ISOTONIC FUNCTIONALS WITH APPLICATIONS

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Abstract. In this paper we obtain some refinements and reverses of the generalized trapezoid inequality for normalized isotonic linear functionals and various classes of functions such as: functions of bounded variation, (δ, Δ)-Lipschitzian functions and lower and upper convex functions. Applications for Jessen and Beesack-Pečarić inequalities for convex functions and isotonic functionals are provided as well. The particular case of Hermite-Hadamard inequality for functionals is also outlined.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised* if
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [22] and [23]). For other inequalities for isotonic functionals see [1], [3]-[17] and [24]-[27].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

We recall Jessen's inequality (see also [14]).

Theorem 1. *Let $\phi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval), be a convex function and $f : E \to I$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\phi(A(f)) \leq A(\phi \circ f). \tag{1.1}$$

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A counterpart of this result was proved by Beesack and Pečarić in [2] for compact intervals $I = [m, M]$. This is the functional version of the Lah-Ribarič inequality [18].

Theorem 2. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $f : E \to [m, M]$ such that $\phi \circ f$, $f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$A(\phi \circ f) \leq \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M). \tag{1.2}$$

Remark 1. *Note that (1.2) is a generalization of the inequality*

$$A(\phi) \leq \frac{M - A(\ell)}{M - m}\phi(m) + \frac{A(\ell) - m}{M - m}\phi(M) \tag{1.3}$$

due to Lupaş [17] *(see for example* [2, Theorem A]*), which assumed $E = [m, M]$, L satisfies (L1), (L2), $A : L \to \mathbb{R}$ satisfies (A1), (A2), $A(\mathbf{1}) = 1$, ϕ is convex on E and $\phi \in L$, $\ell \in L$, where $\ell(x) = x$, $x \in [m, M]$.*

The following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$\varphi\left(\frac{m + M}{2}\right) \leq \frac{1}{M - m}\int_m^M \varphi(t)\,dt \leq \frac{\varphi(m) + \varphi(M)}{2}, \tag{1.4}$$

provided that $\varphi : [m, M] \to \mathbb{R}$ is a convex function. For a monograph on Hermite-Hadamard inequality, see [13].

Using Theorem 1 and Theorem 2, we may state the following generalization of the Hermite-Hadamard inequality for isotonic linear functionals ([23] and [24]).

Theorem 3. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $e : E \to [m, M]$ with e, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional, with $A(e) = \frac{m+M}{2}$, then*

$$\varphi\left(\frac{m + M}{2}\right) \leq A(\phi \circ e) \leq \frac{\varphi(m) + \varphi(M)}{2}. \tag{1.5}$$

In this paper we obtain some refinements and reverses of the generalized trapezoid inequality for normalized isotonic linear functionals and various classes of functions such as: functions of bounded variation, (δ, Δ)-Lipschitzian functions and lower and upper convex functions. Applications for Jessen and Beesack-Pečarić inequalities for convex functions and isotonic functionals are provided as well. The particular case of Hermite-Hadamard inequality for functionals (1.5) is also outlined.

2. Bounds for Bounded Functions and Functions of Bounded Variation

The following simple result [10], which provides a sharp upper bound for the case of bounded functions, has been stated in [8] as an intermediate result needed to obtain a Grüss type inequality.

Lemma 1. *If $\phi : [m, M] \to \mathbb{R}$ is a bounded function with $-\infty < \gamma \leq \phi(t) \leq \Gamma < \infty$ for any $t \in [m, M]$, then*

$$|\Phi_\phi(t)| \leq \Gamma - \gamma, \tag{2.1}$$

where

$$\Phi_{\phi}(t) := \frac{M-t}{M-m}\phi(m) + \frac{t-m}{M-m}\phi(M) - \phi(t). \tag{2.2}$$

The multiplicative constant 1 *in front of* $\Gamma - \gamma$ *cannot be replaced by a smaller quantity.*

Proof. For the sake of completeness, we present a short proof.

Since ϕ is bounded, we have

$$\gamma(M-t) \le (M-t)\phi(m) \le (M-t)\Gamma, \gamma(t-m) \le (t-m)\phi(M) \le (t-m)\Gamma$$

and

$$-(M-m)\Gamma \le -(M-m)\phi(t) \le -(M-m)\gamma,$$

which gives, by addition and division with $M-m$ that

$$-(\Gamma-\gamma) \le \frac{(M-t)\phi(m) + (t-m)\phi(M)}{M-m} - \phi(t) \le \Gamma - \gamma,$$

for each $t \in [m, M]$, i.e., the desired inequality (2.1) holds.

Now, assume that there exists a constant $C > 0$ such that $|\Phi_{\phi}(t)| \le C(\Gamma-\gamma)$ for any ϕ as in the statement of the theorem. Then, for $t = \frac{m+M}{2}$, we should have

$$\left|\frac{\phi(m)+\phi(M)}{2} - \phi\left(\frac{m+M}{2}\right)\right| \le C(\Gamma-\gamma). \tag{2.3}$$

If $\phi : [m, M] \to \mathbb{R}$, $\phi(t) = \left|t - \frac{m+M}{2}\right|$, then $\phi(m) = \phi(M) = \frac{M-m}{2}$, $\phi\left(\frac{m+M}{2}\right) = 0$, $\Gamma = \frac{M-m}{2}$ and $\gamma = 0$ and the inequality (2.3) becomes $\frac{M-m}{2} \le C\frac{M-m}{2}$, which implies that $C \ge 1$. □

Theorem 4. *Let* $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ *be a bounded function with* $-\infty < \gamma \le \phi(t) \le \Gamma < \infty$ *for any* $t \in [m, M]$ *and* $f : E \to [m, M]$ *such that* $\phi \circ f$, $f \in L$. *If* $A : L \to \mathbb{R}$ *is an isotonic linear and normalised functional, then*

$$\left|\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi\circ f)\right| \le \Gamma - \gamma. \tag{2.4}$$

The inequality (2.4) is sharp.

Proof. From (2.1) we have in the order of L that

$$-(\Gamma-\gamma) \le \frac{M-f}{M-m}\phi(m) + \frac{f-m}{M-m}\phi(M) - \phi\circ f \le \Gamma - \gamma.$$

By taking the functional A in this inequality and using its properties of linearity and normality, we get

$$-(\Gamma-\gamma) \le \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi\circ f) \le \Gamma - \gamma,$$

which is equivalent to the desired result (2.4).

Now, assume that the inequality (2.4) is valid with a positive multiplicative constant C in the right hand side, namely

$$\left|\frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi\circ f)\right| \le C(\Gamma-\gamma). \tag{2.5}$$

Consider the bounded function $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \begin{cases} \frac{M-m}{2} \text{ if } t = m, \\ 0 \text{ if } t \in (m, M), \\ \frac{M-m}{2} \text{ if } t = m. \end{cases}$$

We have then

$$\Gamma = \frac{M-m}{2} \text{ and } \gamma = 0.$$

If we write the inequality (2.5) for the isotonic linear and normalised functional

$$A(f) := \frac{1}{M-m} \int_m^M f(t)\, dt$$

and for $f = \ell$ the identity function for the interval $[m, M]$, i.e. $\ell(t) = t$, $t \in [m, M]$, then we get

$$\left| \frac{M - \frac{M+m}{2}}{M-m} \frac{M-m}{2} + \frac{\frac{M+m}{2} - m}{M-m} \frac{M-m}{2} \right| \leq C \frac{M-m}{2}, \tag{2.6}$$

since

$$\frac{1}{M-m} \int_m^M \ell(t)\, dt = \frac{M+m}{2} \text{ and } \frac{1}{M-m} \int_m^M (\phi \circ f)(t)\, dt = 0.$$

Therefore (2.6) is equivalent to $\frac{M-m}{2} \leq C \frac{M-m}{2}$, which implies that $C \geq 1$ showing that $C = 1$ is the best possible constant in (2.4). □

Corollary 1. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function and $e : E \to [m, M]$ with e, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then*

$$0 \leq \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \leq \Gamma - \gamma. \tag{2.7}$$

The inequality (2.7) is sharp.

