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HERMITE-HADAMARD TYPE INEQUALITIES FOR PRODUCT OF SYMMETRIZED CONVEX FUNCTIONS

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Abstract. In this paper we obtain some Hermite-Hadamard type inequalities for product of symmetrized convex functions. Some examples of interest are provided as well.

1. Introduction

In [5], B. G. Pachpatte established two Hermite-Hadamard type inequalities for product of nonnegative convex functions f, $g:[a,b]\rightarrow[0,\infty)$ as follows:

$$
\begin{aligned}
(1.1)\qquad & 2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{6}\left[f(a)g(a)+f(b)g(b)\right] \\
& -\frac{1}{3}\left[f(a)g(b)+f(b)g(a)\right] \\
& \leq\frac{1}{b-a}\int_a^b f(t)g(t)\,dt \\
& \leq\frac{1}{3}\left[f(a)g(a)+f(b)g(b)\right]+\frac{1}{6}\left[f(a)g(b)+f(b)g(a)\right].
\end{aligned}
$$

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by other authors. For instance, in [6], B. G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is established by Kirmaci et. al. in [4]. In [7], M. Z. Sarikaya presented some integral inequalities for two h-convex functions. For recent results and generalizations concerning Hermite-Hadamard type inequality for product of two functions see [7] and the references given therein. For a monograph on Hermite-Hadamard type inequalities see [2].

For a function $f:[a,b]\rightarrow\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$
\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.2}
$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$
\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].
$$

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It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2) &= \frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\ &= \frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\ &\leq \frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\ &= \frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\ &= \alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [1]

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

for any $t\in\mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t)=3(a+b)$, $t\in\mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2}>0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2}<0$. Therefore if $a<0<b$ with $\frac{a+b}{2}>0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [1], see also [3] for an equivalent definition.

Definition 1. *We say that the function $f:[a,b]\to\mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $Con[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $SCon[a,b]$ the class of symmetrized convex functions, then from the above remarks we can conclude that

$$Con[a,b]\subsetneq SCon[a,b]. \tag{1.3}$$

Also, if $[c,d]\subset[a,b]$ and $f\in SCon[a,b]$, then this does not imply in general that $f\in SCon[c,d]$.

We have the following result [1]:

Theorem 1. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(t)\,dt\leq\frac{f(a)+f(b)}{2}. \tag{1.4}$$

We also have [1]:

Theorem 2. *Assume that $f:[a,b]\to\mathbb{R}$ is symmetrized convex on the interval $[a,b]$. Then for any $x\in[a,b]$ we have the bounds*

$$f\left(\frac{a+b}{2}\right)\leq\breve{f}(x)\leq\frac{f(a)+f(b)}{2}. \tag{1.5}$$

Corollary 1. *If $f:[a,b]\to\mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$ and $w:[a,b]\to[0,\infty)$ is integrable on $[a,b]$, then*

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \le \int_a^b w(t)\,\breve{f}(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \tag{1.6}$$

Moreover, if w is symmetric almost everywhere on $[a,b]$, i.e. $w(t)=w(a+b-t)$ for almost every $t\in[a,b]$, then

$$f\left(\frac{a+b}{2}\right)\int_a^b w(t)\,dt \le \int_a^b w(t)\,f(t)\,dt \le \frac{f(a)+f(b)}{2}\int_a^b w(t)\,dt. \tag{1.7}$$

Remark 1. *The inequality (1.7) was obtained by L. Fejér in 1906 for convex functions f and symmetric weights w. It has been shown now that this inequality remains valid for the larger class of symmetrized convex functions f on the interval $[a,b]$.*

For other results, see [1] and [3].

Motivated by the above results, in this paper we establish some Hermite-Hadamard type inequalities for product of symmetrized convex functions. Some examples of interest are provided as well.

2. Inequalities for Product of Symmetrized Convex Functions

The following Hermite-Hadamard type inequalities for the product of two functions hold:

Theorem 3. *Assume that both $f,\ g:[a,b]\to\mathbb{R}$ are symmetrized convex or symmetrized concave and integrable on the interval $[a,b]$. Then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt + f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\&\ge f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt + g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)\,dt,\end{aligned} \tag{2.1}$$

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt + \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}\\&\ge \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt + \frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt\end{aligned} \tag{2.2}$$

$$\begin{aligned}&\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt + g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)\,dt\\&\ge \frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt + \frac{f(a)+f(b)}{2}g\left(\frac{a+b}{2}\right),\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}&\frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt + f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt\\&\ge \frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt + \frac{g(a)+g(b)}{2}f\left(\frac{a+b}{2}\right).\end{aligned} \tag{2.4}$$

Proof. Assume that both of them are symmetrized convex, then by (1.5) we have

$$\left(\breve{f}(t)-f\left(\frac{a+b}{2}\right)\right)\left(\breve{g}(t)-g\left(\frac{a+b}{2}\right)\right)\geq 0$$

for any $t\in[a,b]$.

