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ON SOME INTEGRAL INEQUALITIES FOR SYMMETRIZED SYNCHRONOUS FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish some integral inequalities for the product of symmetrized synchronous/asynchronous functions. Refinements and reverses of Cauchy-Bunyakovsky-Schwarz inequality for one function and some examples for logarithmic and exponential functions are also given.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$.

If f is convex on $[a,b]$, then for any $t_1,\ t_2\in[a,b]$ and $\alpha,\beta\geq 0$ with $\alpha+\beta=1$ we have

$$\begin{aligned}\breve{f}(\alpha t_1+\beta t_2)&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(a+b-\alpha t_1-\beta t_2)\right]\\&=\frac{1}{2}\left[f(\alpha t_1+\beta t_2)+f(\alpha(a+b-t_1)+\beta(a+b-t_2))\right]\\&\leq\frac{1}{2}\left[\alpha f(t_1)+\beta f(t_2)+\alpha f(a+b-t_1)+\beta f(a+b-t_2)\right]\\&=\frac{1}{2}\alpha\left[f(t_1)+f(a+b-t_1)\right]+\frac{1}{2}\beta\left[f(t_2)+f(a+b-t_2)\right]\\&=\alpha\breve{f}(t_1)+\beta\breve{f}(t_2),\end{aligned}$$

which shows that $\breve{f}$ is convex on $[a,b]$.

Consider the real numbers $a<b$ and define the function $f_0:[a,b]\to\mathbb{R}$, $f_0(t)=t^3$. We have [7]

$$\breve{f}_0(t):=\frac{1}{2}\left[t^3+(a+b-t)^3\right]=\frac{3}{2}(a+b)t^2-\frac{3}{2}(a+b)^2t+\frac{1}{2}(a+b)^3$$

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for any $t \in \mathbb{R}$.

Since the second derivative $\left(\breve{f}_0\right)''(t) = 3(a+b)$, $t \in \mathbb{R}$, then $\breve{f}_0$ is strictly convex on $[a,b]$ if $\frac{a+b}{2} > 0$ and strictly concave on $[a,b]$ if $\frac{a+b}{2} < 0$. Therefore if $a < 0 < b$ with $\frac{a+b}{2} > 0$, then we can conclude that f_0 is not convex on $[a,b]$ while $\breve{f}_0$ is convex on $[a,b]$.

We can introduce the following concept of convexity [7], see also [10] for an equivalent definition.

Definition 1. *We say that the function $f : [a,b] \to \mathbb{R}$ is symmetrized convex (concave) on the interval $[a,b]$ if the symmetrical transform $\breve{f}$ is convex (concave) on $[a,b]$.*

Now, if we denote by $\mathcal{C}[a,b]$ the closed convex cone of convex functions defined on $[a,b]$ and by $\mathcal{SC}[a,b]$ the closed convex cone of symmetrized convex functions, then from the above remarks we can conclude that

$$\mathcal{C}[a,b] \subsetneq \mathcal{SC}[a,b]. \tag{1.2}$$

Also, if $[c,d] \subset [a,b]$ and $f \in \mathcal{SC}[a,b]$, then this does not imply in general that $f \in \mathcal{SC}[c,d]$.

We have the following result [7], [10] :

Theorem 1. *Assume that $f : [a,b] \to \mathbb{R}$ is symmetrized convex and integrable on the interval $[a,b]$. Then we have the Hermite-Hadamard inequalities*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}. \tag{1.3}$$

For a monograph on Hermite-Hadamard type inequalities see [9].

In a similar way, we can introduce the following concept as well:

Definition 2. *We say that the function $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ if the anti-symmetrical transform $\tilde{f}$ is monotonic nondecreasing (nonincreasing) on the interval $[a,b]$.*

If f is monotonic nondecreasing on $[a,b]$, then for any $t_1, t_2 \in [a,b]$ we have

$$\begin{aligned}\tilde{f}(t_2) - \tilde{f}(t_1) &= \frac{1}{2}[f(t_2) - f(a+b-t_2)] - \frac{1}{2}[f(t_1) - f(a+b-t_1)] \\ &= \frac{1}{2}[f(t_2) - f(t_1)] + \frac{1}{2}[f(a+b-t_1) - f(a+b-t_2)] \\ &\ge 0,\end{aligned}$$

which shows that $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing on the interval $[a,b]$.