We have the following representation result [10]:

Lemma 2. *If $\phi : [m, M] \to \mathbb{R}$ is bounded on $[m, M]$ and $Q : [m, M]^2 \to \mathbb{R}$ is defined by*

$$Q(t, s) := \begin{cases} t - M & \text{if } m \leq s \leq t \\ \\ t - m & \text{if } t < s \leq M, \end{cases} \tag{2.8}$$

then we have the representation

$$\Phi_\phi(t) = \frac{1}{M-m} \int_m^M Q(t, s)\, d\phi(s), \quad t \in [m, M], \tag{2.9}$$

where the integral in (2.9) is taken in the sense of Riemann-Stieltjes.

Proof. We have:

$$\begin{aligned}\int_m^M Q(t,s)\,d\phi(s) &= \int_m^t (t-M)\,d\phi(s) + \int_t^M (t-m)\,d\phi(s)\\ &= (t-M)\int_m^t d\phi(s) + (t-m)\int_t^M d\phi(s)\\ &= (t-M)\left[\phi(t)-\phi(m)\right] + (t-m)\left[\phi(M)-\phi(t)\right]\\ &= (M-m)\,\Phi_\phi(t)\end{aligned}$$

and the identity is proved. □

The following estimation result holds [10].

Lemma 3. *If $\phi : [m,M] \to \mathbb{R}$ is of bounded variation, then*

$$|\Phi_\phi(t)| \le \left(\frac{M-t}{M-m}\right)\bigvee_m^t(\phi) + \left(\frac{t-m}{M-m}\right)\bigvee_t^M(\phi) \tag{2.10}$$
$$\le \begin{cases} \left[\frac{1}{2} + \left|\frac{t-\frac{m+M}{2}}{M-m}\right|\right]\bigvee_m^M(\phi); \\ \\ \left[\left(\frac{M-t}{M-m}\right)^p + \left(\frac{t-m}{M-m}\right)^p\right]^{\frac{1}{p}}\left[\left(\frac{\bigvee_m^t(\phi)}{M-m}\right)^q + \left(\frac{\bigvee_t^M(\phi)}{M-m}\right)^q\right]^{\frac{1}{q}} \\ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\bigvee_m^M(\phi) + \frac{1}{2}\left|\bigvee_m^t(\phi) - \bigvee_t^M(\phi)\right|. \end{cases}$$

The first inequality in (2.10) is sharp. The constant $\frac{1}{2}$ is best possible in the first and third branches.

Proof. We use the fact that for $p : [\alpha,\beta] \to \mathbb{R}$ continuous and $v : [\alpha,\beta] \to \mathbb{R}$ of bounded variation the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right| \le \sup_{t\in[\alpha,\beta]}|p(t)|\bigvee_\alpha^\beta(v).$$

Then, by the identity (2.9), we have

$$\begin{aligned}|\Phi_\phi(t)| &\le \frac{1}{M-m}\left|(t-M)\int_m^t d\phi(s) + (t-m)\int_t^M d\phi(s)\right|\\ &\le \frac{1}{M-m}\left[(M-t)\left|\int_m^t d\phi(s)\right| + (t-m)\left|\int_t^M d\phi(s)\right|\right]\\ &\le \frac{1}{M-m}\left[(M-t)\bigvee_m^t(\phi) + (t-m)\bigvee_t^M(\phi)\right],\end{aligned}$$

and the first inequality in (2.10) is proved.

Now, by the Hölder inequality, we have

$$(M-t)\bigvee_{m}^{t}(\phi)+(t-m)\bigvee_{t}^{M}(\phi)$$
$$\leq\begin{cases}\max\{M-t,t-m\}\left[\bigvee_{m}^{t}(\phi)+\bigvee_{t}^{M}(\phi)\right];\\[2ex] [(M-t)^{p}+(t-m)^{p}]^{\frac{1}{p}}\left[\left(\bigvee_{m}^{t}(\phi)\right)^{q}+\left(\bigvee_{t}^{M}(\phi)\right)^{q}\right]^{\frac{1}{q}}\\ \hfill\text{if}\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] (M-t+t-m)\max\left\{\bigvee_{m}^{t}(\phi),\bigvee_{t}^{M}(\phi)\right\},\end{cases}$$

which produces the last part of (2.10).

For $t=\frac{1}{2}(m+M)$, (2.10) becomes

$$\left|\phi\left(\frac{m+M}{2}\right)-\frac{\phi(m)+\phi(M)}{2}\right|\leq\frac{1}{2}\bigvee_{m}^{M}(\phi).$$

Assume that there exists a constant $C>0$ such that

$$\left|\phi\left(\frac{m+M}{2}\right)-\frac{\phi(m)+\phi(M)}{2}\right|\leq C\bigvee_{m}^{M}(\phi). \tag{2.11}$$

If in this inequality we choose $\phi:[m,M]\to\mathbb{R}$, $\phi(t)=\left|t-\frac{m+M}{2}\right|$, then we deduce $\frac{M-m}{2}\leq C(M-m)$, which implies that $C\geq\frac{1}{2}$. □

Corollary 2. *If $\phi:[m,M]\to\mathbb{R}$ is L_1-Lipschitzian on $[m,t]$ and L_2-Lipschitzian on $[t,M]$, $L_1,L_2>0$, then*

$$|\Phi_{\phi}(t)|\leq\frac{(M-t)(t-m)}{M-m}(L_1+L_2)\leq\frac{1}{4}(M-m)(L_1+L_2) \tag{2.12}$$

for any $t\in[m,M]$.

In particular, if ϕ is L-Lipschitzian on $[m,M]$, then

$$|\Phi_{\phi}(t)|\leq\frac{2(M-t)(t-m)}{M-m}L\leq\frac{1}{2}(M-m)L. \tag{2.13}$$

The constants $\frac{1}{4}$, 2 and $\frac{1}{2}$ are best possible.

The proof is obvious by Lemma 3 on taking into account that any L-Lipschitzian function is of bounded variation and $\bigvee_{m}^{M}(\phi)\leq(M-m)L$. The sharpness of the constants follows by choosing the function $\phi:[m,M]\to\mathbb{R}$, $\phi(t)=\left|t-\frac{m+M}{2}\right|$ which is Lipschitzian with $L=1$.

The following lemma may be stated (see also [9]).

Lemma 4. *Let $u:[m,M]\to\mathbb{R}$ and $\delta,\Delta\in\mathbb{R}$ with $\Delta>\delta$. The following statements are equivalent:*

(i) *The function $u-\frac{\delta+\Delta}{2}\ell$, where $\ell(t)=t$, $t\in[m,M]$ is $\frac{1}{2}(\Phi-\delta)$-Lipschitzian;*

(ii) *We have the inequality:*

$$\delta \leq \frac{u(t) - u(s)}{t - s} \leq \Delta \quad \text{for each } t, \ s \in [m, M] \quad \text{with } t \neq s. \tag{2.14}$$

(iii) *We have the inequality:*

$$\delta (t - s) \leq u(t) - u(s) \leq \Delta (t - s) \quad \text{for each } t, \ s \in [m, M] \quad \text{with } t > s. \tag{2.15}$$

Following [19], we can introduce the concept:

Definition 1. *The function $u : [m, M] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i) – (iii) is said to be (δ, Δ)-Lipschitzian on $[m, M]$.*

Notice that in [19], the definition was introduced on utilising the statement (iii) and only the equivalence "(i) $\iff$ (iii)" was considered.

