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PERTURBED TRAPEZOID TYPE INEQUALITIES FOR ISOTONIC FUNCTIONALS AND FUNCTIONS OF BOUNDED VARIATION WITH APPLICATIONS

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Abstract. In this paper we obtain some perturbed trapezoid inequality for normalized isotonic linear functionals and various classes of functions such as: functions of bounded variation, (δ, Δ)-Lipschitzian functions and convex functions. Applications for some particular functions of interest are also provided.

1. Introduction

Let L be a *linear class* of real-valued functions $g : E \to \mathbb{R}$ having the properties

(L1) $f, g \in L$ imply $(\alpha f + \beta g) \in L$ for all $\alpha, \beta \in \mathbb{R}$;
(L2) $\mathbf{1} \in L$, i.e., if $f_0(t) = 1$, $t \in E$ then $f_0 \in L$.

An *isotonic linear functional* $A : L \to \mathbb{R}$ is a functional satisfying

(A1) $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for all $f, g \in L$ and $\alpha, \beta \in \mathbb{R}$.
(A2) If $f \in L$ and $f \geq 0$, then $A(f) \geq 0$.
The mapping A is said to be *normalised* if
(A3) $A(\mathbf{1}) = 1$.

Isotonic, that is, order-preserving, linear functionals are natural objects in analysis which enjoy a number of convenient properties. Thus, they provide, for example, Jessen's inequality, which is a functional form of Jensen's inequality (see [2], [22] and [23]). For other inequalities for isotonic functionals see [1], [3]-[19] and [24]-[26].

We note that common examples of such isotonic linear functionals A are given by

$$A(g) = \int_E g d\mu \text{ or } A(g) = \sum_{k \in E} p_k g_k,$$

where μ is a positive measure on E in the first case and E is a subset of the natural numbers $\mathbb{N}$, in the second ($p_k \geq 0, k \in E$).

For a function $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ of bounded variation and $f : E \to [m, M]$, we define

$$\bigvee_m^f (\phi)(s) := \bigvee_m^{f(s)} (\phi) \text{ for } s \in E.$$

In the recent paper [14], we obtained the following trapezoid type inequality for functions of bounded variation and isotonic functionals:

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Theorem 1. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and $f : E \to [m, M]$ such that $\phi \circ f,\ f,\ \bigvee_m^f (\phi),\ (M - f) \bigvee_m^f (\phi),\ (f - m) \bigvee_f^M (\phi) \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \tag{1.1}$$
$$\leq \frac{M - A(f)}{M - m} A\left(\bigvee_m^f (\phi) \right) + \frac{A(f) - m}{M - m} A\left(\bigvee_f^M (\phi) \right)$$
$$\leq \begin{cases} \left[\frac{1}{2} + \left| \frac{A(f) - \frac{m+M}{2}}{M - m} \right| \right] \bigvee_m^M (\phi), \\ \\ \left[\left(\frac{M - A(f)}{M - m} \right)^p + \left(\frac{A(f) - m}{M - m} \right)^p \right]^{1/p} \\ \times \left[\left(\frac{A\left(\bigvee_m^f (\phi) \right)}{M - m} \right)^q + \left(\frac{A\left(\bigvee_f^M (\phi) \right)}{M - m} \right)^q \right]^{1/q} \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{2} \left[\bigvee_m^M (\phi) + \left| A\left(\bigvee_m^f (\phi) \right) - A\left(\bigvee_f^M (\phi) \right) \right| \right]. \end{cases}$$

Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a function of bounded variation and $e : E \to [m, M]$ such that $\phi \circ e,\ e,\ \bigvee_m^e (\phi),\ (M - e) \bigvee_m^e (\phi),\ (e - m) \bigvee_e^M (\phi) \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then

$$\left| \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \right| \leq \frac{1}{2} \bigvee_m^M (\phi). \tag{1.2}$$

The constant $\frac{1}{2}$ in the right hand side of (1.2) is best possible.

The following lemma may be stated (see also [9]).

Lemma 1. *Let $u : [m, M] \to \mathbb{R}$ and $\delta, \Delta \in \mathbb{R}$ with $\Delta > \delta$. The following statements are equivalent:*

(i) *The function $u - \frac{\delta + \Delta}{2} \ell$, where $\ell(t) = t$, $t \in [m, M]$ is $\frac{1}{2}(\Phi - \delta)$-Lipschitzian;*
(ii) *We have the inequality:*

$$\delta \leq \frac{u(t) - u(s)}{t - s} \leq \Delta \quad \text{for each } t,\ s \in [m, M] \quad \text{with } t \neq s. \tag{1.3}$$

(iii) *We have the inequality:*

$$\delta(t - s) \leq u(t) - u(s) \leq \Delta(t - s) \quad \text{for each } t,\ s \in [m, M] \quad \text{with } t > s. \tag{1.4}$$

Following [21], we can introduce the concept:

Definition 1. *The function $u : [m, M] \to \mathbb{R}$ that satisfies one of the equivalent conditions (i) – (iii) is said to be (δ, Δ)-Lipschitzian on $[m, M]$.*

Notice that in [21], the definition was introduced on utilising the statement (iii) and only the equivalence "(i)$\Longleftrightarrow$(iii)" was considered.

Utilising Lagrange's mean value theorem, we can state the following result that provides practical examples of (δ, Δ)-Lipschitzian functions:

Proposition 1. *Let $u : [m, M] \to \mathbb{R}$ be continuous on $[m, M]$ and differentiable on $[m, M]$. If*

$$-\infty < \delta = \inf_{t \in (m,M)} u'(t), \sup_{t \in (m,M)} u'(t) = \Delta < \infty,$$

then u is (Δ, δ)-Lipschitzian on $[m, M]$.

We have the following result for (Δ, δ)-Lipschitzian functions on $[m, M]$, [14]:

Corollary 1. *Assume that $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some δ, $\Delta \in \mathbb{R}$ with $\Delta > \delta$. Then*

$$\begin{aligned} &\left| \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \right| \\ &\le \frac{\Phi - \delta}{M - m} A[(M - f)(f - m)] \le \frac{\Phi - \delta}{M - m} (M - A(f))(A(f) - m) \\ &\le \frac{1}{4}(M - m)(\Phi - \delta), \end{aligned} \tag{1.5}$$

If $e : E \to [m, M]$ is such that $\phi \circ e$, e, $e^2 \in L$ and $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional with $A(e) = \frac{m+M}{2}$, then

$$\begin{aligned} \left| \frac{\phi(m) + \phi(M)}{2} - A(\phi \circ e) \right| &\le \frac{\Phi - \delta}{M - m} A[(M - e)(e - m)] \\ &\le \frac{1}{4}(M - m)(\Phi - \delta), \end{aligned} \tag{1.6}$$

provided that $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some δ, $\Delta \in \mathbb{R}$ with $\Delta > \delta$.

