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ONE PARAMETER BOUNDS FOR AN OPERATOR ASSOCIATED TO HERMITE-HADAMARD INEQUALITY FOR CONVEX FUNCTIONS

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Abstract. In this paper we establish one parameter bounds for the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

in the case of convex functions $f:[a,b]\to\mathbb{R}$. Various weighted Hermite-Hadamard type inequalities are also provided.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [7], the recent survey paper [5], the research papers [1]-[2], [8]-[16] and the references therein.

Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b). We introduce the following operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b). \tag{1.2}$$

We observe that if we take $x=\frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

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and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) := \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

If we change the variable $t = (1-s)a + sx$ for $x \in (a,b)$ then we have

$$\frac{1}{x-a}\int_a^x f(t)\,dt = \int_0^1 f((1-s)a + sx)\,ds$$

and if we change the variable $t = (1-s)x + sb$ for $x \in (a,b)$, then we also have

$$\frac{1}{b-x}\int_x^b f(t)\,dt = \int_0^1 f((1-s)x + sb)\,ds,$$

which gives the representation

$$D_{a+,b-}f(x) = \frac{1}{2}\int_0^1 [f((1-s)a + sx) + f((1-s)x + sb)]\,ds,\ \ x \in (a,b). \tag{1.3}$$

Using the representation (1.3), we observe that the operator $D_{a+,b-}$ is *linear, nonnegative* and *preserves the constant* functions, namely

$$D_{a+,b-}(\alpha f + \beta g) = \alpha D_{a+,b-}(f) + \beta D_{a+,b-}(g)$$

for any complex numbers α, β and integrable functions f, g. If $f \geq 0$ almost everywhere on $[a,b]$ and f is integrable, then $D_{a+,b-}f(x) \geq 0$ for any $x \in (a,b)$. Also, if $f = k$, a constant, then $D_{a+,b-}k(x) = k$ for any $x \in (a,b)$. If we define the function $\mathbf{1}(t) = 1$, $t \in [a,b]$, then, obviously, $D_{a+,b-}\mathbf{1} = \mathbf{1}$.

In this paper we establish one parameter bounds for the operator $D_{a+,b-}f(x)$, $x \in (a,b)$ in the case of convex functions $f : [a,b] \to \mathbb{R}$. Various weighted Hermite-Hadamard type inequalities are also provided.

2. Some Bounds for Convex Functions

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty,\infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(y) + (x-y)\varphi(y) \ \text{ for any } x,\ y \in I. \tag{2.1}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Theorem 1. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a,b] \subset I$. Then for any x, $y \in (a,b)$ we have*

$$f(y) + \varphi(y)\left[\frac{1}{2}\left(x + \frac{a+b}{2}\right) - y\right] \le D_{a+,b-}f(x) \tag{2.2}$$
$$\le \frac{1}{2}f(y) + \frac{1}{2}\frac{(x-y)\left(\frac{a+b}{2} - x\right)}{(x-a)(b-x)}f(x) + \frac{1}{4}\left(\frac{y-a}{x-a}f(a) + \frac{b-y}{b-x}f(b)\right).$$

Proof. From the gradient inequality we have

$$f(t) \ge f(y) + (t-y)\varphi(y)$$

for any t, $y \in I$.

If we denote by e the identity function, namely $e(t) = t$, we have in the function order that

$$f \ge f(y)\mathbf{1} + \varphi(y)(e - y\mathbf{1})$$

for any fixed $y \in I$.

If we take the operator $D_{a+,b-}$ to this inequality and use the facts that it is *linear, nonnegative* and *preserves the constant* functions, then we get

$$\begin{aligned} D_{a+,b-}f &\ge f(y)D_{a+,b-}\mathbf{1} + \varphi(y)(D_{a+,b-}e - yD_{a+,b-}\mathbf{1}) \\ &= f(y)\mathbf{1} + \varphi(y)(D_{a+,b-}e - y\mathbf{1}) \end{aligned}$$

for any fixed $y \in I$.