Utilising Lagrange's mean value theorem, we can state the following result that provides practical examples of (δ, Δ)-Lipschitzian functions:

Proposition 1. *Let $u : [m, M] \to \mathbb{R}$ be continuous on $[m, M]$ and differentiable on $[m, M]$. If*

$$-\infty < \delta = \inf_{t \in (m,M)} u'(t), \sup_{t \in (m,M)} u'(t) = \Delta < \infty,$$

then u is (Δ, δ)-Lipschitzian on $[m, M]$.

Corollary 3. *Assume that $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some $\delta, \ \Delta \in \mathbb{R}$ with $\Delta > \delta$. Then*

$$|\Phi_\phi(t)| \leq \frac{(M - t)(t - m)}{M - m} (\Phi - \delta) \leq \frac{1}{4} (M - m)(\Phi - \delta). \tag{2.16}$$

Proof. It follows by Corollary 2 for the function $\phi - \frac{\delta+\Delta}{2}\ell$ that is $\frac{1}{2}(\Phi - \delta)$-Lipschitzian and taking into account that $\Phi_{\phi - \frac{\delta+\Delta}{2}\ell}(t) = \Phi_\phi(t)$ for any $t \in [m, M]$. □

Corollary 4. *If $\phi : [m, M] \to \mathbb{R}$ is monotonic nondecreasing on $[m, M]$, then*

$$|\Phi_\phi(t)| \leq \left(\frac{M - t}{M - m}\right)[\phi(t) - \phi(m)] + \left(\frac{t - m}{M - m}\right)[\phi(M) - \phi(t)] \tag{2.17}$$

$$\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{t - \frac{m+M}{2}}{M - m}\right|\right][\phi(M) - \phi(m)]; \\ \\ \left[\left(\frac{M-t}{M-m}\right)^p + \left(\frac{t-m}{M-m}\right)^p\right]^{\frac{1}{p}} \\ \times \left[\left[\frac{\phi(t)-\phi(m)}{M-m}\right]^q + \left[\frac{\phi(M)-\phi(t)}{M-m}\right]^q\right]^{\frac{1}{q}} \\ \text{if } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \\ \frac{1}{2}[\phi(M) - \phi(m)] + \frac{1}{2}\left|\phi(t) - \frac{\phi(m)+\phi(M)}{2}\right|. \end{cases}$$

The first inequality and the constant $\frac{1}{2}$ in the first branch of the second inequality are sharp.

The inequality is obvious from (2.10). For $t = \frac{m+M}{2}$, we get in (2.17)

$$\left|\phi\left(\frac{m + M}{2}\right) - \frac{\phi(m) + \phi(M)}{2}\right| \leq \frac{1}{2}[\phi(M) - \phi(m)]. \tag{2.18}$$

In (2.18), the constant $\frac{1}{2}$ is sharp since for the monotonic nondecreasing function $\phi : [m, M] \to \mathbb{R}$

$$\phi(t) = \begin{cases} 0 & \text{if } t \in \left[m, \frac{m+M}{2}\right], \\ \\ 1 & \text{if } t \in \left(\frac{m+M}{2}, M\right], \end{cases}$$

we obtain in both sides of (2.18) the same quantity $\frac{1}{2}$.

For a $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ of bounded variation and $f : E \to [m, M]$, we define

$$\bigvee_m^f (\phi)(s) := \bigvee_m^{f(s)} (\phi) \text{ for } s \in E.$$

We have:

Theorem 5. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and $f : E \to [m, M]$ such that $\phi \circ f$, f, $\bigvee_m^f (\phi)$, $(M - f) \bigvee_m^f (\phi)$, $(f - m) \bigvee_f^M (\phi) \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \tag{2.19}$$
$$\leq \frac{M - A(f)}{M - m} A\left(\bigvee_m^f (\phi)\right) + \frac{A(f) - m}{M - m} A\left(\bigvee_f^M (\phi)\right)$$
$$\leq \begin{cases} \left[\frac{1}{2} + \left|\frac{A(f) - \frac{m+M}{2}}{M - m}\right|\right] \bigvee_m^M (\phi), \\ \\ \left[\left(\frac{M - A(f)}{M - m}\right)^p + \left(\frac{A(f) - m}{M - m}\right)^p\right]^{1/p} \\ \times \left[\left(\frac{A\left(\bigvee_m^f (\phi)\right)}{M - m}\right)^q + \left(\frac{A\left(\bigvee_f^M (\phi)\right)}{M - m}\right)^q\right]^{1/q} \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2}\left[\bigvee_m^M (\phi) + \left|A\left(\bigvee_m^f (\phi)\right) - A\left(\bigvee_f^M (\phi)\right)\right|\right]. \end{cases}$$

Proof. Using Lemma 3 we have in the order of L that

$$-\left[\left(\frac{M - f}{M - m}\right) \bigvee_m^f (\phi) + \left(\frac{f - m}{M - m}\right) \bigvee_f^M (\phi)\right] \tag{2.20}$$
$$\leq \frac{M - f}{M - m} \phi(m) + \frac{f - m}{M - m} \phi(M) - \phi \circ f$$
$$\leq \left(\frac{M - f}{M - m}\right) \bigvee_m^f (\phi) + \left(\frac{f - m}{M - m}\right) \bigvee_f^M (\phi)$$

for $f : E \to [m, M]$.

If we take in this inequality the functional A and use its properties, then we get

$$\begin{aligned} &\left|\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi\circ f)\right| \\ &\leq \frac{1}{M-m}\left[A\left((M-f)\bigvee_m^f(\phi)\right)+A\left((f-m)\bigvee_f^M(\phi)\right)\right]. \end{aligned} \tag{2.21}$$

We use the Čebyšev's inequality for positive functionals, namely

$$A(hg)\geq(\leq)A(h)A(g) \tag{2.22}$$

where h, g are *synchronous (asynchronous)* on E, namely

$$(h(t)-h(s))(g(t)-g(s))\geq(\leq)0$$

for all s, $t\in E$, where $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional and h, g, $hg\in L$.

Since the function $[m,M]\ni t\mapsto\bigvee_m^t(\phi)$ is increasing on $[m,M]$, then $M-f$ and $\bigvee_m^f(\phi)$ are asynchronous on E and by (2.22) we have

$$\begin{aligned} A\left((M-f)\bigvee_m^f(\phi)\right) &\leq A(M-f)A\left(\bigvee_m^f(\phi)\right) \\ &= (M-A(f))A\left(\bigvee_m^f(\phi)\right). \end{aligned} \tag{2.23}$$

Also, because $[m,M]\ni t\mapsto\bigvee_t^M(\phi)$ is decreasing on $[m,M]$, then $f-m$ and $\bigvee_f^M(\phi)$ are asynchronous on E and by (2.22) we have

$$\begin{aligned} A\left((f-m)\bigvee_f^M(\phi)\right) &\leq A(f-m)A\left(\bigvee_f^M(\phi)\right) \\ &= (A(f)-m)A\left(\bigvee_f^M(\phi)\right). \end{aligned} \tag{2.24}$$

Using (2.21), (2.23) and (2.24) we get the first inequality in (2.19).

The last part is obvious by Hölder's inequality

$$\begin{aligned} cd+uv &\leq \begin{cases} \max\{c,u\}(d+v), \\ \\ (c^p+u^p)^{1/p}(d^q+v^q)^{1/q}, \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \end{cases} \\ &= \begin{cases} \left[\frac{1}{2}(c+d)+\frac{1}{2}|c-d|\right](d+v), \\ \\ (c^p+u^p)^{1/p}(d^q+v^q)^{1/q}, \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases} \end{aligned}$$

□

Corollary 5. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and $e : E \to [m, M]$ such that $\phi \circ e$, e, $\bigvee_m^e (\phi)$, $(M - e) \bigvee_m^e (\phi)$, $(e - m) \bigvee_e^M (\phi) \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then*

$$\left| \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \right| \le \frac{1}{2} \bigvee_m^M (\phi) . \tag{2.25}$$

The constant $\frac{1}{2}$ in the right hand side of (2.25) is best possible.