Consider the real numbers $a < b$ and define the function $f_0 : [a,b] \to \mathbb{R}$, $f_0(t) = t^2$. We have

$$\tilde{f}_0(t) := \frac{1}{2}\left[t^2 - (a+b-t)^2\right] = (a+b)t - \frac{1}{2}(a+b)^2$$

and $\left(\tilde{f}_0\right)'(t) = a+b$, therefore $f : [a,b] \to \mathbb{R}$ is asymmetrized monotonic nondecreasing (nonincreasing) on the interval $[a,b]$ provided $\frac{a+b}{2} > 0\,(<0)$. So, if we take $a < 0 < b$ with $\frac{a+b}{2} > 0$, then f is asymmetrized monotonic nondecreasing on $[a,b]$ but not monotonic nondecreasing on $[a,b]$.

If we denote by $\mathcal{M}^{\nearrow}[a,b]$ the closed convex cone of monotonic nondecreasing functions defined on $[a,b]$ and by $\mathcal{AM}^{\nearrow}[a,b]$ the closed convex cone of asymmetrized monotonic nondecreasing functions, then from the above remarks we can conclude that

$$\mathcal{M}^{\nearrow}[a,b] \subsetneq \mathcal{AM}^{\nearrow}[a,b]. \tag{1.4}$$

Also, if $[c,d] \subset [a,b]$ and $f \in \mathcal{AM}^{\nearrow}[a,b]$, then this does not imply in general that $f \in \mathcal{AM}^{\nearrow}[c,d]$.

We recall that the pair of functions (f,g) defined on $[a,b]$ are called *synchronous (asynchronous)* on $[a,b]$ if

$$(f(t)-f(s))(g(t)-g(s)) \geq (\leq)\, 0 \tag{1.5}$$

for any $t,\ s \in [a,b]$. It is clear that if both functions (f,g) are monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are synchronous on $[a,b]$. There are also functions that change monotonicity on $[a,b]$, but as a pair they are still synchronous. For instance if $a<0<b$ and $f,g:[a,b]\to\mathbb{R}$, $f(t)=t^2$ and $g(t)=t^4$, then

$$(f(t)-f(s))(g(t)-g(s)) = \left(t^2-s^2\right)\left(t^4-s^4\right) = \left(t^2-s^2\right)^2\left(t^2+s^2\right) \geq 0$$

for any $t,\ s \in [a,b]$, which show that (f,g) is synchronous.

Definition 3. *We say that the pair of functions (f,g) defined on $[a,b]$ is called asymmetrized synchronous (asynchronous) on $[a,b]$ if the pair of transforms $\left(\tilde{f},\tilde{g}\right)$ is synchronous (asynchronous) on $[a,b]$, namely*

$$\left(\tilde{f}(t)-\tilde{f}(s)\right)(\tilde{g}(t)-\tilde{g}(s)) \geq (\leq)\, 0 \tag{1.6}$$

for any $t,s \in [a,b]$.

It is clear that if f, g are asymmetrized monotonic nondecreasing (nonincreasing) on $[a,b]$ then they are asymmetrized synchronous on $[a,b]$.

In the recent paper we obtained amongst others the following results:

Theorem 2. *Assume that f, g are asymmetrized synchronous (asynchronous) and integrable functions on $[a,b]$. Then*

$$\int_a^b \tilde{f}(t)\,g(t)\,dt \geq (\leq)\, 0. \tag{1.7}$$

If both f, g are asymmetrized monotonic nondecreasing (nonincreasing) and integrable functions on $[a,b]$, then

$$\frac{1}{4}|f(b)-f(a)|\,|g(b)-g(a)| \geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt \geq 0, \tag{1.8}$$

and

$$\begin{aligned}\frac{1}{2}\min&\left\{|f(b)-f(a)|\frac{1}{b-a}\int_a^b |g(t)|\,dt, |g(b)-g(a)|\frac{1}{b-a}\int_a^b |f(t)|\,dt\right\} \\ &\geq \frac{1}{b-a}\int_a^b \tilde{f}(t)\,g(t)\,dt \geq 0.\end{aligned} \tag{1.9}$$