This is equivalent to

$$\breve{f}(t)\breve{g}(t)+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\geq f\left(\frac{a+b}{2}\right)\breve{g}(t)+g\left(\frac{a+b}{2}\right)\breve{f}(t) \tag{2.5}$$

for any $t\in[a,b]$.

Taking the integral mean $\frac{1}{b-a}\int_a^b$ in (2.5) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b\breve{f}(t)\breve{g}(t)\,dt+f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)\\ \geq f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b\breve{g}(t)\,dt+g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt\end{aligned} \tag{2.6}$$

and since

$$\begin{aligned}\frac{1}{b-a}\int_a^b\breve{g}(t)\,dt &= \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{1}{b-a}\int_a^b f(a+b-t)\,dt\right]\\ &= \frac{1}{b-a}\int_a^b f(t)\,dt.\end{aligned}$$

Also, both of them are symmetrized convex, then by (1.5) we have

$$\left(\frac{f(a)+f(b)}{2}-\breve{f}(t)\right)\left(\frac{g(a)+g(b)}{2}-\breve{g}(t)\right)\geq 0$$

for any $t\in[a,b]$, which by the same procedure produces (2.2).

Finally, if both of them are symmetrized convex, then by (1.5) we have

$$\left(\frac{f(a)+f(b)}{2}-\breve{f}(t)\right)\left(\breve{g}(t)-g\left(\frac{a+b}{2}\right)\right)\geq 0$$

for any $t\in[a,b]$, which is equivalent to

$$\frac{f(a)+f(b)}{2}\breve{g}(t)+g\left(\frac{a+b}{2}\right)\breve{f}(t)\geq\breve{g}(t)\breve{f}(t)+\frac{f(a)+f(b)}{2}g\left(\frac{a+b}{2}\right)$$

for any $t\in[a,b]$.

Taking the integral mean $\frac{1}{b-a}\int_a^b$ in this inequality, we get the desired result (2.3).

The inequality (2.4) follows from (2.3) by replacing f with g. □

We notice that, by the inclusion (1.3), the inequalities (2.1)-(2.4) hold *a fortiori* for functions that are both convex or concave on $[a,b]$. Also if the functions have opposite symmetrized convexities, then the inequality (2.1)-(2.4) hold with the reverse sign " $\leq$ ".

Remark 2. *Observe that*

$$\begin{aligned}
\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt &= \frac{1}{4}\int_a^b [f(t)+f(a+b-t)][g(t)+g(a+b-t)]\,dt \\
&= \frac{1}{4}\int_a^b [f(t)g(t)+f(a+b-t)g(t) \\
&+f(t)g(a+b-t)+f(a+b-t)g(a+b-t)]\,dt \\
&= \frac{1}{4}\left[\int_a^b f(t)g(t)\,dt+\int_a^b f(a+b-t)g(t)\,dt\right. \\
&\left.+\int_a^b f(a+b-t)g(t)\,dt+\int_a^b f(t)g(t)\,dt\right] \\
&= \frac{1}{2}\left[\int_a^b f(t)g(t)\,dt+\int_a^b f(a+b-t)g(t)\,dt\right] \\
&= \int_a^b \breve{f}(t)\,g(t)\,dt
\end{aligned} \tag{2.7}$$

since, by the change of variable $s=a+b-t$, $t\in[a,b]$ we have

$$\int_a^b f(a+b-t)g(t)\,dt = \int_a^b f(s)g(a+b-s)\,ds$$

and

$$\int_a^b f(a+b-t)g(a+b-t)\,dt = \int_a^b f(s)g(s)\,ds.$$

From (2.1)-(2.4) we then have

$$\begin{aligned}
\{H_{mix}(f,g;a,b),H_{mix}(g,f;a,b)\} &\geq \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\
&\geq \max\{H_{mid}(f,g;a,b),H_{tra}(f,g;a,b)\}
\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}
&H_{mix}(f,g;a,b) \\
&:= \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt+g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)\,dt \\
&-\frac{f(a)+f(b)}{2}g\left(\frac{a+b}{2}\right),
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&H_{mid}(f,g;a,b) \\
&:= f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt+g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)\,dt \\
&-f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)
\end{aligned} \tag{2.10}$$

and

$$H_{tra}(f,g;a,b) := \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt + \frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}. \tag{2.11}$$