This inequality can be written for any $x \in (a,b)$ as

$$D_{a+,b-}f(x) \ge f(y) + \varphi(y)(D_{a+,b-}e(x) - y). \tag{2.3}$$

Since

$$\begin{aligned} D_{a+,b-}e(x) &= \frac{1}{2}\left[\frac{1}{x-a}\int_a^x t\,dt + \frac{1}{b-x}\int_x^b t\,dt\right] \\ &= \frac{1}{2}\left[\frac{x^2-a^2}{2(x-a)} + \frac{b^2-x^2}{2(b-x)}\right] = \frac{1}{2}\left(\frac{x+a}{2} + \frac{b+x}{2}\right) \\ &= \frac{1}{2}\left(x + \frac{a+b}{2}\right), \end{aligned}$$

then by (2.3) we get the first inequality in (2.2).

From the gradient inequality we also have

$$(t-y)\varphi(t) + f(y) \ge f(t)$$

for any t, $y \in I$.

This can be writen in the function order as

$$(e - y\mathbf{1})\varphi + f(y)\mathbf{1} \ge f$$

for any fixed $y \in I$.

If we take the operator $D_{a+,b-}$ to this inequality, we also get

$$D_{a+,b-}(e\varphi) - yD_{a+,b-}\varphi + f(y)\mathbf{1} \geq D_{a+,b-}f$$

for any fixed $y \in I$.

This inequality can be written for any $x \in (a,b)$ as

$$D_{a+,b-}(e\varphi)(x) - yD_{a+,b-}\varphi(x) + f(y) \geq D_{a+,b-}f(x). \tag{2.4}$$

Since f is convex, then $\varphi(t) = f'(t)$ for almost every $t \in (a,b)$ and we have

$$\begin{aligned}
D_{a+,b-}(e\varphi)(x) &= \frac{1}{2}\left[\frac{1}{x-a}\int_a^x tf'(t)\,dt + \frac{1}{b-x}\int_x^b tf'(t)\,dt\right] \\
&= \frac{1}{2}\left[\frac{1}{x-a}\left(xf(x) - af(a) - \int_a^x f(t)\,dt\right)\right. \\
&\quad \left. + \frac{1}{b-x}\left(bf(b) - xf(x) - \int_x^b f(t)\,dt\right)\right] \\
&= \frac{1}{2}\left(\frac{xf(x) - af(a)}{x-a} + \frac{bf(b) - xf(x)}{b-x}\right) - D_{a+,b-}f(x)
\end{aligned}$$

and

$$\begin{aligned}
D_{a+,b-}\varphi(x) &= \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f'(t)\,dt + \frac{1}{b-x}\int_x^b f'(t)\,dt\right] \\
&= \frac{1}{2}\left[\frac{f(x) - f(a)}{x-a} + \frac{f(b) - f(x)}{b-x}\right].
\end{aligned}$$

Therefore, by (2.4) we get

$$\begin{aligned}
&\frac{1}{2}\left(\frac{xf(x) - af(a)}{x-a} + \frac{bf(b) - xf(x)}{b-x}\right) \\
&- \frac{1}{2}\left[\frac{f(x) - f(a)}{x-a} + \frac{f(b) - f(x)}{b-x}\right]y + f(y) \\
&\geq 2D_{a+,b-}f(x)
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\frac{1}{4}\left(\frac{xf(x) - af(a)}{x-a} + \frac{bf(b) - xf(x)}{b-x}\right) \\
&- \frac{1}{4}\left[\frac{f(x) - f(a)}{x-a} + \frac{f(b) - f(x)}{b-x}\right]y + \frac{1}{2}f(y) \\
&\geq D_{a+,b-}f(x).
\end{aligned} \tag{2.5}$$

Now observe that

$$\begin{aligned}
&\frac{1}{4}\frac{xf(x) - af(a)}{x-a} - \frac{1}{4}\frac{f(x) - f(a)}{x-a}y \\
&= \frac{1}{4(x-a)}\left[xf(x) - af(a) - f(x)y + f(a)y\right] \\
&= \frac{1}{4(x-a)}\left[(x-y)f(x) + f(a)(y-a)\right]
\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{4}\frac{bf(b)-xf(x)}{b-x}-\frac{1}{4}\frac{f(b)-f(x)}{b-x}y\\&=\frac{1}{4(b-x)}\left[bf(b)-xf(x)-f(b)y+f(x)y\right]\\&=\frac{1}{4(b-x)}\left[f(b)(b-y)-f(x)(x-y)\right],\end{aligned}$$