We can introduce the following concept as well:

Definition 4. *We say that the pair of functions* (f,g) *defined on* $[a,b]$ *is called symmetrized synchronous (asynchronous) on* $[a,b]$ *if the pair of symmetrized transforms* $(\breve{f},\breve{g})$ *is synchronous (asynchronous) on* $[a,b]$, *namely*

$$\left(\breve{f}(t)-\breve{f}(s)\right)\left(\breve{g}(t)-\breve{g}(s)\right)\geq(\leq)\,0 \tag{1.10}$$

for any $t,s\in[a,b]$.

Now, assume that the function $x:[a,b]\to I$, where I is an interval of real numbers, and (ϕ,ψ) is a pair of synchronous (asynchronous) functions defined on the interval I. Consider the functions $f,\,g:[a,b]\to\mathbb{R}$ defined by $f=\phi\circ\breve{x}$ and $g=\psi\circ\breve{x}$. Then the functions f and g are symmetrical on $[a,b]$ and $\breve{f}=\phi\circ\breve{x}$ and $\breve{g}=\psi\circ\breve{x}$. Since (ϕ,ψ) is a pair of synchronous (asynchronous) functions, it follows that $(\breve{f},\breve{g})$ is synchronous (asynchronous) on $[a,b]$, namely the pair of functions (f,g) defined on $[a,b]$ is symmetrized synchronous (asynchronous) on $[a,b]$. Therefore, we can give many example of symmetrized synchronous (asynchronous) functions on $[a,b]$. For instance, if (ϕ,ψ) is a pair of synchronous (asynchronous) functions defined on the interval $[0,\infty)$, then the functions $f,\,g:[a,b]\to\mathbb{R}$ defined by $f(t)=\phi\left(\left|t-\frac{a+b}{2}\right|^{p}\right)$ and $g(t)=\psi\left(\left|t-\frac{a+b}{2}\right|^{p}\right)$ with $p>0$ are symmetrized synchronous (asynchronous) on $[a,b]$.

One of the most important results for synchronous (asynchronous) and integrable functions $f,\,g$ on $[a,b]$ is the well-known *Čebyšev's inequality*:

$$\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt\geq(\leq)\,\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.11}$$

For integral inequalities of Čebyšev's type, see [1]-[6], [8], [11]-[20] and the references therein.

Motivated by the above results, we establish in this paper some inequalities for symmetrized synchronous (asynchronous) functions on $[a,b]$. Refinements and reverses of Cauchy-Bunyakovsky-Schwarz inequality for one function and some examples for logarithmic and exponential functions are also given.

2. Main Results

We have the following Čebyšev's type result:

Theorem 3. *Assume that the pair of integrable functions* (f,g) *defined on* $[a,b]$ *is symmetrized synchronous (asynchronous) on* $[a,b]$, *then*

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt\geq(\leq)\,\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{2.1}$$

Proof. Since $(\breve{f},\breve{g})$ is synchronous (asynchronous) on $[a,b]$, then by Čebyšev's inequality (1.11) we have

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt\geq(\leq)\,\frac{1}{b-a}\int_a^b \breve{f}(t)\,dt\frac{1}{b-a}\int_a^b \breve{g}(t)\,dt. \tag{2.2}$$

Observe that, by the change of variable $s = a + b - t$, $t \in [a, b]$ we have

$$\begin{aligned}\int_a^b \breve{f}(t)\,dt &= \frac{1}{2}\int_a^b [f(t) + f(a+b-t)] \\ &= \frac{1}{2}\left[\int_a^b f(t)\,dt + \int_a^b f(a+b-t)\,dt\right] = \int_a^b f(t)\,dt,\end{aligned}$$