Proof. Now, assume that the inequality (2.25) is valid with a positive multiplicative constant C in the right hand side, namely

$$\left| \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \right| \le C \bigvee_m^M (\phi) . \tag{2.26}$$

Consider the function of bounded variation $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ given by

$$\phi(t) = \begin{cases} \frac{M-m}{2} \text{ if } t = m, \\ 0 \text{ if } t \in (m, M), \\ \frac{M-m}{2} \text{ if } t = m. \end{cases}$$

We have then

$$\bigvee_m^M (\phi) = M - m.$$

If we write the inequality (2.26) for the isotonic linear and normalised functional

$$A(f) := \frac{1}{M - m} \int_m^M f(t)\, dt$$

and for $e = \ell$, the identity function for the interval $[m, M]$, i.e. $\ell(t) = t$, $t \in [m, M]$,for which $A(e) = \frac{m+M}{2}$, then we get $\frac{M-m}{2} \le C(M - m)$ implying that $C \ge \frac{1}{2}$, so $\frac{1}{2}$ is best possible in (2.25). □

Remark 2. *We observe that, if f is monotonic nondecreasing, then it is of bounded variation and by (2.19) we get*

$$\begin{aligned} &\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \\ &\le \frac{M - A(f)}{M - m} [A(\phi \circ f) - \phi(m)] + \frac{A(f) - m}{M - m} [\phi(M) - A(\phi \circ f)] \\ &\le \begin{cases} \left[\frac{1}{2} + \left| \frac{A(f) - \frac{m+M}{2}}{M-m} \right| \right] (\phi(M) - \phi(m)), \\ \\ \left[\left(\frac{M-A(f)}{M-m} \right)^p + \left(\frac{A(f)-m}{M-m} \right)^p \right]^{1/p} \\ \times \left[\left(\frac{A(\phi \circ f)-\phi(m)}{M-m} \right)^q + \left(\frac{\phi(M)-A(\phi \circ f)}{M-m} \right)^q \right]^{1/q} \\ \text{if } p,\ q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{\phi(M)-\phi(m)}{2} + \left| A(\phi \circ f) - \frac{\phi(m)+\phi(M)}{2} \right| . \end{cases} \end{aligned} \tag{2.27}$$

Theorem 6. *If $\phi : [m, M] \to \mathbb{R}$ is L-Lipschitzian on $[m, M]$ and $f : E \to [m, M]$ is such that $\phi \circ f$, f, $(M - f)(f - m) \in L$ then for $A : L \to \mathbb{R}$, an isotonic linear and normalised functional,*

$$\begin{aligned} &\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \\ &\leq \frac{2}{M - m} LA[(M - f)(f - m)] \leq \frac{2}{M - m} L(M - A(f))(A(f) - m) \\ &\leq \frac{1}{2} L(M - m). \end{aligned} \tag{2.28}$$

Proof. Using the inequality (2.13) we have in the order of L that

$$\begin{aligned} -\frac{2(M - f)(f - m)}{M - m} L &\leq \frac{M - f}{M - m} \phi(m) + \frac{f - m}{M - m} \phi(M) - \phi \circ f \\ &\leq \frac{2(M - f)(f - m)}{M - m} L. \end{aligned}$$

By taking the functional A in this inequality we get

$$\begin{aligned} &-\frac{2}{M - m} LA[(M - f)(f - m)] \\ &\leq \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \\ &\leq \frac{2}{M - m} LA[(M - f)(f - m)], \end{aligned}$$

which is equivalent to the first inequality in (2.28).

The function $g(t) := (M - t)(t - m)$ is concave on $[m, M]$. By using Jessen's inequality for the concave function g we get

$$A[(M - f)(f - m)] \leq (M - A(f))(A(f) - m)$$

that proves the second inequality in (2.28). □

Corollary 6. *If $\phi : [m, M] \to \mathbb{R}$ is L-Lipschitzian on $[m, M]$ and $e : E \to [m, M]$ is such that $\phi \circ e$, e, $(M - e)(f - e) \in L$ then for $A : L \to \mathbb{R}$, an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$*

$$\begin{aligned} \left| \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \right| &\leq \frac{2}{M - m} LA[(M - e)(e - m)] \\ &\leq \frac{1}{2} L(M - m). \end{aligned} \tag{2.29}$$

Remark 3. *Assume that $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some δ, $\Delta \in \mathbb{R}$ with $\Delta > \delta$. Then by the inequality (2.28) we get*

$$\begin{aligned} &\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \\ &\leq \frac{\Phi - \delta}{M - m} A[(M - f)(f - m)] \leq \frac{\Phi - \delta}{M - m} (M - A(f))(A(f) - m) \\ &\leq \frac{1}{4} (M - m)(\Phi - \delta), \end{aligned} \tag{2.30}$$

while from (2.29) we get

$$\left|\frac{\phi(m)+\phi(M)}{2}-A(\phi\circ e)\right| \le \frac{\Phi-\delta}{M-m}A[(M-e)(e-m)] \le \frac{1}{4}(M-m)(\Phi-\delta). \tag{2.31}$$

3. Bounds for Lower and Upper Convex Functions

We have the following result for convex functions [10]:

Lemma 5. *If $\phi:[m,M]\to\mathbb{R}$ is a convex function on $[m,M]$, then*

$$0\le\Phi_\phi(t)\le\frac{(M-t)(t-m)}{M-m}\left[\phi'_-(M)-\phi'_+(m)\right] \le\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right] \tag{3.1}$$

for any $t\in[m,M]$.

If the lateral derivatives $\phi'_-(M)$ and $\phi'_+(m)$ are finite, then the second inequality and the constant $\frac{1}{4}$ are sharp.

Proof. For the sake of completeness, we present a complete proof of (3.1) below.

Since ϕ is convex, then

$$\frac{t-m}{M-m}\phi(M)+\frac{M-t}{M-m}\phi(m)\ge\phi\left[\frac{(M-t)m+(t-m)M}{M-m}\right]=\phi(t)$$

for any $t\in[m,M]$, i.e., $\Phi(t)\ge 0$ for any $t\in[m,M]$.

If either $\phi'_-(M)$ or $\phi'_+(m)$ are infinite, then the last part of (3.1) is obvious.

Suppose that $\phi'_-(M)$ and $\phi'_+(m)$ are finite. Then, by the convexity of ϕ we have $\phi(t)-\phi(M)\ge\phi'_-(M)(t-M)$ for any $t\in(m,M)$. If we multiply this inequality with $t-m\ge 0$, we deduce

$$(t-m)\phi(t)-(t-m)\phi(M)\ge\phi'_-(M)(t-M)(t-m),\quad t\in(m,M). \tag{3.2}$$

Similarly, we get

$$(M-t)\phi(t)-(M-t)\phi(m)\ge\phi'_+(m)(t-m)(M-t),\quad t\in(m,M). \tag{3.3}$$

Adding (3.2) to (3.3) and dividing by $M-m$, we deduce

$$\phi(t)-\frac{(t-m)\phi(M)+(M-t)\phi(m)}{M-m}\ge\frac{(M-t)(t-m)}{M-m}\left[\phi'_-(M)-\phi'_+(m)\right],$$

for any $t\in(m,M)$, which proves the second inequality for $t\in(m,M)$.

If $t=m$ or $t=M$, the inequality also holds.