and by (2.5) we get

$$\begin{aligned}D_{a+,b-}f(x)&\leq\frac{1}{2}f(y)+\frac{1}{4(x-a)}\left[(x-y)f(x)+f(a)(y-a)\right]\\&\quad+\frac{1}{4(b-x)}\left[f(b)(b-y)-f(x)(x-y)\right]\\&=\frac{1}{2}f(y)+\frac{1}{2}\frac{(x-y)\left(\frac{a+b}{2}-x\right)}{(x-a)(b-x)}f(x)+\frac{1}{4}\left[\frac{y-a}{x-a}f(a)+\frac{b-y}{b-x}f(b)\right],\end{aligned}$$

which proves the second inequality in (2.2). $\square$

Corollary 1. *With the assumptions of Theorem 1 we have that*

$$f(x)+\frac{1}{2}\varphi(x)\left(\frac{a+b}{2}-x\right)\leq D_{a+,b-}f(x)\leq\frac{1}{2}\left(f(x)+\frac{f(a)+f(b)}{2}\right)\tag{2.6}$$

and

$$\begin{aligned}&f\left(\frac{a+b}{2}\right)+\frac{1}{2}\varphi\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)\\&\qquad\leq D_{a+,b-}f(x)\\&\qquad\leq\frac{1}{2}f\left(\frac{a+b}{2}\right)-\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)}f(x)\\&\qquad\qquad+\frac{1}{8}(b-a)\left[\frac{(b-x)f(a)+(x-a)f(b)}{(x-a)(b-x)}\right],\end{aligned}\tag{2.7}$$

for any $x\in(a,b)$.

The proof follows by (2.2) on taking $y=x$ and $y=\frac{a+b}{2}$, respectively.

Corollary 2. *With the assumptions of Theorem 1 we have that*

$$\begin{aligned}&f\left(\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right)\leq D_{a+,b-}f(x)\\&\qquad\leq\frac{1}{2}f\left(\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right)-\frac{1}{4}\frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)}f(x)\\&\qquad\qquad+\frac{1}{8}\left(\frac{\frac{1}{2}\left(x+\frac{a+b}{2}\right)-a}{x-a}f(a)+\frac{b-\frac{1}{2}\left(x+\frac{a+b}{2}\right)}{b-x}f(b)\right)\end{aligned}\tag{2.8}$$

and

$$\frac{2}{b-a}\int_a^b f(t)\,dt-\frac{f(b)\left[b-\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right]+f(a)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-a\right]}{b-a} \tag{2.9}$$
$$\le D_{a+,b-}f(x)$$
$$\le \frac{1}{2}\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)}f(x)$$
$$+\frac{1}{8}(b-a)\left(\frac{(b-x)f(a)+(b-x)f(b)}{(x-a)(b-x)}\right).$$

Proof. The inequality (2.8) follows by (2.2) on taking $y=\frac{1}{2}\left(x+\frac{a+b}{2}\right)$.

If we take the integral mean in (2.2), then we get

$$\frac{1}{b-a}\int_a^b f(y)\,dy+\frac{1}{b-a}\int_a^b \varphi(y)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-y\right]dy\le D_{a+,b-}f(x) \tag{2.10}$$
$$\le \frac{1}{2}\frac{1}{b-a}\int_a^b f(y)\,dy+\frac{1}{2}\frac{\left(x-\frac{1}{b-a}\int_a^b ydy\right)\left(\frac{a+b}{2}-x\right)}{(x-a)(b-x)}f(x)$$
$$+\frac{1}{4}\left(\frac{\frac{1}{b-a}\int_a^b ydy-a}{x-a}f(a)+\frac{b-\frac{1}{b-a}\int_a^b ydy}{b-x}f(b)\right)$$
$$=\frac{1}{2}\frac{1}{b-a}\int_a^b f(y)\,dy+\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)\left(\frac{a+b}{2}-x\right)}{(x-a)(b-x)}f(x)$$
$$+\frac{1}{4}\left(\frac{\frac{a+b}{2}-a}{x-a}f(a)+\frac{b-\frac{a+b}{2}}{b-x}f(b)\right)$$
$$=\frac{1}{2}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)}f(x)$$
$$+\frac{1}{8}(b-a)\left(\frac{(b-x)f(a)+(b-x)f(b)}{(x-a)(b-x)}\right).$$