$$\int_a^b \breve{g}(t)\,dt = \int_a^b g(t)\,dt$$

and

$$\begin{aligned}\int_a^b \breve{f}(t)\,\breve{g}(t)\,dt &= \frac{1}{4}\int_a^b [f(t) + f(a+b-t)][g(t) + g(a+b-t)]\,dt \\ &= \frac{1}{4}\int_a^b [f(t)\,g(t) + f(a+b-t)\,g(t) \\ &\quad + f(t)\,g(a+b-t) + f(a+b-t)\,g(a+b-t)]\,dt \\ &= \frac{1}{4}\left[\int_a^b f(t)\,g(t)\,dt + \int_a^b f(a+b-t)\,g(t)\,dt\right. \\ &\quad \left. + \int_a^b f(a+b-t)\,g(t)\,dt + \int_a^b f(t)\,g(t)\,dt\right] \\ &= \frac{1}{2}\left[\int_a^b f(t)\,g(t)\,dt + \int_a^b f(a+b-t)\,g(t)\,dt\right] \\ &= \int_a^b \breve{f}(t)\,g(t)\,dt\end{aligned} \tag{2.3}$$

since,

$$\int_a^b f(a+b-t)\,g(t)\,dt = \int_a^b f(s)\,g(a+b-s)\,ds$$

and

$$\int_a^b f(a+b-t)\,g(a+b-t)\,dt = \int_a^b f(s)\,g(s)\,ds.$$

By making use of (2.2) we obtain the desired result (2.1). □

In addition, we also have:

Theorem 4. *Assume that the pair of integrable functions (f, g) defined on $[a, b]$ is symmetrized synchronous (asynchronous) on $[a, b]$, then*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt \\ &\geq (\leq)\left(\breve{f}(s) - \frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt - \breve{g}(s)\right)\end{aligned} \tag{2.4}$$

for any $s \in [a, b]$.

In particular,

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.5}$$
$$\geq(\leq)\left(f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)$$
$$\times\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g\left(\frac{a+b}{2}\right)\right)$$

and

$$\frac{1}{b-a}\int_a^b \breve{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt \tag{2.6}$$
$$\geq(\leq)\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)$$
$$\times\left(\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right).$$

Proof. Since $(\check{f},\check{g})$ is synchronous (asynchronous) on $[a,b]$, then

$$\check{f}(t)\,\check{g}(t)-\check{f}(s)\,\check{g}(t)-\check{f}(t)\,\check{g}(s)+\check{f}(s)\,\check{g}(s)\geq(\leq)\,0 \tag{2.7}$$

for any $t,\ s\in[a,b]$.

Taking the integral mean over $t\in[a,b]$ we get

$$\frac{1}{b-a}\int_a^b \check{f}(t)\,\check{g}(t)\,dt-\check{f}(s)\,\frac{1}{b-a}\int_a^b \check{g}(t)\,dt$$
$$-\,\check{g}(s)\,\frac{1}{b-a}\int_a^b \check{f}(t)\,dt+\check{f}(s)\,\check{g}(s)$$
$$\geq(\leq)\,0$$

for any $s\in[a,b]$.

This is equivalent to

$$\frac{1}{b-a}\int_a^b \check{f}(t)\,\check{g}(t)\,dt\geq(\leq)\,\check{f}(s)\,\frac{1}{b-a}\int_a^b \check{g}(t)\,dt \tag{2.8}$$
$$+\,\check{g}(s)\,\frac{1}{b-a}\int_a^b \check{f}(t)\,dt-\check{f}(s)\,\check{g}(s)$$

for any $s\in[a,b]$.

Now, if we subtract in both sides of the inequality (2.8)

$$\frac{1}{b-a}\int_a^b \check{f}(t)\,dt\frac{1}{b-a}\int_a^b \check{g}(t)\,dt,$$

then we get

$$\begin{aligned}(2.9)\qquad & \frac{1}{b-a}\int_a^b \check{f}(t)\,\check{g}(t)\,dt-\frac{1}{b-a}\int_a^b \check{f}(t)\,dt\frac{1}{b-a}\int_a^b \check{g}(t)\,dt\\ &\geq (\leq)\,\check{f}(s)\,\frac{1}{b-a}\int_a^b \check{g}(t)\,dt+\check{g}(s)\,\frac{1}{b-a}\int_a^b \check{f}(t)\,dt-\check{f}(s)\,\check{g}(s)\\ &-\frac{1}{b-a}\int_a^b \check{f}(t)\,dt\frac{1}{b-a}\int_a^b \check{g}(t)\,dt\\ &=\left(\check{f}(s)-\frac{1}{b-a}\int_a^b \check{f}(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b \check{g}(t)\,dt-\check{g}(s)\right)\end{aligned}$$