Now, assume that (3.1) holds with D and E greater than zero, i.e.,

$$\Phi_\phi(t)\le D\cdot\frac{(M-t)(t-m)}{M-m}\left[\phi'_-(M)-\phi'_+(m)\right] \le E(M-m)\left[\phi'_-(M)-\phi'_+(m)\right]$$

for any $t\in[m,M]$. If we choose $t=\frac{m+M}{2}$, then we get

$$\frac{\phi(m)+\phi(M)}{2}-\phi\left(\frac{m+M}{2}\right)\le\frac{1}{4}D(M-m)\left[\phi'_-(M)-\phi'_+(m)\right] \le E(M-m)\left[\phi'_-(M)-\phi'_+(m)\right]. \tag{3.4}$$

Consider $\phi : [m, M] \to \mathbb{R}$, $\phi(t) = \left| t - \frac{m+M}{2} \right|$. Then ϕ is convex, $\phi(m) = \phi(M) = \frac{M-m}{2}$, $\phi\left(\frac{m+M}{2}\right) = 0$, $\phi'_{-}(M) = 1$, $\phi'_{+}(m) = -1$ and by (3.4) we deduce

$$\frac{M-m}{2} \leq \frac{1}{2} D (M-m) \leq 2E (M-m),$$

which implies that $D \geq 1$ and $E \geq \frac{1}{4}$. $\square$

Let $(X, \|\cdot\|)$ be a real or complex normed linear space, $C \subseteq X$ a convex subset of X and $\phi : C \to \mathbb{R}$. As in [5] we can introduce the following concepts. Let ψ, $\Psi \in \mathbb{R}$. The mapping ϕ will be called *ψ-lower convex* on C if $\phi - \frac{\psi}{2} \|\cdot\|^2$ is a convex mapping on C. The mapping ϕ will be called *Ψ-upper convex* on C if $\frac{\Psi}{2} \|\cdot\|^2 - \phi$ is a convex mapping on C. The mapping ϕ will be called *(ψ, Ψ)- convex* on C if it is both ψ-lower convex and Ψ-upper convex on C. Note that if ϕ is (ψ, Ψ)-convex on C, then $\psi \leq \Psi$. Further, assume that c is a positive constant. A function $\phi : C \to \mathbb{R}$ is called: *strongly convex with modulus c* if

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y) - ct(1-t)\|x - y\|^2 \tag{3.5}$$

for all $x, y \in C$ and $t \in [0, 1]$. Also, it is called: *strongly Jensen-convex with modulus c* if (3.5) is assumed only for $t = \frac{1}{2}$, that is

$$\phi\left(\frac{x+y}{2}\right) \leq \frac{\phi(x) + \phi(y)}{2} - \frac{c}{4}\|x - y\|^2, \text{ for all } x,\ y \in C. \tag{3.6}$$

The usual concepts of convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions have been introduced by Polyak [26] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature. Let us only mention the paper [20], which is a survey article devoted to strongly convex functions and related classes of functions.

Denote by $\mathcal{SC}(C, c)$ the class of all functions $\phi : C \to \mathbb{R}$ strongly convex with modulus c and by $\mathcal{LC}(C, \psi)$ the class of all functions $\phi : C \to \mathbb{R}$ that are ψ-lower convex. It is known that [21], if X is an inner product space then $\mathcal{SC}(C, \frac{1}{2}\psi) = \mathcal{LC}(C, \psi)$. *A fortiori,* if the function is defined on an interval of real numbers I, then $\mathcal{SC}(I, \frac{\psi}{2}) = \mathcal{LC}(I, \psi)$ where the norm here is the modulus. However, in arbitrary normed spaces the above classes differ in general [15].

If the function $\phi : I \to \mathbb{R}$ defined on an interval of real numbers I is twice differentiable on the interior of I, denoted $\mathring{I}$, then the ψ-lower convexity is equivalent to $\phi''(t) \geq \psi$ for any $t \in \mathring{I}$ while the Ψ-upper convexity is equivalent to $\Psi \geq \phi''(t)$ for any $t \in \mathring{I}$.

Lemma 6. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ and ψ, $\Psi \in \mathbb{R}$.*

(i) If ϕ is ψ-lower convex on $[m, M]$, then

$$\begin{aligned} &\frac{1}{2}\psi (M-t)(t-m) \\ &\leq \Phi_{\phi}(t) \leq (M-t)(t-m)\left[\frac{\phi'_{-}(M) - \phi'_{+}(m)}{M-m} - \frac{1}{2}\psi\right] \\ &\leq \frac{1}{2}\psi (M-t)(t-m) + \frac{1}{4}(M-m)^2\left[\frac{\phi'_{-}(M) - \phi'_{+}(m)}{M-m} - \psi\right]; \end{aligned} \tag{3.7}$$

(ii) If ϕ is Ψ-upper convex on $[m,M]$, then

$$\begin{aligned}
(3.8)\qquad & \frac{1}{2}\Psi(M-t)(t-m)-\frac{1}{4}(M-m)^2\left[\Psi-\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\right] \\
&\leq\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\Psi\right](M-t)(t-m) \\
&\leq\Phi_\phi(t)\leq\frac{1}{2}\Psi(M-t)(t-m);
\end{aligned}$$

(iii) If ϕ is(ψ,Ψ)-convex on $[m,M]$ (with $\psi<\Psi$), then both inequalities (3.7) and (3.8) hold simultaneously.

Proof. Observe that

$$\begin{aligned}
(3.9)\qquad \Phi_{\ell^2}(t) &:= \frac{M-t}{M-m}\ell^2(m)+\frac{t-m}{M-m}\ell^2(M)-\ell^2(t) \\
&= \frac{(M-t)m^2+(t-m)M^2-(M-m)t^2}{M-m} \\
&= \frac{Mm^2-mM^2+\left(M^2-m^2\right)t-(M-m)t^2}{M-m}=(M-t)(t-m)
\end{aligned}$$

for any $t\in[m,M]$.

We also have

$$\frac{\left(\ell^2\right)'_-(M)-\left(\ell^2\right)'_+(m)}{M-m}=2.$$

(i). If ϕ is ψ-*lower convex* on $[m,M]$, then the function $\phi_\psi:=\phi-\frac{1}{2}\psi\ell^2$ is convex on $[m,M]$ and by (3.1) we get

$$\begin{aligned}
(3.10)\qquad 0\leq\Phi_{\phi_\psi}(t) &\leq\frac{(M-t)(t-m)}{M-m}\left[\phi'_{\psi-}(M)-\phi'_{\psi+}(m)\right] \\
&\leq\frac{1}{4}(M-m)\left[\phi'_{\psi-}(M)-\phi'_{\psi+}(m)\right]
\end{aligned}$$

for any $t\in[m,M]$.

Since

$$\Phi_{\phi_\psi}(t)=\Phi_\phi(t)-\frac{1}{2}\psi\Phi_{\ell^2}(t)=\Phi_\phi(t)-\frac{1}{2}\psi(M-t)(t-m)$$

and

$$\begin{aligned}
\frac{\phi'_{\psi-}(M)-\phi'_{\psi+}(m)}{M-m} &= \frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\frac{\left(\ell^2\right)'_-(M)-\left(\ell^2\right)'_+(m)}{M-m} \\
&= \frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\psi,
\end{aligned}$$

then by (3.10) we get

$$\begin{aligned}
(3.11)\qquad & 0\leq\Phi_\phi(t)-\frac{1}{2}\psi(M-t)(t-m) \\
&\leq(M-t)(t-m)\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\psi\right] \\
&\leq\frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\psi\right]
\end{aligned}$$

for any $t\in[m,M]$. This inequality is of interest in itself.

If we add $\frac{1}{2}\psi(M-t)(t-m)$ in all terms of (3.11) we get (3.7).