Using the integration by parts, we have

$$\int_a^b \varphi(y)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-y\right]dy$$
$$=\int_a^b f'(y)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-y\right]dy$$
$$=f(y)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-y\right]\Big|_a^b+\int_a^b f(y)\,dy$$
$$=f(b)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-b\right]-f(a)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-a\right]+\int_a^b f(y)\,dy$$
$$=-f(b)\left[b-\frac{1}{2}\left(x+\frac{a+b}{2}\right)\right]-f(a)\left[\frac{1}{2}\left(x+\frac{a+b}{2}\right)-a\right]+\int_a^b f(y)\,dy$$

and by (2.10) we get the desired result (2.9). □

3. Some Weighted Integral Inequalities

We have

Theorem 2. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a, b] \subset I$, the interior of I. Then for any $y \in (a, b)$ we have*

$$f(y) + \varphi(y)\left(\frac{a+b}{2} - y\right) \le \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\, dx. \tag{3.1}$$

In particular, we have

$$f\left(\frac{a+b}{2}\right) \le \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\, dx. \tag{3.2}$$

Proof. Taking the integral mean in the first inequality in (2.2) we have

$$f(y) + \varphi(y)\left[\frac{1}{2}\frac{1}{b-a}\int_a^b \left(x + \frac{a+b}{2}\right) dx - y\right] \le \frac{1}{b-a}\int_a^b D_{a+,b-} f(x) \tag{3.3}$$

for any $y \in (a, b)$.

We observe that f is continuous on $[a, b]$. We claim that

$$\int_a^b D_{a+,b-} f(x)\, dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\, dx. \tag{3.4}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\, dt\right) dx &= \int_a^b \left(\int_a^x f(t)\, dt\right) d(\ln(x-a)) \\
&= \ln(x-a)\left(\int_a^x f(t)\, dt\right)\bigg|_{a+}^b - \int_a^b \ln(x-a) f(x)\, dx
\end{aligned}$$

$$= \ln(b-a)\left(\int_a^b f(t)\, dt\right) - \lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\, dt\right)\right] - \int_a^b \ln(x-a) f(x)\, dx.$$

Since

$$\begin{aligned}
\lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\, dt\right)\right] &= \lim_{x\to a+}\left[(x-a)\ln(x-a)\left(\frac{1}{x-a}\int_a^x f(t)\, dt\right)\right] \\
&= \lim_{x\to a+}\left[(x-a)\ln(x-a)\right]\lim_{x\to a+}\left(\frac{1}{x-a}\int_a^x f(t)\, dt\right) = 0 f(a+) = 0,
\end{aligned}$$

hence

$$\begin{aligned}
\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\, dt\right) dx &= \ln(b-a)\left(\int_a^b f(t)\, dt\right) - \int_a^b \ln(x-a) f(x)\, dx \\
&= \int_a^b [\ln(b-a) - \ln(x-a)] f(x)\, dx = \int_a^b \ln\left(\frac{b-a}{x-a}\right) f(x)\, dx
\end{aligned}$$

Also, integrating by parts, we have

$$\int_a^b \left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx = -\int_a^b \left(\int_x^b f(t)\,dt\right) d\left(\ln(b-x)\right)$$

$$= -\ln(b-x)\left(\int_x^b f(t)\,dt\right)\Bigg|_a^{b-} + \int_a^b \ln(b-x)\, d\left(\int_x^b f(t)\,dt\right)$$

$$= -\lim_{x\to b-}\left[\ln(b-x)\left(\int_x^b f(t)\,dt\right)\right] + \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x) f(x)\,dx$$

$$= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x) f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{b-x}\right) f(x)\,dx.$$

Therefore

$$\int_a^b D_{a+,b-} f(x)\,dx = \frac{1}{2}\left[\int_a^b \ln\left(\frac{b-a}{x-a}\right) f(x)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right) f(x)\,dx\right]$$

$$= \frac{1}{2}\int_a^b \ln\left[\frac{(b-a)^2}{(x-a)(b-x)}\right] f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx$$

and the equality (3.4) is obtained.