and since

$$\frac{1}{b-a}\int_a^b \check{f}(t)\,dt=\frac{1}{b-a}\int_a^b f(t)\,dt,\ \ \frac{1}{b-a}\int_a^b \check{g}(t)\,dt=\frac{1}{b-a}\int_a^b g(t)\,dt$$

and, by (2.3)

$$\frac{1}{b-a}\int_a^b \check{f}(t)\,\check{g}(t)\,dt=\frac{1}{b-a}\int_a^b \check{f}(t)\,g(t)\,dt,$$

then by (2.9) we get the desired inequality (2.4).

Since

$$\check{f}\left(\frac{a+b}{2}\right)=f\left(\frac{a+b}{2}\right)\ \text{ and }\ \check{f}(a)=\check{f}(b)=\frac{f(a)+f(b)}{2}$$

and the similar relations for g, hence (2.5) and (2.6) follow from (2.4). □

Remark 1. *We observe that if the pair of integrable functions (f,g) defined on $[a,b]$ is symmetrized synchronous and one is symmetrized convex while the other is symmetrized concave, then we have the following refinements of (2.1)*

$$\begin{aligned}(2.10)\quad & \frac{1}{b-a}\int_a^b \check{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\ &\geq\left(f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\left(\frac{1}{b-a}\int_a^b g(t)\,dt-g\left(\frac{a+b}{2}\right)\right)\geq 0\end{aligned}$$

and

$$\begin{aligned}(2.11)\qquad & \frac{1}{b-a}\int_a^b \check{f}(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\\ &\geq\left(\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right)\\ &\times\left(\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right)\\ &\geq 0.\end{aligned}$$

We have the following refinement of Cauchy-Bunyakovsky-Schwarz integral inequality for a real-valued functions:

Theorem 5. *For any integrable function $f:[a,b]\to\mathbb{R}$, we have*

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt \geq \frac{1}{b-a}\int_a^b \breve{f}(t)\,f(t)\,dt \geq \left(\frac{1}{b-a}\int_a^b f(t)\right)^2 \tag{2.12}$$

or, equivalently

$$\begin{aligned}\frac{1}{b-a}&\int_a^b f^2(t)\,dt \\ &\geq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt + \frac{1}{b-a}\int_a^b f(t)\,f(a+b-t)\,dt\right) \\ &\geq \left(\frac{1}{b-a}\int_a^b f(t)\right)^2.\end{aligned} \tag{2.13}$$

Proof. By taking $g=f$ in (2.1), since for any integrable function f, the pair (f,f) is symmetrized synchronous on $[a,b]$, we get the second inequality in (2.12).

By the Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}\int_a^b f(t)\,f(a+b-t)\,dt &\leq \int_a^b |f(t)\,f(a+b-t)|\,dt \\ &\leq \left(\int_a^b f^2(t)\,dt\right)^{1/2}\left(\int_a^b f^2(a+b-t)\,dt\right)^{1/2} \\ &= \left(\int_a^b f^2(t)\,dt\right)^{1/2}\left(\int_a^b f^2(t)\,dt\right)^{1/2} = \int_a^b f^2(t)\,dt,\end{aligned}$$

which proves the first inequality in (2.13). $\square$

The following reverse inequality also holds:

Theorem 6. *Assume that the measurable function $f:[a,b]\to\mathbb{R}$ satisfies the conditions*

$$-\infty < m \leq f(t) \leq M < \infty \text{ for a.e. } t\in[a,b]$$

and

$$-\infty < m \leq \breve{m} \leq \breve{f}(t) \leq \breve{M} \leq M < \infty \text{ for a.e. } t\in[a,b]$$

for the constants m, $\breve{m}$, M, $\breve{M}$. Then

$$\begin{aligned}0 &\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt + \frac{1}{b-a}\int_a^b f(t)\,f(a+b-t)\,dt\right) \\ &\quad - \left(\frac{1}{b-a}\int_a^b f(t)\right)^2 \\ &\leq \frac{1}{4}(M-m)\left(\breve{M}-\breve{m}\right) \leq \frac{1}{4}(M-m)^2.\end{aligned} \tag{2.14}$$