If $\phi : [m, M] \to \mathbb{R}$ is convex and $\phi_+(m)$, $\phi_-(M)$ are finite, then by (1.5) we obtain the following reverse of Beesack-Pečarić result established in [18]

$$\begin{aligned} 0 &\le \frac{M - A(f)}{M - m} \phi(m) + \frac{A(f) - m}{M - m} \phi(M) - A(\phi \circ f) \\ &\le \frac{\phi_-(M) - \phi_+(m)}{M - m} A[(M - f)(f - m)] \\ &\le \frac{\phi_-(M) - \phi_+(m)}{M - m} (M - A(f))(A(f) - m) \\ &\le \frac{1}{4}(M - m)\big(\phi_-(M) - \phi_+(m)\big), \end{aligned} \tag{1.7}$$

provided that $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional and f, f^2, $\phi \circ f \in L$. This inequality was obtained for the discrete case in 2008, see [10, Proposition 8.2].

2. Some Preliminary Facts

We start with the following representation result:

Lemma 2. *Let $\phi : [m, M] \to \mathbb{R}$ be a function of bounded variation on $[m, M]$. Then for any $x \in [m, M]$ and $\lambda \in \mathbb{R}$, δ, $\gamma \in \mathbb{R}$ we have*

$$\phi(x) = (1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)(x-m)\delta - \lambda(M-x)\gamma + R_\lambda(x, m, M; \delta, \gamma), \tag{2.1}$$

where the remainder $R_\lambda(x, m, M; \delta, \gamma)$ is given by

$$R_\lambda(x, m, M; \delta, \gamma) := (1-\lambda)\int_m^x d(\phi(s) - \delta\ell(s)) + \lambda\int_x^M d(\gamma\ell(s) - \phi(s)), \tag{2.2}$$

while ℓ is the identity function on $[m, M]$, namely $\ell(s) = s$, $s \in [m, M]$.

Proof. Since $\phi : [m, M] \to \mathbb{R}$ is a function of bounded variation on $[m, M]$, then for any $x \in [m, M]$ the *Riemann-Stieltjes integrals*

$$\int_m^x d(\phi(s) - \delta\ell(s)) \text{ and } \int_x^M d(\gamma\ell(s) - \phi(s))$$

exist and we have

$$\int_m^x d(\phi(s) - \delta\ell(s)) = \phi(x) - \delta\ell(x) - [\phi(m) - \delta\ell(m)]$$

and

$$\int_x^M d(\gamma\ell(s) - \phi(s)) = \gamma\ell(M) - \phi(M) - [\gamma\ell(x) - \phi(x)].$$

Therefore

$$\begin{aligned} &R_\lambda(x, m, M; \delta, \gamma) \\ &= (1-\lambda)\int_m^x d(\phi(s) - \delta\ell(s)) + \lambda\int_x^M d(\gamma\ell(s) - \phi(s)) \\ &= (1-\lambda)(\phi(x) - \delta\ell(x) - [\phi(m) - \delta\ell(m)]) \\ &+ \lambda(\gamma\ell(M) - \phi(M) - [\gamma\ell(x) - \phi(x)]) \\ &= (1-\lambda)(\phi(x) - \phi(m) - \delta(x-m)) + \lambda(\gamma(M-x) - \phi(M) + \phi(x)) \\ &= (1-\lambda)\phi(x) - (1-\lambda)\phi(m) - (1-\lambda)(x-m)\delta \\ &- \lambda\phi(M) + \lambda\phi(x) + \lambda(M-x)\gamma \\ &= \phi(x) - (1-\lambda)\phi(m) - \lambda\phi(M) - (1-\lambda)(x-m)\delta + \lambda(M-x)\gamma, \end{aligned} \tag{2.3}$$

which is clearly equivalent to (2.1). □

Corollary 2. *Let $\phi : [m, M] \to \mathbb{R}$ be a function of bounded variation on $[m, M]$. Then for any $x \in [m, M]$ and δ, $\gamma \in \mathbb{R}$ we have*

$$\begin{aligned} \phi(x) &= \frac{1}{M-m}[(M-x)\phi(m) + (x-m)\phi(M)] \\ &+ \frac{(M-x)(x-m)}{M-m}(\delta - \gamma) + R_1(x, m, M; \delta, \gamma), \end{aligned} \tag{2.4}$$

where the remainder $R_1(x,m,M;\delta,\gamma)$ *is given by*

$$R_1(x,m,M;\delta,\gamma) := \frac{M-x}{M-m}\int_m^x d(\phi(s)-\delta\ell(s)) + \frac{x-m}{M-m}\int_x^M d(\gamma\ell(s)-\phi(s)). \tag{2.5}$$

Alternatively, we have

$$\phi(x) = \frac{1}{M-m}\left[(x-m)\phi(m)+(M-x)\phi(M)\right] + \frac{1}{M-m}\left[(x-m)^2\delta-(M-x)^2\gamma\right] + R_2(x,m,M;\delta,\gamma), \tag{2.6}$$

where the remainder $R_2(x,m,M;\delta,\gamma)$ *is given by*

$$R_2(x,m,M;\delta,\gamma) := \frac{x-m}{M-m}\int_m^x d(\phi(s)-\delta\ell(s)) + \frac{M-x}{M-m}\int_x^M d(\gamma\ell(s)-\phi(s)). \tag{2.7}$$

Proof. Follows by Lemma 2 on taking $\lambda=\frac{x-m}{M-m}$ and $\lambda=\frac{M-x}{M-m}$, respectively. □

The following particular case is of interest as well:

Corollary 3. *Let* $\phi:[m,M]\to\mathbb{R}$ *be a function of bounded variation on* $[m,M]$*. Then for any* $\lambda\in[0,1]$ *and* $\delta,\gamma\in\mathbb{R}$ *we have*

$$\phi((1-\lambda)m+\lambda M) = (1-\lambda)\phi(m)+\lambda\phi(M)+(1-\lambda)\lambda(M-m)(\delta-\gamma) + R_{1,\lambda}(m,M;\delta,\gamma), \tag{2.8}$$

where the remainder $R_{1,\lambda}(m,M;\delta,\gamma)$ *is given by*

$$R_{1,\lambda}(m,M;\delta,\gamma) := (1-\lambda)\int_m^{(1-\lambda)m+\lambda M} d(\phi(s)-\delta\ell(s)) + \lambda\int_{(1-\lambda)m+\lambda M}^M d(\gamma\ell(s)-\phi(s)). \tag{2.9}$$

Alternatively, we have

$$\phi(\lambda m+(1-\lambda)M) = (1-\lambda)\phi(m)+\lambda\phi(M) + (M-m)\left[(1-\lambda)^2\delta-\lambda^2\gamma\right] + R_{2,\lambda}(m,M;\delta,\gamma), \tag{2.10}$$

where the remainder $R_{2,\lambda}(m,M;\delta,\gamma)$ *is given by*

$$R_{2,\lambda}(m,M;\delta,\gamma) := (1-\lambda)\int_m^{\lambda m+(1-\lambda)M} d(\phi(s)-\delta\ell(s)) + \lambda\int_{\lambda m+(1-\lambda)M}^M d(\gamma\ell(s)-\phi(s)). \tag{2.11}$$