(ii). If ϕ is Ψ-*upper convex* on $[m,M]$, then the function $\phi_{\Psi} := \frac{1}{2}\Psi\ell^2 - \phi$ is convex on $[m,M]$ and by (3.1) we get

$$\begin{aligned} 0 &\le \Phi_{\phi_{\Psi}}(t) \le \frac{(M-t)(t-m)}{M-m}\left[\phi'_{\Psi-}(M) - \phi'_{\Psi+}(m)\right] \\ &\le \frac{1}{4}(M-m)\left[\phi'_{\Psi-}(M) - \phi'_{\Psi+}(m)\right] \end{aligned} \tag{3.12}$$

for any $t \in [m,M]$.

Since

$$\Phi_{\phi_{\Psi}}(t) = \frac{1}{2}\Psi\Phi_{\ell^2}(t) - \Phi_{\phi}(t) = \frac{1}{2}\Psi(M-t)(t-m) - \Phi_{\phi}(t)$$

and

$$\begin{aligned} \frac{\phi'_{\Psi-}(M) - \phi'_{\Psi+}(m)}{M-m} &= \frac{1}{2}\Psi\frac{\left(\ell^2\right)'_-(M) - \left(\ell^2\right)'_+(m)}{M-m} - \frac{\phi'_-(M) - \phi'_+(m)}{M-m} \\ &= \Psi - \frac{\phi'_-(M) - \phi'_+(m)}{M-m}, \end{aligned}$$

then by (3.12) we get

$$\begin{aligned} 0 &\le \frac{1}{2}\Psi(M-t)(t-m) - \Phi_{\phi}(t) \\ &\le (M-t)(t-m)\left[\Psi - \frac{\phi'_-(M) - \phi'_+(m)}{M-m}\right] \\ &\le \frac{1}{4}(M-m)^2\left[\Psi - \frac{\phi'_-(M) - \phi'_+(m)}{M-m}\right] \end{aligned} \tag{3.13}$$

for any $t \in [m,M]$. This inequality is of interest in itself and is equivalent to (3.8). □

Corollary 7. *With the assumptions of Lemma 6 and if ϕ is ψ-lower convex on $[m,M]$, then*

$$\begin{aligned} \frac{1}{8}\psi(M-m)^2 &\le \frac{\phi(m) + \phi(M)}{2} - \phi\left(\frac{m+M}{2}\right) \\ &\le \frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M) - \phi'_+(m)}{M-m} - \frac{1}{2}\psi\right]. \end{aligned} \tag{3.14}$$

If ϕ is Ψ-upper convex on $[m,M]$, then

$$\begin{aligned} &\frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M) - \phi'_+(m)}{M-m} - \frac{1}{2}\Psi\right] \\ &\le \frac{\phi(m) + \phi(M)}{2} - \phi\left(\frac{m+M}{2}\right) \le \frac{1}{8}\Psi(M-m)^2; \end{aligned} \tag{3.15}$$

If ϕ is(ψ,Ψ)-convex on $[m,M]$ (with $\psi < \Psi$), then both inequalities (3.14) and (3.15) hold simultaneously.

We have:

Theorem 7. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$, ψ, $\Psi \in \mathbb{R}$, $f : E \to [m, M]$ such that $\phi \circ f$, $f \in L$ and $A : L \to \mathbb{R}$ an isotonic linear and normalised functional on L.*

(i) If ϕ is ψ-lower convex on $[m, M]$, then

$$\begin{aligned}
(3.16) \qquad & \frac{1}{2}\psi A\left[(M-f)(f-m)\right] \\
& \leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi \circ f) \\
& \leq \left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - \frac{1}{2}\psi\right] A\left[(M-f)(f-m)\right] \\
& \leq \frac{1}{2}\psi A\left[(M-f)(f-m)\right] + \frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - \psi\right];
\end{aligned}$$

(ii) If ϕ is Ψ-upper convex on $[m, M]$, then

$$\begin{aligned}
(3.17) \qquad & \frac{1}{2}\Psi A\left[(M-f)(f-m)\right] - \frac{1}{4}(M-m)^2\left[\Psi - \frac{\phi'_-(M)-\phi'_+(m)}{M-m}\right] \\
& \leq \left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - \frac{1}{2}\Psi\right] A\left[(M-f)(f-m)\right] \\
& \leq \frac{M-A(f)}{M-m}\phi(m) + \frac{A(f)-m}{M-m}\phi(M) - A(\phi \circ f) \\
& \leq \frac{1}{2}\Psi A\left[(M-f)(f-m)\right];
\end{aligned}$$

(iii) If ϕ is(ψ, Ψ)-convex on $[m, M]$ (with $\psi < \Psi$), then both inequalities (3.16) and (3.17) hold simultaneously.

Proof. Follows by making use of Lemma 6 and utilising the monotonicity, linearity and normality of the functional A. We omit the details. $\square$

Corollary 8. *Let $e : E \to [m, M]$ with e, e^2, $\phi \circ e \in L$. If $A \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then*

$$\begin{aligned}
(3.18) \qquad & \frac{1}{2}\psi\left[\frac{1}{2}\left(m^2+M^2\right) - A\left(e^2\right)\right] \\
& \leq \frac{\phi(m)+\phi(M)}{2} - A(\phi \circ e) \\
& \leq \left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m} - \frac{1}{2}\psi\right]\left[\frac{1}{2}\left(m^2+M^2\right) - A\left(e^2\right)\right] \\
& \leq \frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right] + \frac{1}{2}\psi\left[mM - A\left(e^2\right)\right]
\end{aligned}$$

if ϕ is ψ-lower convex on $[m,M]$ and

$$\begin{aligned}
&\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right]+\frac{1}{2}\Psi\left[mM-A\left(e^2\right)\right] \\
&\leq\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\Psi\right]\left[\frac{1}{2}\left(m^2+M^2\right)-A\left(e^2\right)\right] \\
&\leq\frac{\phi(m)+\phi(M)}{2}-A(\phi\circ e) \\
&\leq\frac{1}{2}\Psi\left[\frac{1}{2}\left(m^2+M^2\right)-A\left(e^2\right)\right]
\end{aligned}\tag{3.19}$$

if ϕ is Ψ-upper convex on $[m,M]$.

If ϕ is (ψ,Ψ)-convex on $[m,M]$ (with $\psi<\Psi$), then both inequalities (3.18) and (3.19) hold simultaneously.

Remark 4. *Since the function $g(t):=(M-t)(t-m)$ is concave on $[m,M]$, then by using Jessen's inequality for the concave function g we get*

$$A[(M-f)(f-m)]\leq(M-A(f))(A(f)-m).$$

If ϕ is ψ-lower convex on $[m,M]$ with $\psi\geq0$, then by (3.16),

$$\begin{aligned}
0&\leq\frac{1}{2}\psi A[(M-f)(f-m)] \\
&\leq\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi\circ f) \\
&\leq\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right]A[(M-f)(f-m)] \\
&\leq\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right](M-A(f))(A(f)-m) \\
&\leq\frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right].
\end{aligned}\tag{3.20}$$

If $\psi=0$, namely ϕ is convex, then we obtain from (3.20) the following reverse of Beesack-Pečarić result established in [16]

$$\begin{aligned}
0&\leq\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi\circ f) \\
&\leq\frac{\phi'_-(M)-\phi'_+(m)}{M-m}A[(M-f)(f-m)] \\
&\leq\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}\right](M-A(f))(A(f)-m) \\
&\leq\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right].
\end{aligned}\tag{3.21}$$

This inequality was obtained for the discrete case in 2008, see [10, Proposition 8.2].