Proof. We use G. Grüss' inequality [12], who showed that

$$|T(h,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{2.15}$$

provided $m,\ M, n,\ N$ are real numbers with the property

$$-\infty < m \leq h \leq M < \infty,\ -\infty < n \leq g \leq N < \infty \text{ a.e. on } [a,b], \tag{2.16}$$

where the Čebyšev functional $T(h,g)$ is defined by

$$T(h,g) := \frac{1}{b-a}\int_a^b h(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt. \tag{2.17}$$

By taking in (2.15) $h = f$ and $g = \breve{f}$ we get (2.14). □

Another less well known inequality for $T(h,g)$ was derived in 1882 by Čebyšev [4] under the assumption that f', g' exist and are bounded in (a,b) and is given by

$$|T(h,g)| \leq \frac{1}{12}\|h'\|_\infty \|g'\|_\infty (b-a)^2, \tag{2.18}$$

where $\|h'\|_\infty := \sup_{t\in(a,b)} |h'(t)|$. The constant $\frac{1}{12}$ cannot be improved in the general case. This inequality can be extended for absolutely continuos functions $f,\ g : [a,b] \to \mathbb{R}$ for which the derivatives are essentially bounded, namely f', $g' \in L_\infty[a,b]$.

Theorem 7. *Assume that the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous and $f' \in L_\infty[a,b]$. Then we have*

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt + \frac{1}{b-a}\int_a^b f(t)\,f(a+b-t)\,dt\right) \\ &\quad - \left(\frac{1}{b-a}\int_a^b f(t)\right)^2 \\ &\leq \frac{1}{24}\|f' - f'(a+b-\cdot)\|_\infty \|f'\|_\infty (b-a)^2 \leq \frac{1}{12}\|f'\|_\infty^2 (b-a)^2. \end{aligned} \tag{2.19}$$

The proof follows by Čebyšev's inequality (2.18) for the functions $h = \breve{f}(t)$ and $g = f$ and by observing that $h' = \frac{1}{2}(f' - f'(a+b-\cdot))$.

Corollary 1. *Assume that the function $f : [a,b] \to \mathbb{R}$ is absolutely continuous and f' is H-r-Hölder continuous, i.e.*

$$|f'(t) - f'(s)| \leq H|t-s|^r \text{ for any } t,s \in [a,b] \tag{2.20}$$

for some $H > 0$ and $r \in (0,1]$. Then we have

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt + \frac{1}{b-a}\int_a^b f(t)\,f(a+b-t)\,dt\right) \\ &\quad - \left(\frac{1}{b-a}\int_a^b f(t)\right)^2 \\ &\leq \frac{1}{24}H\|f'\|_\infty (b-a)^{2+r}. \end{aligned} \tag{2.21}$$

Proof. From (2.20) we have

$$|f'(t)-f'(a+b-t)| \le H|t-a-b+t|^r = 2^r H\left|t-\frac{a+b}{2}\right|^r$$
$$\le 2^r H\frac{(b-a)^r}{2^r} = H(b-a)^r$$

for any $t\in[a,b]$. Taking the supremum over $t\in[a,b]$, we get

$$\|f'-f'(a+b-\cdot)\|_\infty \le H(b-a)^r$$

and by (2.19) we get (2.21). □

Remark 2. *If $f:[a,b]\to\mathbb{R}$ is absolutely continuous and f' is K-Lipschitzian, i.e. $r=1$ and $H=K>0$, then from (2.21) we have*

$$0 \le \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f^2(t)\,dt+\frac{1}{b-a}\int_a^b f(t)\,f(a+b-t)\,dt\right) - \left(\frac{1}{b-a}\int_a^b f(t)\right)^2 \le \frac{1}{24}K\|f'\|_\infty (b-a)^3. \tag{2.22}$$