Since

$$\frac{1}{b-a}\int_a^b \left(x + \frac{a+b}{2}\right) dx = a+b,$$

then by (3.3) we get the desired result (3.1). □

We have:

Theorem 3. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a,b] \subset I$, the interior of I. Assume also that $w \geq 0$ a.e. on $[a,b]$ and $\int_a^b w(t)\,dt > 0$, then we have*

(3.5)
$$\frac{1}{2}\left(\frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x)\left(x - \frac{a+b}{2}\right) w'(x)\,dx - \frac{(f(a)w(a) + f(b)w(b))(b-a)}{2}\right)$$
$$+ \frac{3}{2}\frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x) w(x)\,dx$$
$$\leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b D_{a+,b-} f(x) w(x)\,dx$$
$$\leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x) w(x)\,dx + \frac{f(a)+f(b)}{2}.$$

Proof. From (2.6) to get

$$f(x)w(x)+\frac{1}{2}\varphi(x)\left(\frac{a+b}{2}-x\right)w(x)\le D_{a+,b-}f(x)w(x) \le \frac{1}{2}\left(f(x)w(x)+\frac{f(a)+f(b)}{2}w(x)\right) \tag{3.6}$$

for a.e. $x\in[a,b]$.

Integrating this on $[a,b]$ we get

$$\begin{aligned}&\int_a^b f(x)w(x)\,dx+\frac{1}{2}\int_a^b f'(x)\left(\frac{a+b}{2}-x\right)w(x)\,dx\\ &\le\int_a^b D_{a+,b-}f(x)w(x)\,dx\\ &\le\frac{1}{2}\left(\int_a^b f(x)w(x)\,dx+\frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx\right)\end{aligned}$$

namely

$$\begin{aligned}&\frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x)w(x)\,dx\\ &\qquad+\frac{1}{2}\frac{1}{\int_a^b w(x)\,dx}\int_a^b f'(x)\left(\frac{a+b}{2}-x\right)w(x)\,dx\\ &\qquad\le\frac{1}{\int_a^b w(x)\,dx}\int_a^b D_{a+,b-}f(x)w(x)\,dx\\ &\qquad\le\frac{1}{2}\left(\frac{1}{\int_a^b w(x)\,dx}\int_a^b f(x)w(x)\,dx+\frac{f(a)+f(b)}{2}\right).\end{aligned}\tag{3.7}$$

Using the integration by parts formula, we have

$$\begin{aligned}&\int_a^b f'(x)\left(\frac{a+b}{2}-x\right)w(x)\,dx\\ &=f(x)\left(\frac{a+b}{2}-x\right)w(x)\Big|_a^b-\int_a^b f(x)\left(\left(\frac{a+b}{2}-x\right)w(x)\right)'dx\\ &=f(b)\left(\frac{a+b}{2}-b\right)w(b)-f(a)\left(\frac{a+b}{2}-a\right)w(a)\\ &-\int_a^b f(x)\left(-w(x)+\left(\frac{a+b}{2}-x\right)w'(x)\right)dx\\ &=-\frac{f(a)w(a)+f(b)w(b)}{2}(b-a)+\int_a^b f(x)w(x)\,dx\\ &+\int_a^b f(x)\left(x-\frac{a+b}{2}\right)w'(x)\,dx.\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^b f(x)\,w(x)\,dx+\frac{1}{2}\int_a^b f'(x)\left(\frac{a+b}{2}-x\right)w(x)\,dx\\
&=\int_a^b f(x)\,w(x)\,dx+\frac{1}{2}\int_a^b f(x)\,w(x)\,dx\\
&+\frac{1}{2}\left(\int_a^b f(x)\left(x-\frac{a+b}{2}\right)w'(x)\,dx-\frac{f(a)\,w(a)+f(b)\,w(b)}{2}(b-a)\right)\\
&=\frac{3}{2}\int_a^b f(x)\,w(x)\,dx\\
&+\frac{1}{2}\left(\int_a^b f(x)\left(x-\frac{a+b}{2}\right)w'(x)\,dx-\frac{f(a)\,w(a)+f(b)\,w(b)}{2}(b-a)\right)
\end{aligned}$$

and by (3.7) we get (3.5). □

The following representation holds:

Lemma 1. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b). Then we have*

$$\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx=\frac{1}{4}\int_a^b\left[(x-a)^2+(b-x)^2\right]f(x)\,dx. \tag{3.8}$$

Proof. We have

$$\begin{aligned}
&\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx\\
&=\frac{1}{2}\left[\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right)dx+\int_a^b (x-a)\left(\int_x^b f(t)\,dt\right)dx\right].
\end{aligned} \tag{3.9}$$

Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right)dx\\
&=-\int_a^b\left(\int_a^x f(t)\,dt\right)d\left(\frac{(b-x)^2}{2}\right)\\
&=-\left(\int_a^x f(t)\,dt\right)\frac{(b-x)^2}{2}\Bigg|_{a+}^{b}+\int_a^b\frac{(b-x)^2}{2}d\left(\int_a^x f(t)\,dt\right)\\
&=\frac{1}{2}\int_a^b (b-x)^2 f(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b (x-a)\left(\int_x^b f(t)\,dt\right)dx \\
&= \int_a^b \left(\int_x^b f(t)\,dt\right) d\left(\frac{(x-a)^2}{2}\right) \\
&= \frac{(x-a)^2}{2}\int_x^b f(t)\,dt\Bigg|_a^{b-} - \int_a^b \frac{(x-a)^2}{2} d\left(\int_x^b f(t)\,dt\right) \\
&= \frac{1}{2}\int_a^b (x-a)^2 f(x)\,dx,
\end{aligned}$$

which, by (3.9) produces the desired result (3.8). □

Corollary 3. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a, b] \subset I$, the interior of I. Then*

$$\begin{aligned}
&\frac{3}{2}\int_a^b f(x)(x-a)(b-x)\,dx - \int_a^b f(x)\left(x-\frac{a+b}{2}\right)^2 dx \\
&\le \frac{1}{4}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx \\
&\le \int_a^b f(x)(x-a)(b-x)\,dx + \frac{f(a)+f(b)}{12}(b-a)^3 .
\end{aligned} \tag{3.10}$$

Proof. If we take $w(x) = (x-a)(b-x)$, $x \in [a, b]$ in (3.5), then we have

$$\begin{aligned}
&\frac{1}{\int_a^b (x-a)(b-x)\,dx}\int_a^b f(x)\left(x-\frac{a+b}{2}\right)\left(\frac{a+b}{2}-x\right)dx \\
&+ \frac{3}{2}\frac{1}{\int_a^b (x-a)(b-x)\,dx}\int_a^b f(x)(x-a)(b-x)\,dx \\
&\le \frac{1}{\int_a^b (x-a)(b-x)\,dx}\int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx \\
&\le \frac{1}{\int_a^b (x-a)(b-x)\,dx}\int_a^b f(x)(x-a)(b-x)\,dx + \frac{f(a)+f(b)}{2},
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\frac{3}{2}\int_a^b f(x)(x-a)(b-x)\,dx - \int_a^b f(x)\left(x-\frac{a+b}{2}\right)^2 dx \\
&\le \int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx \\
&\le \int_a^b f(x)(x-a)(b-x)\,dx + \frac{f(a)+f(b)}{2}\int_a^b (x-a)(b-x)\,dx,
\end{aligned} \tag{3.11}$$

Since $\int_a^b (x-a)(b-x)\,dx = \frac{1}{6}(b-a)^3$,then by Lemma 1 and (3.11) we get (3.10). □

We also have:

Theorem 4. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a,b] \subset I$, the interior of I. Assume also that $w \geq 0$ a.e. on $[a,b]$ and $\int_a^b w(t)\,dt > 0$, then we have*

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) + \frac{1}{2}\varphi\left(\frac{a+b}{2}\right)\frac{1}{\int_a^b w(x)\,dx}\int_a^b \left(x - \frac{a+b}{2}\right) w(x)\,dx \\
&\leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b D_{a+,b-}f(x)\,w(x)\,dx \\
&\leq \frac{1}{2}f\left(\frac{a+b}{2}\right) - \frac{1}{2}\frac{1}{\int_a^b w(x)\,dx}\int_a^b \frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)} w(x) f(x)\,dx \\
&+ \frac{1}{8}(b-a)\frac{1}{\int_a^b w(x)\,dx}\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{(x-a)(b-x)} w(x)\,dx.
\end{aligned} \tag{3.12}$$