Remark 1. *Let* ϕ *be as in Lemma 2, then for any* $\lambda\in\mathbb{R}$, $\delta,\gamma\in\mathbb{R}$ *we have*

$$\phi\left(\frac{m+M}{2}\right) = (1-\lambda)\phi(m)+\lambda\phi(M)+\frac{1}{2}(M-m)\left[(1-\lambda)\delta-\lambda\gamma\right] + R_\lambda(m,M;\delta,\gamma), \tag{2.12}$$

where the remainder $R_\lambda(m,M;\delta,\gamma)$ *is given by*

$$R_\lambda(m,M;\delta,\gamma) := (1-\lambda)\int_m^{\frac{m+M}{2}} d(\phi(s)-\delta\ell(s)) + \lambda\int_{\frac{m+M}{2}}^M d(\gamma\ell(s)-\phi(s)). \tag{2.13}$$

The case $\delta=\gamma=0$ in (2.1) produces the following simple identities for each $x\in[m,M]$ and $\lambda\in\mathbb{R}$

$$\phi(x) = (1-\lambda)\phi(m)+\lambda\phi(M)+R_\lambda(x,m,M), \tag{2.14}$$

where the remainder $R_\lambda(x,m,M)$ is given by

$$R_\lambda(x,m,M) := (1-\lambda)\int_m^x d\phi(s) - \lambda\int_x^M d\phi(s). \tag{2.15}$$

We then have for each $x\in[m,M]$ that

$$\phi(x) = \frac{1}{M-m}\left[(M-x)\phi(m)+(x-m)\phi(M)\right]+U(x,m,M), \tag{2.16}$$

where

$$U(x,m,M) := \frac{M-x}{M-m}\int_m^x d\phi(s) - \frac{x-m}{M-m}\int_x^M d\phi(s) \tag{2.17}$$

and

$$\phi(x) = \frac{1}{M-m}\left[(x-m)\phi(m)+(M-x)\phi(M)\right]+V(x,m,M), \tag{2.18}$$

where

$$V(x,m,M) := \frac{x-m}{M-m}\int_m^x d\phi(s) - \frac{M-x}{M-m}\int_x^M d\phi(s). \tag{2.19}$$

We also have

$$\phi((1-\lambda)m+\lambda M) = (1-\lambda)\phi(m)+\lambda\phi(M)+U_\lambda(m,M), \tag{2.20}$$

where the remainder $U_\lambda(m,M)$ is given by

$$U_\lambda(m,M) := (1-\lambda)\int_m^{(1-\lambda)m+\lambda M} d\phi(s) - \lambda\int_{(1-\lambda)m+\lambda M}^M d\phi(s) \tag{2.21}$$

and

$$\phi((1-\lambda)M+\lambda m) = (1-\lambda)\phi(m)+\lambda\phi(M)+V_\lambda(m,M), \tag{2.22}$$

where the remainder $V_\lambda(m,M)$ is given by

$$V_\lambda(m,M) := (1-\lambda)\int_m^{(1-\lambda)M+\lambda m} d\phi(s) - \lambda\int_{(1-\lambda)M+\lambda m}^M d\phi(s). \tag{2.23}$$

Moreover, if we take in (2.14) $x=\frac{m+M}{2}$ then for each $\lambda\in\mathbb{R}$, we have

$$\phi\left(\frac{m+M}{2}\right) = (1-\lambda)\phi(m)+\lambda\phi(M)+S_\lambda(m,M), \tag{2.24}$$

where the remainder $S_\lambda(m,M)$ is given by

$$S_\lambda(m,M) := (1-\lambda)\int_m^{\frac{m+M}{2}} d\phi(s) - \lambda\int_{\frac{m+M}{2}}^M d\phi(s). \tag{2.25}$$

In particular, for $\lambda = \frac{1}{2}$ we have

$$\phi\left(\frac{m+M}{2}\right) = \frac{\phi(m)+\phi(M)}{2} + S(m,M), \tag{2.26}$$

where

$$S(m,M) := \frac{1}{2}\left(\int_m^{\frac{m+M}{2}} d\phi(s) - \int_{\frac{m+M}{2}}^M d\phi(s)\right). \tag{2.27}$$

The following estimate result holds:

Lemma 3. *Let $\phi : [m,M] \to \mathbb{R}$ be a function of bounded variation on $[m,M]$. Then for any $x \in [m,M]$ and $\lambda \in \mathbb{R}$, δ, $\gamma \in \mathbb{R}$ we have*

$$\begin{aligned} &|(1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)(x-m)\delta - \lambda(M-x)\gamma - \phi(x)| \\ &\leq |1-\lambda| \bigvee_m^x (\phi - \delta\ell) + |\lambda| \bigvee_x^M (\gamma\ell - \phi) \\ &\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\bigvee_m^x (\phi-\delta\ell) + \bigvee_x^M (\gamma\ell - \phi)\right), \\ \\ (|1-\lambda|^p + |\lambda|^p)^{1/p} \left(\left(\bigvee_m^x (\phi-\delta\ell)\right)^q + \left(\bigvee_x^M (\gamma\ell-\phi)\right)^q\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ (|1-\lambda|+|\lambda|) \max\left\{\bigvee_m^x (\phi-\delta\ell), \bigvee_x^M (\gamma\ell-\phi)\right\}. \end{cases} \end{aligned} \tag{2.28}$$

Proof. We use the fact that for $p : [\alpha,\beta] \to \mathbb{R}$ continuous and $v : [\alpha,\beta] \to \mathbb{R}$ of bounded variation the Riemann-Stieltjes integral $\int_\alpha^\beta p(t)\,dv(t)$ exists and

$$\left|\int_\alpha^\beta p(t)\,dv(t)\right| \leq \sup_{t\in[\alpha,\beta]} |p(t)| \bigvee_\alpha^\beta (v). \tag{2.29}$$

Using the identity (2.1) and the property (2.29) we have

$$\begin{aligned} &|(1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)(x-m)\delta - \lambda(M-x)\gamma - \phi(x)| \\ &= \left|(1-\lambda)\int_m^x d(\phi(s) - \delta\ell(s)) + \lambda\int_x^M d(\gamma\ell(s) - \phi(s))\right| \end{aligned}$$

$$
\begin{aligned}
&\leq |1-\lambda| \left| \int_m^x d\left(\phi(s) - \delta\ell(s)\right) \right| + |\lambda| \left| \int_x^M d\left(\gamma\ell(s) - \phi(s)\right) \right| \\
&\leq |1-\lambda| \bigvee_m^x (\phi - \delta\ell) + |\lambda| \bigvee_m^x (\gamma\ell - \phi) \\
&\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \left(\bigvee_m^x (\phi - \delta\ell) + \bigvee_m^x (\gamma\ell - \phi) \right), \\ \\ \left(|1-\lambda|^p + |\lambda|^p\right)^{1/p} \left(\left(\bigvee_m^x (\phi - \delta\ell) \right)^q + \left(\bigvee_m^x (\gamma\ell - \phi) \right)^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left(|1-\lambda| + |\lambda|\right) \max\left\{ \bigvee_m^x (\phi - \delta\ell), \bigvee_m^x (\gamma\ell - \phi) \right\}. \end{cases}
\end{aligned}
$$

The last part is obvious by Hölder's inequality

$$
cd + uv \leq \begin{cases} \max\{c, u\} (d + v) \\ \\ (c^p + u^p)^{1/p} (d^q + v^q)^{1/q}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}
$$