3. Some Examples

Consider the function $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ given by $f(t)=\ln t$. We have

$$\breve{f}(t)=\frac{1}{2}[\ln t+\ln(a+b-t)]$$

and

$$\left(\breve{f}(t)\right)'=\frac{1}{2}\left(\frac{1}{t}-\frac{1}{a+b-t}\right)=\frac{\frac{a+b}{2}-t}{t(a+b-t)},\ t\in(a,b)$$

and

$$\left(\breve{f}(t)\right)''=-\frac{1}{2}\left(\frac{1}{t^2}+\frac{1}{(a+b-t)^2}\right),\ t\in(a,b).$$

These shows that $\breve{f}$ is strictly increasing on $\left(a,\frac{a+b}{2}\right)$ and strictly decreasing on $\left(\frac{a+b}{2},b\right)$ and strictly concave on (a,b). Therefore

$$\ln G(a,b)\le \breve{f}(t)\le \ln A(a,b)\ \text{ for any } t\in(a,b),$$

where $G(a,b):=\sqrt{ab}$ is the *geometric mean* and $A(a,b):=\frac{1}{2}(a+b)$ is the *arithmetic mean* of positive numbers a, b.

Since

$$\frac{1}{b-a}\int_a^b \ln t\,dt=\ln I(a,b)$$

where $I(a,b)$ is the *identric mean* defined by

$$I(a,b):=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}.$$

By using the inequality (2.14) for $f(t) = \ln t$, $t \in [a,b]$, we have

$$0 \leq \frac{1}{2}\left(\frac{1}{b-a}\int_a^b (\ln t)^2\,dt + \frac{1}{b-a}\int_a^b \ln t \ln(a+b-t)\,dt\right) - (I(a,b))^2 \tag{3.1}$$

$$\leq \frac{1}{4}\ln\left(\frac{b}{a}\right)\ln\left(\frac{A(a,b)}{G(a,b)}\right) \leq \frac{1}{4}\left(\ln\left(\frac{b}{a}\right)\right)^2.$$

Consider the function $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $f(t) = \exp(\alpha t)$ for $\alpha > 0$. We have

$$\breve{f}(t) = \frac{1}{2}\left[\exp(\alpha t) + \exp(\alpha(a+b-t))\right],$$

$$\left(\breve{f}(t)\right)' = \frac{1}{2}\alpha\left[\exp(\alpha t) - \exp(\alpha(a+b-t))\right]$$

and

$$\left(\breve{f}(t)\right)' = \frac{1}{2}\alpha^2\left[\exp(\alpha t) + \exp(\alpha(a+b-t))\right]$$

for any $t \in [a,b]$.

These shows that $\breve{f}$ is strictly decreasing on $\left(a, \frac{a+b}{2}\right)$ and strictly increasing on $\left(\frac{a+b}{2}, b\right)$ and strictly convex on (a,b). Therefore

$$\exp(\alpha A(a,b)) \leq \breve{f}(t) \leq A(\exp(\alpha a), \exp(\alpha b))$$

for any $t \in [a,b]$.

If we define the exponential mean $E(u,v)$ for $u \neq v$ by

$$E(u,v) := \frac{\exp u - \exp v}{u - v},$$

then

$$\frac{1}{b-a}\int_a^b f^2(t)\,dt = \frac{1}{b-a}\int_a^b \exp(2\alpha t)\,dt = E(2\alpha b, 2\alpha a),$$

$$\frac{1}{b-a}\int_a^b f(t) f(a+b-t)\,dt = \frac{1}{b-a}\int_a^b \exp(2\alpha t)\exp[2\alpha(a+b-t)]\,dt$$

$$= \exp[2\alpha(a+b)]$$

and

$$\frac{1}{b-a}\int_a^b f(t) = \frac{1}{b-a}\int_a^b \exp(\alpha t)\,dt = E(\alpha b, \alpha a).$$

By using the inequality (2.14) for $f(t) = \exp(\alpha t)$, $t \in [a,b]$, we have

$$0 \leq \frac{1}{2}\left(E(2\alpha b, 2\alpha a) + \exp[2\alpha(a+b)]\right) - [E(\alpha b, \alpha a)]^2 \tag{3.2}$$

$$\leq \frac{1}{8}\left[\exp(\alpha b) - \exp(\alpha a)\right]\left[\exp\left(\frac{\alpha}{2}b\right) - \exp\left(\frac{\alpha}{2}a\right)\right]^2.$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa