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HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT OF CONVEX AND SYMMETRIZED CONVEX FUNCTIONS

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Abstract. In this paper we establish some Hermite-Hadamard type inequalities for the product of a convex function with a symmetrized convex function and also for two convex functions. Some examples for weighted integrals of interest are also provided.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leq\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [1]

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

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for any $t \in \mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t) = 3(a+b)$, $t \in \mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [1], see also [3] for an equivalent definition.

Definition 1. *We say that the function $f : [a,b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b] \subsetneq SCon[a,b]. \tag{1.2}$$

Also, if $[c,d] \subset [a,b]$ and $f \in SCon[a,b]$, then this does not imply in general that $f \in SCon[c,d]$.

We have the following result [1], [3]:

Theorem 1. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

The following result holds :

Theorem 2. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x \in [a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right) \le \breve{f}(x) \le \frac{f(a)+f(b)}{2}. \tag{1.4}$$

For other results, see [1] and [3].

In [5], B. G. Pachpatte established two Hermite-Hadamard type inequalities for products of nonnegative convex functions $f, g : [a,b] \to [0,\infty)$ as follows:

$$\begin{aligned} &2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}\left[f(a)g(a)+f(b)g(b)\right] \\ &-\frac{1}{3}\left[f(a)g(b)+f(b)g(a)\right] \\ &\le \frac{1}{b-a}\int_a^b f(t)g(t)\,dt \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)g(t)\,dt \\ &\le \frac{1}{3}\left[f(a)g(a)+f(b)g(b)\right] + \frac{1}{6}\left[f(a)g(b)+f(b)g(a)\right]. \end{aligned} \tag{1.6}$$

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by other authors. For instance, in [6], B. G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is established by Kirmaci et. al. in [4]. In [7], M. Z. Sarikaya presented some integral inequalities for two h-convex functions. For recent results and generalizations concerning Hermite-Hadamard type inequality for product of two functions see [7] and the references given therein. For a monograph on Hermite-Hadamard type inequalities see [2].

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for the product of a convex function with a symmetrized convex function and also for two convex functions that can take negative values as well. Some examples for weighted integrals of interest are also provided.

2. The Case of One Convex and the Other Symmetrized Convex Functions

In this section we analyze the case in which one function is convex (concave) in the classical sense and the other is symmetrized convex (concave) on an interval $[a,b]$.

Theorem 3. *Assume that $g:[a,b]\to\mathbb{R}$ is convex (concave) and $f:[a,b]\to\mathbb{R}$ is symmetrized convex (symmetrized concave) and integrable on the interval $[a,b]$. Then we have*

$$\begin{aligned}\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt\\ -\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\\ \leq\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt\\ \leq\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt+f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt\\ -f\left(\frac{a+b}{2}\right)\frac{g(a)+g(b)}{2}.\end{aligned}\tag{2.2}$$

Proof. Assume that $g:[a,b]\to\mathbb{R}$ is convex and $f:[a,b]\to\mathbb{R}$ is symmetrized convex on $[a,b]$, then for any $\lambda\in[0,1]$

$$(1-\lambda)g(a)+\lambda g(b)\geq g((1-\lambda)a+\lambda b)\tag{2.3}$$

and

$$\frac{f(a)+f(b)}{2}\geq\breve{f}((1-\lambda)a+\lambda b)\geq f\left(\frac{a+b}{2}\right),\tag{2.4}$$

where, by (1.1)

$$\breve{f}((1-\lambda)a+\lambda b)=\frac{1}{2}\left[f((1-\lambda)a+\lambda b)+f(\lambda a+(1-\lambda)b)\right],\ \lambda\in[0,1].$$

By (2.3) and (2.4) we have

$$\begin{aligned}
0 &\leq [(1-\lambda) g(a) + \lambda g(b) - g((1-\lambda) a + \lambda b)] \\
&\times \left[\frac{f(a)+f(b)}{2} - \breve{f}((1-\lambda) a + \lambda b)\right] \\
&= [(1-\lambda) g(a) + \lambda g(b)] \frac{f(a)+f(b)}{2} - \frac{f(a)+f(b)}{2} g((1-\lambda) a + \lambda b) \\
&- [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) \\
&+ \breve{f}((1-\lambda) a + \lambda b) g((1-\lambda) a + \lambda b)
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&[(1-\lambda) g(a) + \lambda g(b)] \frac{f(a)+f(b)}{2} + \breve{f}((1-\lambda) a + \lambda b) g((1-\lambda) a + \lambda b) \\
&\geq \frac{f(a)+f(b)}{2} g((1-\lambda) a + \lambda b) + [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b)
\end{aligned}$$

for any $\lambda \in [0,1]$.

Integrating over λ on $[0,1]$, we get

$$\begin{aligned}
&\frac{f(a)+f(b)}{2} \int_0^1 [(1-\lambda) g(a) + \lambda g(b)] d\lambda \\
&+ \int_0^1 \breve{f}((1-\lambda) a + \lambda b) g((1-\lambda) a + \lambda b) d\lambda \\
&\geq \frac{f(a)+f(b)}{2} \int_0^1 g((1-\lambda) a + \lambda b) d\lambda \\
&+ \int_0^1 [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) d\lambda.
\end{aligned} \tag{2.5}$$

Observe that

$$\int_0^1 [(1-\lambda) g(a) + \lambda g(b)] d\lambda = \frac{g(a)+g(b)}{2},$$

$$\int_0^1 g((1-\lambda) a + \lambda b) d\lambda = \frac{1}{b-a} \int_a^b g(t) dt$$

and

$$\int_0^1 \breve{f}((1-\lambda) a + \lambda b) g((1-\lambda) a + \lambda b) d\lambda = \frac{1}{b-a} \int_a^b \breve{f}(t) g(t) dt.$$

Also

$$\begin{aligned}
&\int_0^1 [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) d\lambda \\
&= g(a) \int_0^1 (1-\lambda) \breve{f}((1-\lambda) a + \lambda b) d\lambda + g(b) \int_0^1 \lambda \breve{f}((1-\lambda) a + \lambda b) d\lambda.
\end{aligned} \tag{2.6}$$

Since $\breve{f}$ is symmetric, then

$$\int_0^1 (1-\lambda) \breve{f}((1-\lambda) a + \lambda b) d\lambda = \int_0^1 (1-\lambda) \breve{f}(\lambda a + (1-\lambda) b) d\lambda.$$

By changing the variable $s=1-\lambda,\ \lambda\in[0,1]$, we have

$$\int_0^1 (1-\lambda)\,\breve{f}(\lambda a+(1-\lambda)b)\,d\lambda=\int_0^1 s\breve{f}((1-s)a+sb)\,ds$$

and by (2.6) we get

$$\int_0^1 [(1-\lambda)g(a)+\lambda g(b)]\,\breve{f}((1-\lambda)a+\lambda b)\,d\lambda \tag{2.7}$$
$$=g(a)\int_0^1 s\breve{f}((1-s)a+sb)\,ds+g(b)\int_0^1 \lambda\breve{f}((1-\lambda)a+\lambda b)\,d\lambda$$
$$=[g(a)+g(b)]\int_0^1 \lambda\breve{f}((1-\lambda)a+\lambda b)\,d\lambda.$$

Further,

$$\int_0^1 \lambda\breve{f}((1-\lambda)a+\lambda b)\,d\lambda \tag{2.8}$$
$$=\frac{1}{2}\int_0^1 \lambda[f((1-\lambda)a+\lambda b)+f(\lambda a+(1-\lambda)b)]\,d\lambda$$
$$=\frac{1}{2}\left(\int_0^1 \lambda f((1-\lambda)a+\lambda b)\,d\lambda+\int_0^1 \lambda f(\lambda a+(1-\lambda)b)\,d\lambda\right).$$

Using the change of variable $s=1-\lambda,\ \lambda\in[0,1]$, we have

$$\int_0^1 \lambda f(\lambda a+(1-\lambda)b)\,d\lambda=\int_0^1 (1-s)\,f((1-s)a+sb)\,ds$$

and by (2.8)

$$\int_0^1 \lambda\breve{f}((1-\lambda)a+\lambda b)\,d\lambda \tag{2.9}$$
$$=\frac{1}{2}\left(\int_0^1 \lambda f((1-\lambda)a+\lambda b)\,d\lambda+\int_0^1 (1-\lambda)\,f((1-\lambda)a+\lambda b)\,d\lambda\right)$$
$$=\frac{1}{2}\int_0^1 f((1-\lambda)a+\lambda b)\,d\lambda=\frac{1}{2}\frac{1}{b-a}\int_a^b f(t)\,dt.$$

By the inequality (2.5) we then get

$$\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}+\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt$$
$$\geq\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt$$

and the inequality (2.1) is proved.

By (2.3) and (2.4) we also have

$$0\leq[(1-\lambda)g(a)+\lambda g(b)-g((1-\lambda)a+\lambda b)]\left[\breve{f}((1-\lambda)a+\lambda b)-f\left(\frac{a+b}{2}\right)\right]$$
$$=[(1-\lambda)g(a)+\lambda g(b)]\,\breve{f}((1-\lambda)a+\lambda b)+g((1-\lambda)a+\lambda b)\,f\left(\frac{a+b}{2}\right)$$
$$-[(1-\lambda)g(a)+\lambda g(b)]\,f\left(\frac{a+b}{2}\right)-g((1-\lambda)a+\lambda b)\,\breve{f}((1-\lambda)a+\lambda b)$$

for any $\lambda \in [0,1]$, which is equivalent to

$$[(1-\lambda) g(a) + \lambda g(b)] f\left(\frac{a+b}{2}\right) + g((1-\lambda) a + \lambda b) \breve{f}((1-\lambda) a + \lambda b)$$
$$\leq [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) + g((1-\lambda) a + \lambda b) f\left(\frac{a+b}{2}\right)$$

for any $\lambda \in [0,1]$.

Taking the integral over $\lambda \in [0,1]$, we get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \int_0^1 [(1-\lambda) g(a) + \lambda g(b)] d\lambda \\ & + \int_0^1 g((1-\lambda) a + \lambda b) \breve{f}((1-\lambda) a + \lambda b) d\lambda \\ & \leq \int_0^1 [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) d\lambda \\ & + f\left(\frac{a+b}{2}\right) \int_0^1 g((1-\lambda) a + \lambda b) d\lambda. \end{aligned} \tag{2.10}$$

Since

$$\int_0^1 [(1-\lambda) g(a) + \lambda g(b)] d\lambda = \frac{g(a)+g(b)}{2},$$

$$\int_0^1 \breve{f}((1-\lambda) a + \lambda b) g((1-\lambda) a + \lambda b) d\lambda = \frac{1}{b-a} \int_a^b \breve{f}(t) g(t) dt$$

$$\int_0^1 g((1-\lambda) a + \lambda b) d\lambda = \frac{1}{b-a} \int_a^b g(t) dt$$

and, as above

$$\int_0^1 [(1-\lambda) g(a) + \lambda g(b)] \breve{f}((1-\lambda) a + \lambda b) d\lambda = \frac{g(a)+g(b)}{2} \frac{1}{b-a} \int_a^b f(t) dt.$$

From (2.10) we then get

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \frac{g(a)+g(b)}{2} + \frac{1}{b-a} \int_a^b \breve{f}(t) g(t) d \\ & \leq \frac{g(a)+g(b)}{2} \frac{1}{b-a} \int_a^b f(t) dt + f\left(\frac{a+b}{2}\right) \frac{1}{b-a} \int_a^b g(t) dt, \end{aligned}$$

and the inequality (2.2) is proved. □

We observe that the inequalities (2.1) and (2.2) can be written in an equivalent for as

$$\begin{aligned} & \frac{1}{b-a} \int_a^b f(t) dt \frac{1}{b-a} \int_a^b g(t) dt - \frac{1}{b-a} \int_a^b \breve{f}(t) g(t) dt \\ & \leq \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt\right] \left[\frac{g(a)+g(b)}{2} - \frac{1}{b-a} \int_a^b g(t) dt\right] \end{aligned} \tag{2.11}$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \tag{2.12}$$
$$\geq -\left[\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right].$$

Remark 1. *Since any convex function on $[a,b]$ is symmetrized convex and integrable, then the inequalities (2.1) and (2.2) hold, a fortiori, for both f, g that are convex (concave) on $[a,b]$. If f is symmetrical, i.e. $f(t)=f(a+b-t)$ for any $t\in[a,b]$, then from (2.1) and (2.2) we get the simpler forms*

$$\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt+\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt \tag{2.13}$$
$$-\frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}$$
$$\leq \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \tag{2.14}$$
$$\leq \frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt+f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt$$
$$-f\left(\frac{a+b}{2}\right)\frac{g(a)+g(b)}{2}.$$

These are equivalent to

$$\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \tag{2.15}$$
$$\leq \left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]\left[\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right]$$

and

$$\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \tag{2.16}$$
$$\geq -\left[\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right].$$

We observe that, if $g:[a,b]\to\mathbb{R}$ is concave (convex) and $f:[a,b]\to\mathbb{R}$ is symmetrized convex (symmetrized concave) and integrable on the interval $[a,b]$, then by (2.11) and (2.12) for $g\leftrightarrow -g$ we get

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.17}$$
$$\leq \left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right]\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right]$$

and

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.18}$$
$$\geq -\left[\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right]\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right].$$

3. The Case of Two Convex Functions

We have the following results for two functions of same convexity on $[a,b]$:

Theorem 4. *Assume that $f,\ g:[a,b]\to\mathbb{R}$ are both convex (concave) on $[a,b]$. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right]g(t)\,dt\\&+\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}g(a)+\frac{t-a}{b-a}g(b)\right]f(t)\,dt\\&\leq\frac{1}{3}\left[f(a)\,g(a)+f(b)\,g(b)\right]+\frac{1}{6}\left[f(a)\,g(b)+f(b)\,g(a)\right]\\&+\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\end{aligned}\tag{3.1}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right]g(a+b-t)\,dt\\&+\frac{1}{b-a}\int_a^b\left[\frac{t-a}{b-a}g(a)+\frac{b-t}{b-a}g(b)\right]f(t)\,dt\\&\leq\frac{1}{3}\left[f(a)\,g(b)+f(b)\,g(a)\right]+\frac{1}{6}\left[f(a)\,g(a)+f(b)\,g(b)\right]\\&+\frac{1}{b-a}\int_a^b f(t)\,g(a+b-t)\,dt.\end{aligned}\tag{3.2}$$

If one function is convex and the other concave, then the inequalities in (3.1) and (3.2) reverse.

Proof. By the convexity of the functions $f,\ g:[a,b]\to\mathbb{R}$ on $[a,b]$ we have

$$\begin{aligned}0&\leq\left[(1-\lambda)f(a)+\lambda f(b)-f((1-\lambda)a+\lambda b)\right]\\&\times\left[(1-\lambda)g(a)+\lambda g(b)-g((1-\lambda)a+\lambda b)\right]\\&=\left[(1-\lambda)f(a)+\lambda f(b)\right]\left[(1-\lambda)g(a)+\lambda g(b)\right]\\&+f((1-\lambda)a+\lambda b)\,g((1-\lambda)a+\lambda b)\\&-\left[(1-\lambda)f(a)+\lambda f(b)\right]g((1-\lambda)a+\lambda b)\\&-\left[(1-\lambda)g(a)+\lambda g(b)\right]f((1-\lambda)a+\lambda b)\end{aligned}$$

for any $\lambda\in[0,1]$.

This is equivalent to

$$\begin{aligned}
&[(1-\lambda) f(a)+\lambda f(b)] g((1-\lambda) a+\lambda b) \\
&+[(1-\lambda) g(a)+\lambda g(b)] f((1-\lambda) a+\lambda b) \\
&\leq[(1-\lambda) f(a)+\lambda f(b)][(1-\lambda) g(a)+\lambda g(b)] \\
&+f((1-\lambda) a+\lambda b) g((1-\lambda) a+\lambda b) \\
&=(1-\lambda)^2 f(a) g(a)+\lambda^2 f(b) g(b) \\
&+\lambda(1-\lambda) f(b) g(a)+(1-\lambda) \lambda f(a) g(b) \\
&+f((1-\lambda) a+\lambda b) g((1-\lambda) a+\lambda b)
\end{aligned}$$

for any $\lambda \in [0,1]$.

Integrating over $\lambda \in [0,1]$ we get

$$\begin{aligned}
&\int_0^1 [(1-\lambda) f(a)+\lambda f(b)] g((1-\lambda) a+\lambda b) d\lambda \\
&+\int_0^1 [(1-\lambda) g(a)+\lambda g(b)] f((1-\lambda) a+\lambda b) d\lambda \\
&\leq f(a) g(a) \int_0^1 (1-\lambda)^2 d\lambda+f(b) g(b) \int_0^1 \lambda^2 d\lambda \\
&+f(b) g(a) \int_0^1 \lambda(1-\lambda) d\lambda+f(a) g(b) \int_0^1 (1-\lambda) \lambda d\lambda \\
&+\int_0^1 f((1-\lambda) a+\lambda b) g((1-\lambda) a+\lambda b) d\lambda .
\end{aligned} \tag{3.3}$$

Observe that

$$\int_0^1 \lambda^2 d\lambda=\int_0^1 (1-\lambda)^2 d\lambda=\frac{1}{3} \text{ and } \int_0^1 \lambda(1-\lambda) d\lambda=\frac{1}{6}.$$

Using the change of variable $t=(1-\lambda) a+\lambda b$, $\lambda \in [0,1]$ we have $\lambda=\frac{t-a}{b-a}$, $d\lambda=\frac{1}{b-a} dt$ and then

$$\int_0^1 f((1-\lambda) a+\lambda b) g((1-\lambda) a+\lambda b) d\lambda=\frac{1}{b-a} \int_a^b f(t) g(t) dt,$$

$$\begin{aligned}
&\int_0^1 [(1-\lambda) f(a)+\lambda f(b)] g((1-\lambda) a+\lambda b) d\lambda \\
&=\frac{1}{b-a} \int_a^b \left[\frac{b-t}{b-a} f(a)+\frac{t-a}{b-a} f(b)\right] g(t) dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 [(1-\lambda) g(a)+\lambda g(b)] f((1-\lambda) a+\lambda b) d\lambda \\
&=\frac{1}{b-a} \int_a^b \left[\frac{b-t}{b-a} g(a)+\frac{t-a}{b-a} g(b)\right] f(t) dt .
\end{aligned}$$

By making use of (3.3) we get (3.1).

Consider the function $h:[a,b]\to\mathbb{R}$ defined by $h(t):=g(a+b-t)$, $t\in[a,b]$. If g is convex, then h is convex and by (3.1) we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right]h(t)\,dt\\&+\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}h(a)+\frac{t-a}{b-a}h(b)\right]f(t)\,dt\\&\le\frac{1}{3}\left[f(a)h(a)+f(b)h(b)\right]+\frac{1}{6}\left[f(b)h(a)+f(a)h(b)\right]\\&+\frac{1}{b-a}\int_a^b f(t)h(t)\,dt\end{aligned}$$

that is equivalent to (3.2). □

Remark 2. *Assume that $f,g:[a,b]\to\mathbb{R}$ are both convex (concave) on $[a,b]$. Then by adding the inequalities (3.1) and (3.2) we get*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right]\breve{g}(t)\,dt+\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt\\&\le\frac{g(a)+g(b)}{2}\frac{f(a)+f(b)}{2}+\frac{1}{b-a}\int_a^b f(t)\breve{g}(t)\,dt\end{aligned}\tag{3.4}$$

and since, as in the proof of Theorem 3, we have

$$\frac{1}{b-a}\int_a^b\left[\frac{b-t}{b-a}f(a)+\frac{t-a}{b-a}f(b)\right]\breve{g}(t)\,dt=\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt$$

and by (3.4) we recapture the inequality (2.1) for $f\leftrightarrow g$.

4. Some Examples

Consider the symmetrical function $f:[a,b]\to\mathbb{R}$, $f(t)=\left|t-\frac{a+b}{2}\right|^p$ for $p\ge1$. Then $f\left(\frac{a+b}{2}\right)=0$, $\frac{f(a)+f(b)}{2}=\frac{(b-a)^p}{2^p}$ and

$$\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|^p dt=\frac{(b-a)^p}{(p+1)2^p}.$$

Assume that g is convex, then by (2.1) and (2.2) we have

$$\begin{aligned}&\frac{(b-a)^p}{2^p}\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{p}{p+1}\frac{g(a)+g(b)}{2}\right]\\&\le\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|^p g(t)\,dt\le\frac{(b-a)^p}{(p+1)2^p}\frac{g(a)+g(b)}{2}.\end{aligned}\tag{4.1}$$

If we take in (4.1) $p=1$, then we get

$$\begin{aligned}\frac{b-a}{2}\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{4}\right]&\le\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|g(t)\,dt\\&\le(b-a)\frac{g(a)+g(b)}{8}.\end{aligned}\tag{4.2}$$

If we take in (4.1) $p=2$, then we get

$$\frac{(b-a)^2}{4}\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{3}\right] \le \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right)^2 g(t)\,dt \le \frac{g(a)+g(b)}{24}(b-a)^2. \tag{4.3}$$

Consider the function $f:[0,\pi]\to[0,1]$, $f(x)=\sin x$. The function is concave and symmetrical on $[0,\pi]$. If we use the inequalities (2.1) and (2.2) for the concave function $g:[0,\pi]\to\mathbb{R}$, then we have:

$$\frac{g(0)+g(\pi)}{\pi}\le\frac{1}{\pi}\int_0^\pi g(t)\sin t dt\le\frac{1}{\pi}\int_0^\pi g(t)\,dt-\frac{\pi-2}{2\pi}(g(0)+g(\pi)). \tag{4.4}$$

Also, if we consider the function $f:[\pi,2\pi]\to[0,1]$, $f(x)=\sin x$. The function is convex and symmetrical on $[\pi,2\pi]$. If we use the inequalities (2.1) and (2.2) for the convex function $g:[\pi,2\pi]\to\mathbb{R}$, then we have:

$$-\frac{g(\pi)+g(2\pi)}{\pi}\le\frac{1}{\pi}\int_\pi^{2\pi} g(t)\sin t dt \le\frac{\pi-2}{2\pi}(g(\pi)+g(2\pi))-\frac{1}{\pi}\int_\pi^{2\pi} g(t)\,dt. \tag{4.5}$$

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