Proof. From the inequality (2.7) we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) w(x) &+ \frac{1}{2}\varphi\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) w(x) \\
&\leq D_{a+,b-}f(x)\,w(x) \\
&\leq \frac{1}{2}f\left(\frac{a+b}{2}\right) w(x) - \frac{1}{2}\frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)} w(x) f(x) \\
&\quad + \frac{1}{8}(b-a)\left[\frac{(b-x)f(a) + (x-a)f(b)}{(x-a)(b-x)} w(x)\right],
\end{aligned}$$

which by integration we get

$$\begin{aligned}
f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx &+ \frac{1}{2}\varphi\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right) w(x)\,dx \\
&\leq \int_a^b D_{a+,b-}f(x)\,w(x)\,dx \\
&\leq \frac{1}{2}f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx - \frac{1}{2}\int_a^b \frac{\left(x-\frac{a+b}{2}\right)^2}{(x-a)(b-x)} w(x) f(x)\,dx \\
&\quad + \frac{1}{8}(b-a)\int_a^b \frac{(b-x)f(a) + (x-a)f(b)}{(x-a)(b-x)} w(x)\,dx,
\end{aligned}$$

that is equivalent to (3.12). □

Remark 1. *If w is symmetric, i.e. $w(a+b-x) = w(x)$ for any $x \in (a,b)$, then the function $g(x) := \left(x - \frac{a+b}{2}\right) w(x)$ is assymetric and thus $\int_a^b g(x)\,dx = 0$. By the first inequality in (3.12) we then get*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b D_{a+,b-}f(x)\,w(x)\,dx. \tag{3.13}$$

We have:

Corollary 4. *Assume that $f : I \to \mathbb{R}$ is a convex function on the interval of real numbers I and a, b real numbers such that $[a,b] \subset I$, the interior of I. Then*

$$\frac{1}{6}(b-a)^3 f\left(\frac{a+b}{2}\right) \le \frac{1}{4}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx \tag{3.14}$$
$$\le \frac{1}{12}(b-a)^3 f\left(\frac{a+b}{2}\right) - \frac{1}{2}\int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx + \frac{1}{8}(b-a)^3 \frac{f(a)+f(b)}{2}$$
$$\le \frac{1}{8}(b-a)^3\left[\frac{1}{3}f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

Proof. From the inequality (3.12) we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_a^b (x-a)(b-x)\,dx \\ & \le \int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx \\ & \le \frac{1}{2}f\left(\frac{a+b}{2}\right)\int_a^b (x-a)(b-x)\,dx - \frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx \\ & + \frac{1}{8}(b-a)\int_a^b [(b-x)f(a)+(x-a)f(b)]\,dx. \end{aligned} \tag{3.15}$$

and since

$$\int_a^b [(b-x)f(a)+(x-a)f(b)]\,dx = \frac{f(a)+f(b)}{2}(b-a)^2$$

then (3.15) is equivalent to

$$\begin{aligned} & \frac{1}{6}(b-a)^3 f\left(\frac{a+b}{2}\right) \\ & \le \int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx \\ & \le \frac{1}{12}(b-a)^3 f\left(\frac{a+b}{2}\right) - \frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx \\ & + \frac{1}{8}(b-a)^3\frac{f(a)+f(b)}{2}. \end{aligned}$$

By making use of Lemma 1 we get the first two inequalities in (3.14).

By Fejér's inequality

$$h\left(\frac{a+b}{2}\right)\int_a^b g(t)\,dt \le \int_a^b h(t)g(t)\,dx \le \frac{h(a)+h(b)}{2}\int_a^b g(t)\,dt. \tag{3.16}$$

that holds for the convex function $h : [a,b] \to \mathbb{R}$ and the positive symmetric function $g : [a,b] \to \mathbb{R}$, we also have

$$\begin{aligned} \int_a^b \left(x-\frac{a+b}{2}\right)^2 f(x)\,dx &\ge f\left(\frac{a+b}{2}\right)\int_a^b\left(x-\frac{a+b}{2}\right)^2 dx \\ &= \frac{1}{12}(b-a)^3 f\left(\frac{a+b}{2}\right) \end{aligned}$$

giving that

$$\begin{aligned}
&\frac{1}{12}(b-a)^3 f\left(\frac{a+b}{2}\right)-\frac{1}{2}\int_a^b\left(x-\frac{a+b}{2}\right)^2 f(x)\,dx \\
&\leq \frac{1}{12}(b-a)^3 f\left(\frac{a+b}{2}\right)-\frac{1}{24}(b-a)^3 f\left(\frac{a+b}{2}\right) \\
&= \frac{1}{24}(b-a)^3 f\left(\frac{a+b}{2}\right),
\end{aligned}$$

which proves the last part of (3.14). □

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