□

For any $x \in [m, M]$ and $\lambda \in \mathbb{R}$, $\delta \in \mathbb{R}$ we have

$$
\begin{aligned}
(2.30) \quad & \left| (1-\lambda)\phi(m) + \lambda\phi(M) + [x - (1-\lambda)m - \lambda M]\delta - \phi(x) \right| \\
&\leq |1-\lambda| \bigvee_m^x (\phi - \delta\ell) + |\lambda| \bigvee_x^M (\phi - \delta\ell) \\
&\leq \begin{cases} \max\{|1-\lambda|, |\lambda|\} \bigvee_m^M (\phi - \delta\ell), \\ \\ \left(|1-\lambda|^p + |\lambda|^p\right)^{1/p} \left(\left(\bigvee_m^x (\phi - \delta\ell) \right)^q + \left(\bigvee_x^M (\phi - \delta\ell) \right)^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left(|1-\lambda| + |\lambda|\right) \max\left\{ \bigvee_m^x (\phi - \delta\ell), \bigvee_x^M (\phi - \delta\ell) \right\}. \end{cases}
\end{aligned}
$$

Taking into (2.30) $\lambda \in [0,1]$, then we get

$$\begin{aligned} &\left|(1-\lambda)\phi(m)+\lambda\phi(M)+[x-(1-\lambda)m-\lambda M]\delta-\phi(x)\right| \\ &\leq (1-\lambda)\bigvee_{m}^{x}(\phi-\delta\ell)+\lambda\bigvee_{x}^{M}(\phi-\delta\ell) \\ &\leq \begin{cases} \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left(\bigvee_{m}^{M}(\phi-\delta\ell)\right), \\ \\ \left((1-\lambda)^{p}+\lambda^{p}\right)^{1/p}\left(\left(\bigvee_{m}^{x}(\phi-\delta\ell)\right)^{q}+\left(\bigvee_{x}^{M}(\phi-\delta\ell)\right)^{q}\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}\left[\bigvee_{m}^{M}(\phi-\delta\ell)+\left|\bigvee_{m}^{x}(\phi-\delta\ell)-\bigvee_{x}^{M}(\phi-\delta\ell)\right|\right]. \end{cases} \end{aligned} \tag{2.31}$$

Moreover, if we take in (2.31) $\delta=0$, then we get

$$\begin{aligned} &\left|(1-\lambda)\phi(m)+\lambda\phi(M)-\phi(x)\right| \\ &\leq (1-\lambda)\bigvee_{m}^{x}(\phi)+\lambda\bigvee_{x}^{M}(\phi) \\ &\leq \begin{cases} \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\left(\bigvee_{m}^{M}(\phi)\right), \\ \\ \left((1-\lambda)^{p}+\lambda^{p}\right)^{1/p}\left(\left(\bigvee_{m}^{x}(\phi)\right)^{q}+\left(\bigvee_{x}^{M}(\phi)\right)^{q}\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}\left[\bigvee_{m}^{M}(\phi)+\left|\bigvee_{m}^{x}(\phi)-\bigvee_{x}^{M}(\phi)\right|\right]. \end{cases} \end{aligned} \tag{2.32}$$

If $x \in [m, M]$ and if we take $\lambda = \frac{x-m}{M-m}$ in (2.32), then we get

$$\left| \left(\frac{M-x}{M-m} \right) \phi(m) + \left(\frac{x-m}{M-m} \right) \phi(M) - \phi(x) \right| \tag{2.33}$$
$$\leq \left(\frac{M-x}{M-m} \right) \bigvee_m^x (\phi) + \left(\frac{x-m}{M-m} \right) \bigvee_x^M (\phi)$$
$$\leq \begin{cases} \left(\frac{1}{2} + \left| \frac{x - \frac{m+M}{2}}{M-m} \right| \right) \left(\bigvee_m^M (\phi) \right), \\ \left(\left(\frac{M-x}{M-m} \right)^p + \left(\frac{x-m}{M-m} \right)^p \right)^{1/p} \left(\left(\frac{\bigvee_m^x (\phi)}{M-m} \right)^q + \left(\frac{\bigvee_x^M (\phi)}{M-m} \right)^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{2} \left[\bigvee_m^M (\phi) + \left| \bigvee_m^x (\phi) - \bigvee_x^M (\phi) \right| \right], \end{cases}$$

which was obtained in [11, Theorem 3.2].

For other related results, see [11] and [12].

Corollary 4. *Assume that $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some δ, $\Delta \in \mathbb{R}$ with $\Delta > \delta$. Then*

$$\left| (1-\lambda) \phi(m) + \lambda \phi(M) + [x - (1-\lambda) m - \lambda M] \frac{\delta + \Delta}{2} - \phi(x) \right| \tag{2.34}$$
$$\leq \frac{1}{2} (\Delta - \delta) [|1-\lambda| (x-m) + |\lambda| (M-x)]$$

for any $x \in [m, M]$ and $\lambda \in \mathbb{R}$.

Proof. We have by (2.30), for any $x \in [m, M]$ and $\lambda \in \mathbb{R}$, that

$$\left| (1-\lambda) \phi(m) + \lambda \phi(M) + [x - (1-\lambda) m - \lambda M] \frac{\delta + \Delta}{2} - \phi(x) \right|$$
$$\leq |1-\lambda| \bigvee_m^x \left(\phi - \frac{\delta + \Delta}{2} \ell \right) + |\lambda| \bigvee_x^M \left(\phi - \frac{\delta + \Delta}{2} \ell \right)$$
$$\leq \frac{1}{2} |1-\lambda| (\Delta - \delta) (x-m) + \frac{1}{2} |\lambda| (\Delta - \delta) (M-x)$$
$$= \frac{1}{2} (\Delta - \delta) [|1-\lambda| (x-m) + |\lambda| (M-x)],$$

which proves (2.34). □

Corollary 5. *Assume that $\phi : [m, M] \to \mathbb{R}$ is convex and $\phi_+(m)$, $\phi_-(M)$ are finite, then*

$$\left| (1-\lambda) \phi(m) + \lambda \phi(M) + \frac{1}{2} [x - (1-\lambda) m - \lambda M] [\phi_+(m) + \phi_-(M)] \right. \tag{2.35}$$
$$\left. - \phi(x) \right|$$
$$\leq \frac{1}{2} (\phi_-(M) - \phi_+(m)) [|1-\lambda| (x-m) + |\lambda| (M-x)]$$

for any $x \in [m, M]$ *and* $\lambda \in \mathbb{R}$.

If follows by Corollary 4 for $\delta = \phi_+ (m)$ and $\Delta = \phi_- (M)$, since by the gradient inequality we have

$$\phi_+ (m) (t - s) \le \phi_+ (s) (t - s) \le \phi (t) - \phi (s) \le \phi_- (t) (t - s) \le \phi_- (M) (t - s)$$

for each $t, s \in (m, M)$ with $t > s$.