If ϕ is Ψ-upper convex on $[m,M]$ with $\Psi \geq 0$, then by (3.17),

$$\begin{aligned} &\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-A(\phi\circ f) \\ &\leq \frac{1}{2}\Psi A[(M-f)(f-m)] \leq \frac{1}{2}\Psi(M-A(f))(A(f)-m) \\ &\leq \frac{1}{8}\Psi(M-m)^2. \end{aligned} \tag{3.22}$$

Moreover, if $e: E \to [m,M]$ with $e,\ e^2,\ \phi\circ e \in L$ and $A \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e)=\frac{m+M}{2}$, then by (3.20) we get

$$\begin{aligned} 0 &\leq \frac{1}{2}\psi A[(M-e)(e-m)] \leq \frac{\phi(m)+\phi(M)}{2}-A(\phi\circ e) \\ &\leq \left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right]A[(M-e)(e-m)] \\ &\leq \frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right] \end{aligned} \tag{3.23}$$

provided ϕ is ψ-lower convex on $[m,M]$ with $\psi \geq 0$, and by (3.22)

$$\frac{\phi(m)+\phi(M)}{2}-A(\phi\circ e) \leq \frac{1}{2}\Psi A[(M-e)(e-m)] \leq \frac{1}{8}\Psi(M-m)^2, \tag{3.24}$$

provided ϕ is Ψ-upper convex on $[m,M]$ with $\Psi \geq 0$.

4. Applications for Jessen's Inequality

We have the following reverse of Jessen's inequality:

Theorem 8. *Assume that $\phi:[m,M]\to\mathbb{R}$ is convex and $f:E\to[m,M]$ such that $\phi\circ f,\ f\in L$. If $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$0 \leq A(\phi\circ f)-\phi(A(f)) \leq \left(\frac{M-A(f)}{M-m}\right)\bigvee_m^{A(f)}(\phi)+\left(\frac{A(f)-m}{M-m}\right)\bigvee_{A(f)}^{M}(\phi) \tag{4.1}$$

$$\leq \begin{cases} \left[\frac{1}{2}+\left|\frac{A(f)-\frac{m+M}{2}}{M-m}\right|\right]\bigvee_m^M(\phi); \\ \\ \left[\left(\frac{M-A(f)}{M-m}\right)^p+\left(\frac{A(f)-m}{M-m}\right)^p\right]^{\frac{1}{p}}\left[\left(\frac{\bigvee_m^{A(f)}(\phi)}{M-m}\right)^q+\left(\frac{\bigvee_{A(f)}^M(\phi)}{M-m}\right)^q\right]^{\frac{1}{q}} \\ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{2}\bigvee_m^M(\phi)+\frac{1}{2}\left|\bigvee_m^{A(f)}(\phi)-\bigvee_{A(f)}^M(\phi)\right|. \end{cases}$$

Proof. By the inequality (1.2) we have

$$0 \leq A(\phi\circ f)-\phi(A(f)) \leq \frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-\phi(A(f)). \tag{4.2}$$

If we write the inequality (2.10) for $t = A(f) \in [m, M]$, then we have

$$
\begin{aligned}
(4.3)\qquad & \frac{M - A(f)}{M - m}\phi(m) + \frac{A(f) - m}{M - m}\phi(M) - \phi(A(f)) \\
&\leq \left(\frac{M - A(f)}{M - m}\right) \bigvee_{m}^{A(f)}(\phi) + \left(\frac{A(f) - m}{M - m}\right) \bigvee_{A(f)}^{M}(\phi) \\
&\leq \begin{cases}
\left[\frac{1}{2} + \left|\frac{A(f) - \frac{m+M}{2}}{M-m}\right|\right] \bigvee_{m}^{M}(\phi); \\[2ex]
\left[\left(\frac{M-A(f)}{M-m}\right)^{p} + \left(\frac{A(f)-m}{M-m}\right)^{p}\right]^{\frac{1}{p}} \left[\left(\frac{\bigvee_{m}^{A(f)}(\phi)}{M-m}\right)^{q} + \left(\frac{\bigvee_{A(f)}^{M}(\phi)}{M-m}\right)^{q}\right]^{\frac{1}{q}} \\
\text{if } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\frac{1}{2}\bigvee_{m}^{M}(\phi) + \frac{1}{2}\left|\bigvee_{m}^{A(f)}(\phi) - \bigvee_{A(f)}^{M}(\phi)\right|.
\end{cases}
\end{aligned}
$$

By making use of (4.2) and (4.3) we get (4.1). □

Corollary 9. *Assume that $\phi : [m, M] \to \mathbb{R}$ is convex and $e : E \to [m, M]$ such that $\phi \circ e$, $e \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then*

$$
(4.4)\qquad 0 \leq A(\phi \circ e) - \phi\left(\frac{m+M}{2}\right) \leq \frac{1}{2}\bigvee_{m}^{M}(\phi).
$$

We observe that if $\phi : [m, M] \to \mathbb{R}$ is convex and monotonic on $[m, M]$, then by (4.1) we have

$$
\begin{aligned}
(4.5)\quad & 0 \leq A(\phi \circ f) - \phi(A(f)) \\
&\leq \left(\frac{M - A(f)}{M - m}\right)|\phi(A(f)) - \phi(m)| + \left(\frac{A(f) - m}{M - m}\right)|\phi(A(f)) - \phi(M)| \\
&\leq \begin{cases}
\left[\frac{1}{2} + \left|\frac{A(f) - \frac{m+M}{2}}{M-m}\right|\right] |\phi(M) - \phi(m)|; \\[2ex]
\left[\left(\frac{M-A(f)}{M-m}\right)^{p} + \left(\frac{A(f)-m}{M-m}\right)^{p}\right]^{\frac{1}{p}} \\
\times \left[\left(\frac{|\phi(A(f)) - \phi(m)|}{M-m}\right)^{q} + \left(\frac{|\phi(A(f)) - \phi(M)|}{M-m}\right)^{q}\right]^{\frac{1}{q}} \\
\text{if } \ p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\[2ex]
\frac{1}{2}|\phi(M) - \phi(m)| + \left|\phi(A(f)) - \frac{\phi(M) + \phi(m)}{2}\right|
\end{cases}
\end{aligned}
$$

and by (4.4) we have

$$
(4.6)\qquad 0 \leq A(\phi \circ e) - \phi\left(\frac{m+M}{2}\right) \leq \frac{1}{2}|\phi(M) - \phi(m)|.
$$

Using the inequality (3.1) for $t=A(f)\in[m,M]$, we have

$$\begin{aligned}(4.7)\qquad &\frac{M-A(f)}{M-m}\phi(m)+\frac{A(f)-m}{M-m}\phi(M)-\phi(A(f))\\ &\leq\frac{(M-A(f))(A(f)-m)}{M-m}\left[\phi'_-(M)-\phi'_+(m)\right]\\ &\leq\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right].\end{aligned}$$

This inequality together with (4.2) provides the following reverse of Jessen's inequality established in [16]

$$\begin{aligned}(4.8)\qquad 0&\leq A(\phi\circ f)-\phi(A(f))\leq\frac{(M-A(f))(A(f)-m)}{M-m}\left[\phi'_-(M)-\phi'_+(m)\right]\\ &\leq\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right].\end{aligned}$$

The integral version of this inequality was obtained in 2011 in [11].