The following particular case when one of the functions is symmetrical on $[a,b]$ holds:

Corollary 2. *Assume that $f:[a,b]\to\mathbb{R}$ is convex (concave) and symmetrical, i.e. $f(a+b-t)=f(t)$ for any $t\in[a,b]$. If $g:[a,b]\to\mathbb{R}$ is symmetrized convex (symmetrized concave) and integrable on the interval $[a,b]$, then*

$$\begin{aligned} &\min\{H_{mix}(f,g;a,b), H_{mix}(g,f;a,b)\} \\ &\geq \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt \\ &\geq \max\{H_{mid}(f,g;a,b), H_{tra}(f,g;a,b)\}. \end{aligned} \tag{2.12}$$

This inequality may be used to provide many interesting weighted inequalities in the case that when the weight is symmetrical.

Remark 3. *We also observe that the inequalities (2.1)-(2.4) can be written in an equivalent form as*

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\leq \left[\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right]\left[\frac{1}{b-a}\int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right)\right] \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\leq \left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right]\left[\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt\right] \end{aligned} \tag{2.14}$$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\geq -\left[\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt\right]\left[\frac{1}{b-a}\int_a^b g(t)\,dt - g\left(\frac{a+b}{2}\right)\right] \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\geq -\left[\frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right)\right]\left[\frac{g(a)+g(b)}{2} - \frac{1}{b-a}\int_a^b g(t)\,dt\right]. \end{aligned} \tag{2.16}$$

The case when the functions are positive provides some simpler inequalities as follows:

Theorem 4. *Assume that both $f,\ g:[a,b]\to[0,\infty)$ are symmetrized convex (symmetrized concave) and integrable on the interval $[a,b]$. Then we have*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\le (\ge)\, f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt \\ &\le (\ge)\, \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\le (\ge)\, \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt \\ &\le (\ge)\, \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2} \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\le (\ge)\, g\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, \frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt \\ &\le (\ge)\, \frac{g(a)+g(b)}{2}\frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le (\ge)\, \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}. \end{aligned} \tag{2.18}$$

Proof. If $f,\ g:[a,b]\to(0,\infty)$ are symmetrized convex, then by (1.5) we have

$$0\le f\left(\frac{a+b}{2}\right)\le \breve{f}(t)\le \frac{f(a)+f(b)}{2} \tag{2.19}$$

and

$$0\le g\left(\frac{a+b}{2}\right)\le \breve{g}(t)\le \frac{g(a)+g(b)}{2}, \tag{2.20}$$

for any $t\in[a,b]$.

If we multiply (2.19) by $\breve{g}(t)$, then we get

$$0\le f\left(\frac{a+b}{2}\right)\breve{g}(t)\le \breve{f}(t)\,\breve{g}(t)\le \frac{f(a)+f(b)}{2}\breve{g}(t),$$

for any $t\in[a,b]$.

If we take the integral mean in this inequality, then we get

$$\begin{aligned} 0&\le f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b \breve{g}(t)\,dt\le \frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt \\ &\le \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b \breve{g}(t)\,dt, \end{aligned}$$

namely

$$\begin{aligned} 0&\le f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt\le \frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt \\ &\le \frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt. \end{aligned} \tag{2.21}$$

Since, by (1.4) we have

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)\,dt \leq \frac{g(a)+g(b)}{2},$$

then

$$f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leq f\left(\frac{a+b}{2}\right)\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.22}$$

and

$$\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b g(t)\,dt \leq \frac{f(a)+f(b)}{2}\frac{g(a)+g(b)}{2}. \tag{2.23}$$

By utilising (2.21)-(2.23) we get (2.17).