We have:

Corollary 6. *With the assumptions of Lemma 3 for the function* ϕ, *we have for any* $\lambda \in [0, 1]$ *that*

$$\begin{aligned} (2.36) \quad & |(1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)\lambda(M-m)(\delta-\gamma) \\ & \qquad -\phi((1-\lambda)m + \lambda M)| \\ & \le (1-\lambda) \bigvee_{m}^{(1-\lambda)m+\lambda M} (\phi - \delta\ell) + \lambda \bigvee_{(1-\lambda)m+\lambda M}^{M} (\gamma\ell - \phi) \\ & \le \begin{cases} \left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right) \left(\bigvee_{m}^{(1-\lambda)m+\lambda M} (\phi - \delta\ell) + \bigvee_{(1-\lambda)m+\lambda M}^{M} (\gamma\ell - \phi) \right), \\ ((1-\lambda)^p + \lambda^p)^{1/p} \\ \times \left(\left(\bigvee_{m}^{(1-\lambda)m+\lambda M} (\phi - \delta\ell) \right)^q + \left(\bigvee_{(1-\lambda)m+\lambda M}^{M} (\gamma\ell - \phi) \right)^q \right)^{1/q}, \\ p, q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\ \max \left\{ \bigvee_{m}^{(1-\lambda)m+\lambda M} (\phi - \delta\ell), \bigvee_{(1-\lambda)m+\lambda M}^{M} (\gamma\ell - \phi) \right\} \end{cases} \end{aligned}$$

for any $\delta, \gamma \in \mathbb{R}$.

In particular,

$$|(1-\lambda)\phi(m)+\lambda\phi(M)-\phi((1-\lambda)m+\lambda M)| \tag{2.37}$$
$$\leq (1-\lambda)\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi-\delta\ell)+\lambda\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi-\delta\ell)$$
$$\leq \begin{cases} \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\bigvee_{m}^{M}(\phi-\delta\ell), \\ \\ ((1-\lambda)^{p}+\lambda^{p})^{1/p} \\ \times\left(\left(\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi-\delta\ell)\right)^{q}+\left(\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi-\delta\ell)\right)^{q}\right)^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi-\delta\ell),\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi-\delta\ell)\right\}, \end{cases}$$

for any $\delta\in\mathbb{R}$.

We observe that, with the assumptions of Corollary 6 we have from (2.38) that

$$|(1-\lambda)\phi(m)+\lambda\phi(M)-\phi((1-\lambda)m+\lambda M)| \tag{2.38}$$
$$\leq (1-\lambda)\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi)+\lambda\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi)$$
$$\leq \begin{cases} \left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\bigvee_{m}^{M}(\phi), \\ \\ ((1-\lambda)^{p}+\lambda^{p})^{1/p}\left[\left(\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi)\right)^{q}+\left(\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi)\right)^{q}\right]^{1/q}, \\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \max\left\{\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi),\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi)\right\}. \end{cases}$$

If ϕ is convex on $[m,M]$, then from (2.38) we get

$$0\leq(1-\lambda)\phi(m)+\lambda\phi(M)-\phi((1-\lambda)m+\lambda M) \tag{2.39}$$
$$\leq (1-\lambda)\bigvee_{m}^{(1-\lambda)m+\lambda M}(\phi)+\lambda\bigvee_{(1-\lambda)m+\lambda M}^{M}(\phi)$$
$$\leq\left(\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right)\bigvee_{m}^{M}(\phi)$$

for any $\lambda\in[0,1]$.

Also, if $\phi : [m, M] \to \mathbb{R}$ is (Δ, δ)-Lipschitzian on $[m, M]$ for some δ, $\Delta \in \mathbb{R}$ with $\Delta > \delta$, then by (2.34) we get

$$\begin{aligned} &|(1-\lambda)\phi(m) + \lambda\phi(M) - \phi((1-\lambda)m + \lambda M)| \\ &\leq (\Delta - \delta)(1-\lambda)\lambda(M-m) \end{aligned} \tag{2.40}$$

for any $\lambda \in [0, 1]$.

Moreover, if ϕ is convex on $[m, M]$, then by (2.35) we get

$$\begin{aligned} 0 &\leq (1-\lambda)\phi(m) + \lambda\phi(M) - \phi((1-\lambda)m + \lambda M) \\ &\leq \left(\phi_-(M) - \phi_+(m)\right)(1-\lambda)\lambda(M-m) \\ &\leq \frac{1}{4}\left(\phi_-(M) - \phi_+(m)\right)(M-m) \end{aligned} \tag{2.41}$$

for any $\lambda \in [0, 1]$.

Remark 2. *If there exists the constants δ, $\Delta \in \mathbb{R}$ such that*

$$\bigvee_m^M \left(\phi - \frac{\delta + \Delta}{2}\ell\right) \leq \frac{1}{2}|\Delta - \delta|(M-m), \tag{2.42}$$

then by (2.30) for $x \in [m, M]$

$$\begin{aligned} &\left|(1-\lambda)\phi(m) + \lambda\phi(M) + [x - (1-\lambda)m - \lambda M]\frac{\delta + \Delta}{2} - \phi(x)\right| \\ &\leq |1-\lambda|\bigvee_m^x \left(\phi - \frac{\delta + \Delta}{2}\ell\right) + |\lambda|\bigvee_x^M \left(\phi - \frac{\delta + \Delta}{2}\ell\right) \\ &\leq \frac{1}{2}|\Delta - \delta|\max\{|1-\lambda|, |\lambda|\}(M-m). \end{aligned} \tag{2.43}$$

In particular, if $\lambda \in [0, 1]$, then

$$\begin{aligned} &|(1-\lambda)\phi(m) + \lambda\phi(M) - \phi((1-\lambda)m + \lambda M)| \\ &\leq \frac{1}{2}|\Delta - \delta|\left(\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right)(M-m), \end{aligned} \tag{2.44}$$

giving that

$$\left|\frac{\phi(m) + \phi(M)}{2} - \phi\left(\frac{m+M}{2}\right)\right| \leq \frac{1}{4}|\Delta - \delta|(M-m). \tag{2.45}$$

We observe that if $\phi : [m, M] \to \mathbb{R}$ is a absolutely continuous function on $[m, M]$ then we have

$$\bigvee_m^M \left(\phi - \frac{\delta + \Delta}{2}\ell\right) \leq \int_m^M \left|\phi'(s) - \frac{\delta + \Delta}{2}\right| ds.$$

If ϕ satisfies the condition $k \leq \phi'(s) \leq K$, for a.e. $t \in I$ and for some real constants k, K, then we have that

$$\bigvee_m^M \left(\phi - \frac{k + K}{2}\ell\right) \leq \int_m^M \left|\phi'(s) - \frac{k + K}{2}\right| ds \leq \frac{1}{2}(K - k)(M-m).$$

This provides many examples of functions satisfying the condition (2.42).

3. Inequalities for Functionals

For a function $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ of bounded variation and $f : E \to [m, M]$, we define

$$\bigvee_{m}^{f} (\phi)(s) := \bigvee_{m}^{f(s)} (\phi) \text{ for } s \in E.$$

Theorem 2. *Let $\phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a function of bounded variation, $\delta, \gamma \in \mathbb{R}$ and $f : E \to [m, M]$ such that $\phi \circ f,\ f,\ \bigvee_{m}^{f} (\phi - \delta\ell), \bigvee_{f}^{M} (\gamma\ell - \phi) \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned}(3.1)\quad & |(1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)(A(f) - m)\delta - \lambda(M - A(f))\gamma \\ & \qquad - A(\phi \circ f)| \\ & \le |1-\lambda| A\left(\bigvee_{m}^{f} (\phi - \delta\ell)\right) + |\lambda| A\left(\bigvee_{f}^{M} (\gamma\ell - \phi)\right)\end{aligned}$$

for any $\lambda \in \mathbb{R}$,

In particular, if $\delta \in \mathbb{R}$, $\phi \circ f,\ f,\ \bigvee_{m}^{f} (\phi - \delta\ell) \in L$, then for any $\lambda \in \mathbb{R}$

$$\begin{aligned}(3.2)\quad & |(1-\lambda)\phi(m) + \lambda\phi(M) + [A(f) - (1-\lambda)m - \lambda M]\delta - A(\phi \circ f)| \\ & \le |1-\lambda| A\left(\bigvee_{m}^{f} (\phi - \delta\ell)\right) + |\lambda| A\left(\bigvee_{f}^{M} (\phi - \delta\ell)\right) \\ & \le \max\{|1-\lambda|, |\lambda|\} \bigvee_{m}^{M} (\phi - \delta\ell).\end{aligned}$$

Proof. We have from (2.28) in the order of L that

$$\begin{aligned}(3.3)\quad & -\left(|1-\lambda| \bigvee_{m}^{f} (\phi - \delta\ell) + |\lambda| \bigvee_{f}^{M} (\gamma\ell - \phi)\right) \\ & \le (1-\lambda)\phi(m) + \lambda\phi(M) + (1-\lambda)(f - m)\delta - \lambda(M - f)\gamma - \phi \circ f \\ & \le |1-\lambda| \bigvee_{m}^{f} (\phi - \delta\ell) + |\lambda| \bigvee_{f}^{M} (\gamma\ell - \phi)\end{aligned}$$

where $f : E \to [m, M]$ is such that $\phi \circ f,\ f,\ \bigvee_{m}^{f} (\phi) \in L$.

If we take the functional A in (3.3) and use its properties, we get the desired result (3.1).

We have

$$\begin{aligned}
&|1-\lambda|\, A\left(\bigvee_m^f(\phi-\delta\ell)\right)+|\lambda|\, A\left(\bigvee_f^M(\phi-\delta\ell)\right)\\
&\le \max\{|1-\lambda|,|\lambda|\}\left[A\left(\bigvee_m^f(\phi-\delta\ell)\right)+A\left(\bigvee_f^M(\phi-\delta\ell)\right)\right]\\
&= \max\{|1-\lambda|,|\lambda|\}\left[A\left(\bigvee_m^f(\phi-\delta\ell)+\bigvee_f^M(\phi-\delta\ell)\right)\right]\\
&= \max\{|1-\lambda|,|\lambda|\}\left[A\left(\bigvee_m^M(\phi-\delta\ell)\right)\right]\\
&= \max\{|1-\lambda|,|\lambda|\}\bigvee_m^M(\phi-\delta\ell)
\end{aligned}$$

and the inequality (3.2) is proved. □

Corollary 7. *Let $\phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a function of bounded variation and $f:E\to[m,M]$ such that $\phi\circ f,\ f,\ \bigvee_m^f(\phi)\in L$. If $A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then*

$$\begin{aligned}
&|(1-\lambda)\,\phi(m)+\lambda\phi(M)-A(\phi\circ f)|\\
&\le |1-\lambda|\, A\left(\bigvee_m^f(\phi)\right)+|\lambda|\, A\left(\bigvee_f^M(\phi)\right)\le \max\{|1-\lambda|,|\lambda|\}\bigvee_m^M(\phi),
\end{aligned}\tag{3.4}$$

for any $\lambda\in\mathbb{R}$.

If $\lambda\in[0,1]$, then

$$\begin{aligned}
&|(1-\lambda)\,\phi(m)+\lambda\phi(M)-A(\phi\circ f)|\\
&\le (1-\lambda)\, A\left(\bigvee_m^f(\phi)\right)+\lambda A\left(\bigvee_f^M(\phi)\right)\\
&\le \left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]\bigvee_m^M(\phi).
\end{aligned}\tag{3.5}$$

We have the following result:

Theorem 3. *Assume that $\phi:[m,M]\to\mathbb{R}$ is (Δ,δ)-Lipschitzian on $[m,M]$ for some $\delta,\ \Delta\in\mathbb{R}$ with $\Delta>\delta$. Let $f:E\to[m,M]$ such that $\phi\circ f,\ f\in L$. If*

$A:L\to\mathbb{R}$ is an isotonic linear and normalised functional, then

$$\left|(1-\lambda)\phi(m)+\lambda\phi(M)+[A(f)-(1-\lambda)m-\lambda M]\frac{\delta+\Delta}{2}-A(\phi\circ f)\right| \tag{3.6}$$

$$\leq\frac{1}{2}(\Delta-\delta)[|1-\lambda|(A(f)-m)+|\lambda|(M-A(f))]$$

$$\leq\frac{1}{2}(\Delta-\delta)\begin{cases}\max\{|1-\lambda|,|\lambda|\}(M-m),\\ \\ (|1-\lambda|^q+|\lambda|^q)^{1/q}[(A(f)-m)^p+(M-A(f))^p]^{1/p},\\ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ (|1-\lambda|+|\lambda|)\left[\frac{1}{2}(M-m)+\left|A(f)-\frac{m+M}{2}\right|\right],\end{cases}$$

for any $\lambda\in\mathbb{R}$.

Proof. From the inequality (2.34) we have, for $f:E\to[m,M]$ such that $\phi\circ f$, $f\in L$ in the order of L, that

$$-\frac{1}{2}(\Delta-\delta)[|1-\lambda|(f-m)+|\lambda|(M-f)] \tag{3.7}$$

$$\leq(1-\lambda)\phi(m)+\lambda\phi(M)+[f-(1-\lambda)m-\lambda M]\frac{\delta+\Delta}{2}-\phi(f)$$

$$\leq\frac{1}{2}(\Delta-\delta)[|1-\lambda|(f-m)+|\lambda|(M-f)]$$

for any $\lambda\in\mathbb{R}$.

If we take the functional A in (3.7) and use its properties, we get the first inequality in (3.6).

The last part is obvious by Hölder's inequality

$$cd+uv\leq\begin{cases}\max\{c,u\}(d+v),\\ \\ (c^p+u^p)^{1/p}(d^q+v^q)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

$$=\begin{cases}\left[\frac{1}{2}(c+d)+\frac{1}{2}|c-d|\right](d+v),\\ \\ (c^p+u^p)^{1/p}(d^q+v^q)^{1/q},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

□

Remark 3. *If $\lambda\in[0,1]$, then the inequality (3.6) can be written simpler as*

$$\left|(1-\lambda)\phi(m)+\lambda\phi(M)+[A(f)-(1-\lambda)m-\lambda M]\frac{\delta+\Delta}{2}-A(\phi\circ f)\right| \tag{3.8}$$

$$\leq\frac{1}{2}(\Delta-\delta)[(1-\lambda)(A(f)-m)+\lambda(M-A(f))]$$

$$\leq\frac{1}{2}(\Delta-\delta)\begin{cases}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right](M-m),\\ \\ ((1-\lambda)^q+\lambda^q)^{1/q}[(A(f)-m)^p+(M-A(f))^p]^{1/p},\\ \text{if } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \left[\frac{1}{2}(M-m)+\left|A(f)-\frac{m+M}{2}\right|\right].\end{cases}$$

In particular, for $\lambda = \frac{1}{2}$ *we get*

$$\left| \frac{\phi(m) + \phi(M)}{2} + \left(A(f) - \frac{m+M}{2} \right) \frac{\delta + \Delta}{2} - A(\phi \circ f) \right| \leq \frac{1}{4} (\Delta - \delta)(M - m). \tag{3.9}$$

Corollary 8. *Assume that* $\phi : [m, M] \to \mathbb{R}$ *is convex and* $\phi_+(m)$, $\phi_-(M)$ *are finite. Let* $f : E \to [m, M]$ *such that* $\phi \circ f$, $f \in L$. *If* $A : L \to \mathbb{R}$ *is an isotonic linear and normalised functional, then*

$$\begin{aligned} &\Big| (1-\lambda)\phi(m) + \lambda\phi(M) + \frac{1}{2}[A(f) - (1-\lambda)m - \lambda M] \\ &\qquad \times [\phi_+(m) + \phi_-(M)] - A(\phi \circ f) \Big| \\ &\leq \frac{1}{2}(\phi_-(M) - \phi_+(m))[|1-\lambda|(A(f) - m) + |\lambda|(M - A(f))] \\ &\leq \frac{1}{2}(\phi_-(M) - \phi_+(m)) \\ &\quad \times \begin{cases} \max\{|1-\lambda|, |\lambda|\}(M-m), \\ \\ (|1-\lambda|^q + |\lambda|^q)^{1/q} [(A(f) - m)^p + (M - A(f))^p]^{1/p}, \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ (|1-\lambda| + |\lambda|) \left[\frac{1}{2}(M-m) + \left|A(f) - \frac{m+M}{2}\right|\right], \end{cases} \end{aligned} \tag{3.10}$$

for any $\lambda \in \mathbb{R}$.

Remark 4. *If* $\lambda \in [0, 1]$, *then the inequality (3.10) can be written simpler as*

$$\begin{aligned} &\Big| (1-\lambda)\phi(m) + \lambda\phi(M) + \frac{1}{2}[A(f) - (1-\lambda)m - \lambda M] \\ &\qquad \times [\phi_+(m) + \phi_-(M)] - A(\phi \circ f) \Big| \\ &\leq \frac{1}{2}(\phi_-(M) - \phi_+(m))[(1-\lambda)(A(f) - m) + \lambda(M - A(f))] \\ &\leq \frac{1}{2}(\phi_-(M) - \phi_+(m)) \\ &\quad \times \begin{cases} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](M-m), \\ \\ ((1-\lambda)^q + \lambda^q)^{1/q} [(A(f) - m)^p + (M - A(f))^p]^{1/p}, \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2}(M-m) + \left|A(f) - \frac{m+M}{2}\right|\right], \end{cases} \end{aligned} \tag{3.11}$$

and, in particular

$$\left| \frac{\phi(m) + \phi(M)}{2} + \left(A(f) - \frac{m+M}{2} \right) \frac{\phi_+(m) + \phi_-(M)}{2} - A(\phi \circ f) \right| \leq \frac{1}{4}(\phi_-(M) - \phi_+(m))(M - m). \tag{3.12}$$

4. Some Examples

Assume that $\phi : [m, M] \to \mathbb{R}$ is a function of bounded variation and let $f : E \to [m, M]$ such that $\phi \circ f,\ f \in L$. If $A : L \to \mathbb{R}$ is an isotonic linear and normalised functional, then from (3.5), we have

$$|(1-\lambda)\phi(m) + \lambda\phi(M) - A(\phi \circ f)| \leq \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \bigvee_m^M (\phi) \tag{4.1}$$

for any $\lambda \in [0, 1]$. In particular,

$$\left|\frac{\phi(m) + \phi(M)}{2} - A(\phi \circ f)\right| \leq \frac{1}{2} \bigvee_m^M (\phi). \tag{4.2}$$

Assume that $[m, M] \subset (0, \infty)$. If we take in (4.1) the function $\phi(t) = \ln t$ and assume that $f : E \to [m, M],\ f,\ \ln f \in L$, then we get

$$|(1-\lambda)\ln m + \lambda \ln M - A(\ln f)| \leq \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \ln\left(\frac{M}{m}\right)$$

and in particular

$$\left|\frac{\ln m + \ln M}{2} - A(\ln f)\right| \leq \frac{1}{2} \bigvee_m^M \ln\left(\frac{M}{m}\right). \tag{4.3}$$

Consider the function $g(t) = t \ln t$, $t > 0$. We have $g'(t) = \ln t + 1$, $t > 0$, which shows that the function g is strictly decreasing on $\left(0, e^{-1}\right)$ and strictly increasing on $\left(e^{-1}, \infty\right)$. Therefore

$$\bigvee_m^M (g) = \begin{cases} \ln\left(\frac{m^m}{M^M}\right) \text{ if } 0 < m < M < e^{-1}, \\ \ln\left(m^m M^M\right) + 2e^{-1} \text{ if } 0 < m \leq e^{-1} \leq M < \infty, \\ \ln\left(\frac{M^M}{m^m}\right) \text{ if } e^{-1} < m < M < \infty. \end{cases}$$

If we take in (4.1) the function $\phi(t) = g(t) = t \ln t$ and assume that $f : E \to [m, M],\ f,\ f \ln f \in L$, then we have

$$\begin{aligned} &|(1-\lambda)\, m \ln m + \lambda M \ln M - A(f \ln f)| \\ &\leq \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \begin{cases} \ln\left(\frac{m^m}{M^M}\right) \text{ if } 0 < m < M < e^{-1}, \\ \ln\left(m^m M^M\right) + 2e^{-1} \text{ if } 0 < m \leq e^{-1} \leq M < \infty, \\ \ln\left(\frac{M^M}{m^m}\right) \text{ if } e^{-1} < m < M < \infty. \end{cases} \end{aligned} \tag{4.4}$$

for any $\lambda \in [0, 1]$. In particular,

$$\begin{aligned} &\left|\frac{m \ln m + M \ln M}{2} - A(f \ln f)\right| \\ &\leq \frac{1}{2} \begin{cases} \ln\left(\frac{m^m}{M^M}\right) \text{ if } 0 < m < M < e^{-1}, \\ \ln\left(m^m M^M\right) + 2e^{-1} \text{ if } 0 < m \leq e^{-1} \leq M < \infty, \\ \ln\left(\frac{M^M}{m^m}\right) \text{ if } e^{-1} < m < M < \infty. \end{cases} \end{aligned} \tag{4.5}$$

Consider the function $h(t) = (t-1)\ln t$, $t > 0$. We have $h'(t) = \ln t + 1 - \frac{1}{t}$ and $h''(t) = \frac{1}{t} + \frac{1}{t^2}$, $t > 0$, which shows that the function h is strictly decreasing on

$(0,1)$, strictly increasing on $(1,\infty)$ and strictly convex on $(0,\infty)$. Therefore

$$\bigvee_m^M (h) = \begin{cases} \ln\left(\frac{m^{m-1}}{M^{M-1}}\right) \text{ if } 0 < m < M < 1, \\ \ln\left(m^{m-1}M^{M-1}\right) \text{ if } 0 < m \le 1 \le M < \infty, \\ \ln\left(\frac{M^{M-1}}{m^{m-1}}\right) \text{ if } 1 < m < M < \infty. \end{cases}$$

If we take in (4.1) the function $\phi(t) = h(t) = (t-1)\ln t$ and assume that $f: E \to [m,M]$, $f, (f-1)\ln f \in L$, then we have

$$\begin{aligned} &\left|(1-\lambda)(m-1)\ln m + \lambda(M-1)\ln M - A((f-1)\ln f)\right| \\ &\le \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right] \begin{cases} \ln\left(\frac{m^{m-1}}{M^{M-1}}\right) \text{ if } 0 < m < M < 1, \\ \ln\left(m^{m-1}M^{M-1}\right) \text{ if } 0 < m \le 1 \le M < \infty, \\ \ln\left(\frac{M^{M-1}}{m^{m-1}}\right) \text{ if } 1 < m < M < \infty \end{cases} \end{aligned} \tag{4.6}$$

for any $\lambda \in [0,1]$. In particular,

$$\begin{aligned} &\left|\frac{(m-1)\ln m + (M-1)\ln M}{2} - A((f-1)\ln f)\right| \\ &\le \frac{1}{2} \begin{cases} \ln\left(\frac{m^{m-1}}{M^{M-1}}\right) \text{ if } 0 < m < M < 1, \\ \ln\left(m^{m-1}M^{M-1}\right) \text{ if } 0 < m \le 1 \le M < \infty, \\ \ln\left(\frac{M^{M-1}}{m^{m-1}}\right) \text{ if } 1 < m < M < \infty. \end{cases} \end{aligned} \tag{4.7}$$

Moreover, if we write the inequalities (3.11) and (3.12) for the convex function $\phi(t) = -\ln t$, $t \in [m,M] \subset (0,\infty)$ and assume that $f: E \to [m,M]$, $f, \ln f \in L$, then we get

$$\begin{aligned} &\left|A(\ln f) - \ln\left(m^{1-\lambda}M^{\lambda}\right) - \frac{m+M}{2mM}\left[A(f) - (1-\lambda)m - \lambda M\right]\right| \\ &\le \frac{M-m}{2mM}\left[(1-\lambda)(A(f)-m) + \lambda(M - A(f))\right] \\ &\le \frac{M-m}{2mM} \begin{cases} \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](M-m), \\ \\ \left((1-\lambda)^q + \lambda^q\right)^{1/q}\left[(A(f)-m)^p + (M-A(f))^p\right]^{1/p}, \\ \text{if } p,q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\frac{1}{2}(M-m) + \left|A(f) - \frac{m+M}{2}\right|\right], \end{cases} \end{aligned} \tag{4.8}$$

for any $\lambda \in [0,1]$, and, in particular

$$\left|A(\ln f) - \ln\left(\sqrt{mM}\right) - \frac{m+M}{2mM}\left(A(f) - \frac{m+M}{2}\right)\right| \le \frac{(M-m)^2}{4mM}. \tag{4.9}$$

If we write the inequalities (3.11) and (3.12) for the convex function $\phi(t) = t\ln t$, $t \in [m, M] \subset (0, \infty)$ and assume that $f : E \to [m, M]$, f, $f\ln f \in L$, then we get

$$\left|\ln\left(m^{(1-\lambda)m}M^{\lambda M}\right) + \ln\left(e\sqrt{Mm}\right)[A(f) - (1-\lambda)m - \lambda M] - A(f\ln f)\right| \tag{4.10}$$
$$\leq \frac{1}{2}\ln\left(\frac{M}{m}\right)[(1-\lambda)(A(f) - m) + \lambda(M - A(f))]$$
$$\leq \frac{1}{2}\ln\left(\frac{M}{m}\right)\begin{cases}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](M-m), \\ \left((1-\lambda)^q + \lambda^q\right)^{1/q}\left[(A(f) - m)^p + (M - A(f))^p\right]^{1/p}, \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2}(M-m) + \left|A(f) - \frac{m+M}{2}\right|\right],\end{cases}$$

for any $\lambda \in [0, 1]$, and, in particular

$$\left|\ln\left(\sqrt{m^m M^M}\right) + \ln\left(e\sqrt{Mm}\right)\left(A(f) - \frac{m+M}{2}\right) - A(f\ln f)\right| \tag{4.11}$$
$$\leq \frac{1}{4}\ln\left(\frac{M}{m}\right)(M-m).$$

Consider the function $\phi : [m, M] \subset (0, \infty) \to (0, \infty)$, $\phi(t) = t^p$, $p \neq 0$. Then by (4.1) we have

$$|(1-\lambda)m^p + \lambda M^p - A(f^p)| \leq \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]\begin{cases}(M^p - m^p) \text{ if } p > 0, \\ \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \text{ if } p < 0\end{cases} \tag{4.12}$$

for $\lambda \in [0, 1]$ and provided that $f : E \to [m, M]$, f, $f^p \in L$. In particular,

$$\left|\frac{m^p + M^p}{2} - A(f^p)\right| \leq \frac{1}{2}\begin{cases}(M^p - m^p) \text{ if } p > 0, \\ \frac{M^{-p} - m^{-p}}{M^{-p}m^{-p}} \text{ if } p < 0.\end{cases} \tag{4.13}$$

If we write the inequalities (3.11) and (3.12) for the convex function $\phi(t) = t^p$, $t \in [m, M] \subset (0, \infty)$, $p \in (-\infty, 0) \cup [1, \infty)$

$$\left|(1-\lambda)m^p + \lambda M^p + p\left(\frac{m^{p-1} + M^{p-1}}{2}\right)[A(f) - A_\lambda(m, M)] - A(f^p)\right| \tag{4.14}$$
$$\leq \frac{1}{2}p\left(M^{p-1} - m^{p-1}\right)[(1-\lambda)(A(f) - m) + \lambda(M - A(f))]$$
$$\leq \frac{1}{2}p\left(M^{p-1} - m^{p-1}\right)\begin{cases}\left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right](M-m), \\ \left((1-\lambda)^q + \lambda^q\right)^{1/q} \\ \times\left[(A(f) - m)^p + (M - A(f))^p\right]^{1/p}, \\ \text{if } p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1, \\ \left[\frac{1}{2}(M-m) + \left|A(f) - \frac{m+M}{2}\right|\right],\end{cases}$$

where $A_\lambda(m,M) := (1-\lambda)m + \lambda M$ is the weighted arithmetic mean, and, in particular

$$\left|\frac{m^p+M^p}{2}+\frac{1}{2}p\left(m^{p-1}+M^{p-1}\right)\left(A(f)-\frac{m+M}{2}\right)-A(f^p)\right| \le \frac{1}{4}p\left(M^{p-1}-m^{p-1}\right)(M-m). \tag{4.15}$$

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