If $\phi:[m,M]\to\mathbb{R}$ is convex and $e:E\to[m,M]$ such that $\phi\circ e$, $e\in L$, then for $A:L\to\mathbb{R}$ an isotonic linear and normalised functional with $A(e)=\frac{m+M}{2}$,

$$(4.9)\qquad 0\leq A(\phi\circ e)-\phi\left(\frac{m+M}{2}\right)\leq\frac{1}{4}(M-m)\left[\phi'_-(M)-\phi'_+(m)\right].$$

Theorem 9. *Let $\phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$, ψ, $\Psi\in\mathbb{R}$, $f:E\to[m,M]$ be such that $\phi\circ f$, $f\in L$ and $A:L\to\mathbb{R}$ an isotonic linear and normalised functional on L.*

(i) If ϕ is ψ-lower convex on $[m,M]$ with $\psi\geq0$, then

$$\begin{aligned}(4.10)\qquad 0&\leq\frac{1}{2}\psi\left(A\left(f^2\right)-(A(f))^2\right)\\ &\leq A(\phi\circ f)-\phi(A(f))\\ &\leq(M-A(f))(A(f)-m)\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right]\\ &\leq\frac{1}{4}(M-m)^2\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right];\end{aligned}$$

(ii) If ϕ is convex and Ψ-upper convex on $[m,M]$, then

$$(4.11)\quad 0\leq A(\phi\circ f)-\phi(A(f))\leq\frac{1}{2}\Psi(M-A(f))(A(f)-m)\leq\frac{1}{8}\Psi(M-m)^2;$$

(iii) If ϕ is(ψ,Ψ)-convex on $[m,M]$ with $0\leq\psi<\Psi$, then both inequalities (4.10) and (4.11) hold simultaneously.

Proof. (i) If we take in the second inequality of (3.7) $t=A(f)\in[m,M]$, then we get

$$\Phi_\phi(A(f))\leq(M-A(f))(A(f)-m)\left[\frac{\phi'_-(M)-\phi'_+(m)}{M-m}-\frac{1}{2}\psi\right],$$

which together with (4.2) produces the first two inequalities in (4.10). The last part follows by the fact that

$$(M-A(f))(A(f)-m)\leq\frac{1}{4}(M-m)^2.$$

Now, since $\phi_\psi := \phi - \frac{1}{2}\psi\ell^2$ is convex, then by Jessen's inequality for ϕ_ψ we get

$$\begin{aligned}
0 &\le A\left(\phi_\psi \circ f\right) - \phi_\psi\left(A\left(f\right)\right) \\
&= A\left(\phi \circ f - \frac{1}{2}\psi f^2\right) - \left(\phi\left(A\left(f\right)\right) - \frac{1}{2}\psi\left(A\left(f\right)\right)^2\right) \\
&= A\left(\phi \circ f\right) - \phi\left(A\left(f\right)\right) - \frac{1}{2}\psi\left(A\left(f^2\right) - \left(A\left(f\right)\right)^2\right),
\end{aligned}$$

which proves the first part of (4.10).

(ii) If we take in the last inequality in (3.8) $t = A\left(f\right) \in [m, M]$, then we also get

$$\Phi_\phi\left(A\left(f\right)\right) \le \frac{1}{2}\Psi\left(M - A\left(f\right)\right)\left(A\left(f\right) - m\right),$$

which together with (4.2) proves the second inequality in (4.11). □

We have the following results related to the Hermite-Hadamard inequalities:

Corollary 10. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$, ψ, $\Psi \in \mathbb{R}$, $e : E \to [m, M]$ be such that $\phi \circ e$, $e \in L$ and $A : L \to \mathbb{R}$ an isotonic linear and normalised functional with $A\left(e\right) = \frac{m+M}{2}$.*

(i) If ϕ is ψ-lower convex on $[m, M]$ with $\psi \ge 0$, then

$$\begin{aligned}
0 &\le \frac{1}{2}\psi\left(A\left(e^2\right) - \left(\frac{m+M}{2}\right)^2\right) \le A\left(\phi \circ e\right) - \phi\left(\frac{m+M}{2}\right) \\
&\le \frac{1}{4}\left(M - m\right)^2\left[\frac{\phi'_-\left(M\right) - \phi'_+\left(m\right)}{M - m} - \frac{1}{2}\psi\right];
\end{aligned} \tag{4.12}$$

(ii) If ϕ is convex and Ψ-upper convex on $[m, M]$, then

$$0 \le A\left(\phi \circ e\right) - \phi\left(\frac{m+M}{2}\right) \le \frac{1}{8}\Psi\left(M - m\right)^2. \tag{4.13}$$

5. Inequalities for Logarithm

In order to compare the various upper bounds in the Hermite-Hadamard inequalities for general isotonic functionals obtained above we consider the case of logarithmic function $\phi : [m, M] \subset (0, \infty) \to \mathbb{R}$, $\phi\left(t\right) = -\ln t$. For this function we have $\phi'\left(t\right) = -\frac{1}{t}$ and $\phi''\left(t\right) = \frac{1}{t^2}$. Therefore $\phi'\left(t\right) \in \left[-\frac{1}{m}, -\frac{1}{M}\right]$ and $\phi''\left(t\right) \in \left[\frac{1}{M^2}, \frac{1}{m^2}\right]$ for $t \in [m, M]$.

Let $g : E \to [m, M]$ be such that $\phi \circ g$, $g \in L$ and $A : L \to \mathbb{R}$ an isotonic linear and normalised functional with $A\left(g\right) = \frac{m+M}{2}$. From (2.25) we get

$$0 \le A\left(\ln g\right) - \ln G\left(m, M\right) \le \frac{1}{2}\ln\left(\frac{M}{m}\right), \tag{5.1}$$

from (3.17)

$$0 \le A\left(\ln g\right) - \ln G\left(m, M\right) \le \frac{1}{Mm}A\left[\left(M - g\right)\left(g - m\right)\right] \le \frac{1}{4mM}\left(M - m\right)^2, \tag{5.2}$$

from (3.23) we have

$$\begin{aligned}
0 &\le \frac{1}{2M^2}A\left[\left(M - g\right)\left(g - m\right)\right] \le A\left(\ln g\right) - \ln G\left(m, M\right) \\
&\le \frac{2M - m}{2mM^2}A\left[\left(M - g\right)\left(g - m\right)\right] \le \frac{2M - m}{8mM^2}\left(M - m\right)^2
\end{aligned} \tag{5.3}$$

and from (3.24) we have

$$0 \le A(\ln g) - \ln G(m, M) \le \frac{1}{2m^2} A[(M-g)(g-m)] \le \frac{1}{8m^2}(M-m)^2. \tag{5.4}$$

Also, from (4.4) we have

$$0 \le \ln\left(\frac{m+M}{2}\right) - A(\ln g) \le \frac{1}{2}\ln\left(\frac{M}{m}\right), \tag{5.5}$$

from (4.9)

$$0 \le \ln\left(\frac{m+M}{2}\right) - A(\ln g) \le \frac{1}{4mM}(M-m)^2, \tag{5.6}$$

from (4.12)

$$\begin{aligned} \frac{1}{2M^2}\left(A\left(g^2\right) - \left(\frac{m+M}{2}\right)^2\right) &\le \ln\left(\frac{m+M}{2}\right) - A(\ln g) \\ &\le \frac{2M-m}{8mM^2}(M-m)^2 \end{aligned} \tag{5.7}$$

and from (4.13)

$$0 \le \ln\left(\frac{m+M}{2}\right) - A(\ln g) \le \frac{1}{8m^2}(M-m)^2. \tag{5.8}$$

We observe that if $0 < m < M < \infty$, then

$$\frac{1}{4mM} - \frac{2M-m}{8mM^2} = \frac{1}{8M^2} > 0$$

and

$$\frac{1}{8m^2} - \frac{2M-m}{8mM^2} = \frac{(M-m)^2}{8m^2M^2} > 0,$$

which show that the upper bound in (5.3) is better than either of the upper bounds from (5.2) and (5.4). Also, the upper bound in (5.7) is better than either of the upper bounds from (5.6) and (5.8).

If we consider the difference

$$D(m, M) := \frac{1}{2}\ln\left(\frac{M}{m}\right) - \frac{2M-m}{8mM^2}(M-m)^2$$

on the domain $\Delta := \{(m, M) \mid 0.1 \le m \le M \le 1\}$, then the $3D$ plot $D(m, M)$ shows that it takes both positive and negative values, meaning that neither of the bounds (5.1) or (5.3) is always best. The same conclusion applies for (5.5) and (5.7).

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