The inequality (2.18) follows in a similar way by making use of (2.20) and a similar procedure. The details are omitted. □

3. Some Examples

Consider the symmetric convex function $f:[a,b]\to\mathbb{R}$ defined by $f(t)=\left|t-\frac{a+b}{2}\right|$. Then $f(a)=f(b)=\frac{b-a}{2}$, $f\left(\frac{a+b}{2}\right)=0$, and $\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{b-a}{4}$. Therefore, for $\ell(t)=t$, $t\in[a,b]$, we have

$$H_{\text{mix}}\left(\left|\ell-\frac{a+b}{2}\right|, g; a, b\right) = \frac{1}{2}\int_a^b g(t)\,dt - \frac{b-a}{4}g\left(\frac{a+b}{2}\right),$$

$$H_{\text{mix}}\left(g, \left|\ell-\frac{a+b}{2}\right|; a, b\right) = \frac{g(a)+g(b)}{8}(b-a),$$

$$H_{\text{mid}}\left(\left|\ell-\frac{a+b}{2}\right|, g; a, b\right) = g\left(\frac{a+b}{2}\right)\frac{b-a}{4}$$

and

$$H_{\text{tra}}\left(\left|\ell-\frac{a+b}{2}\right|, g; a, b\right) = \frac{1}{2}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{8}(b-a).$$

From (2.12) we get

$$\begin{aligned}
&\min\left\{\frac{1}{2}\int_a^b g(t)\,dt - \frac{b-a}{4}g\left(\frac{a+b}{2}\right), \frac{g(a)+g(b)}{8}(b-a)\right\} \\
&\geq \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| g(t)\,dt \\
&\geq \max\left\{g\left(\frac{a+b}{2}\right)\frac{b-a}{4}, \frac{1}{2}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{8}(b-a)\right\},
\end{aligned} \tag{3.1}$$

provided that $g:[a,b]\to\mathbb{R}$ is symmetrized convex and integrable on $[a,b]$.

If h is a convex function on $[a,b]$, then the following result that is known in literature as *Bullen's inequality* holds:

$$\frac{1}{b-a}\int_a^b h(t)\,dt \leq \frac{1}{2}\left[h\left(\frac{a+b}{2}\right) + \frac{h(a)+h(b)}{2}\right].$$

We observe that, if $g:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then

$$\frac{1}{2}\int_a^b g(t)\,dt - \frac{b-a}{4}g\left(\frac{a+b}{2}\right) \leq \frac{g(a)+g(b)}{8}(b-a),$$

which gives that

$$\min\left\{\frac{1}{2}\int_a^b g(t)\,dt-\frac{b-a}{4}g\left(\frac{a+b}{2}\right),\frac{g(a)+g(b)}{8}(b-a)\right\}$$
$$=\frac{1}{2}(b-a)\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{1}{2}g\left(\frac{a+b}{2}\right)\right].$$

Also

$$g\left(\frac{a+b}{2}\right)\frac{b-a}{4}\geq\frac{1}{2}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{8}(b-a),$$

which gives that

$$\max\left\{g\left(\frac{a+b}{2}\right)\frac{b-a}{4},\frac{1}{2}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{8}(b-a)\right\}=g\left(\frac{a+b}{2}\right)\frac{b-a}{4}.$$

Therefore, if $g:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then

$$\begin{aligned}\frac{1}{2}(b-a)&\left[\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{1}{2}g\left(\frac{a+b}{2}\right)\right]\\&\geq\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|g(t)\,dt\geq\frac{b-a}{4}g\left(\frac{a+b}{2}\right).\end{aligned}\tag{3.2}$$

This inequality is equivalent to

$$\begin{aligned}\frac{1}{2}(b-a)&\left[\frac{1}{b-a}\int_a^b g(t)\,dt-g\left(\frac{a+b}{2}\right)\right]\\&\geq\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|g(t)\,dt-\frac{b-a}{4}g\left(\frac{a+b}{2}\right)\geq 0,\end{aligned}\tag{3.3}$$

where $g:[a,b]\to\mathbb{R}$ is convex on $[a,b]$.

Consider the symmetric convex function $f:[a,b]\to\mathbb{R}$ defined by $f(t)=\left(t-\frac{a+b}{2}\right)^2$. Then $f(a)=f(b)=\frac{(b-a)^2}{4}$, $f\left(\frac{a+b}{2}\right)=0$ and $\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2dt=\frac{(b-a)^2}{12}$. Therefore

$$H_{\mathrm{mix}}\left(\left(\ell-\frac{a+b}{2}\right)^2,g;a,b\right)=\frac{1}{2}(b-a)\left[\frac{1}{2}\int_a^b g(t)\,dt-\frac{1}{3}(b-a)\,g\left(\frac{a+b}{2}\right)\right],$$

$$H_{\mathrm{mix}}\left(g,\left(\ell-\frac{a+b}{2}\right)^2;a,b\right)=\frac{(b-a)^2}{24}[g(a)+g(b)],$$

$$H_{\mathrm{mid}}\left(\left(\ell-\frac{a+b}{2}\right)^2,g;a,b\right)=\frac{(b-a)^2}{12}g\left(\frac{a+b}{2}\right)$$

and

$$H_{\mathrm{tra}}\left(\left(\ell-\frac{a+b}{2}\right)^2,g;a,b\right)=\frac{1}{4}(b-a)\left[\int_a^b g(t)\,dt-\frac{b-a}{3}[g(a)+g(b)]\right]$$

From (2.12) we get

$$\frac{1}{2}(b-a)\min\left\{\frac{1}{2}\int_a^b g(t)\,dt-\frac{1}{3}(b-a)\,g\left(\frac{a+b}{2}\right),\frac{b-a}{12}[g(a)+g(b)]\right\}\tag{3.4}$$
$$\geq\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)^2 g(t)\,dt$$
$$\geq\frac{1}{4}(b-a)\max\left\{\frac{b-a}{3}g\left(\frac{a+b}{2}\right),\int_a^b g(t)\,dt-\frac{b-a}{3}[g(a)+g(b)]\right\},$$

provided that $g:[a,b]\to\mathbb{R}$ is symmetrized convex and integrable on $[a,b]$.

Consider the symmetric concave function $f:[a,b]\to\mathbb{R}$ defined by $f(t)=(b-t)(t-a)$. Then $f(a)=f(b)=0$, $f\left(\frac{a+b}{2}\right)=\frac{(b-a)^2}{4}$ and $\frac{1}{b-a}\int_a^b(b-t)(t-a)\,dt=\frac{(b-a)^2}{6}$. Therefore

$$H_{\mathrm{mix}}((b-\ell)(\ell-a),g;a,b):=\frac{(b-a)^2}{6}g\left(\frac{a+b}{2}\right),$$

$$H_{\mathrm{mix}}(g,(b-\ell)(\ell-a);a,b)=\frac{1}{4}(b-a)\left[\int_a^b g(t)\,dt-\frac{1}{6}(b-a)[g(a)+g(b)]\right],$$

$$H_{\mathrm{mid}}((b-\ell)(\ell-a),g;a,b)=\frac{1}{4}(b-a)\left[\int_a^b g(t)\,dt-\frac{1}{3}(b-a)\,g\left(\frac{a+b}{2}\right)\right]$$

and

$$H_{\mathrm{tra}}((b-\ell)(\ell-a),g;a,b)=\frac{(b-a)^2}{12}[g(a)+g(b)].$$

Then by (2.12) we have

$$\frac{1}{2}(b-a)\tag{3.5}$$
$$\times\min\left\{\frac{b-a}{3}g\left(\frac{a+b}{2}\right),\frac{1}{2}\left[\int_a^b g(t)\,dt-\frac{1}{6}(b-a)[g(a)+g(b)]\right]\right\}$$
$$\geq\frac{1}{b-a}\int_a^b(b-t)(t-a)\,g(t)\,dt$$
$$\geq\frac{1}{4}(b-a)$$
$$\times\max\left\{\int_a^b g(t)\,dt-\frac{1}{3}(b-a)\,g\left(\frac{a+b}{2}\right),\frac{b-a}{3}[g(a)+g(b)]\right\},$$

provided that $g:[a,b]\to\mathbb{R}$ is symmetrized concave and integrable on $[a,b]$.

Now, consider the positive, convex and symmetric function $f(t)=\exp\left[k\left|t-\frac{a+b}{2}\right|\right]$, $k>0$. Then, by (2.17), we have for any integrable symmetrized convex function $g:[a,b]\subset\mathbb{R}\to[0,\infty)$ that

$$g\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b g(t)\,dt\leq\frac{1}{b-a}\int_a^b\exp\left[k\left|t-\frac{a+b}{2}\right|\right]g(t)\,dt\tag{3.6}$$
$$\leq\frac{1}{2}\exp\left[k\left(\frac{b-a}{2}\right)\right]\frac{1}{b-a}\int_a^b g(t)\,dt\leq\frac{1}{2}\exp\left[k\left(\frac{b-a}{2}\right)\right]\frac{g(a)+g(b)}{2}.$$

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