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THE RIGHT RIEAMAN-LIOUVILLE FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, with a new approach, a new fractional Hermite-Hadamard type inequality for convex functions is obtained by using only the right Riemann-Liouville fractional integral. Also, to have new fractional trapezoid and midpoint type inequalities for the differentiable convex functions, two new equalities are proved. Our results generalise the studies [1, 4, 6]. We expect that this study will be lead to the new fractional integration studies for Hermite-Hadamard type inequalities.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2, 3].

In [1, 6], the authors used the following equality to obtain trapezoid type inequalities and some applications:

Lemma 1. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° is the interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{b-a}{2}\int_0^1 (1-2t)\, f'(ta+(1-t)b)\,dt. \tag{1.2}$$

In [4], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 2. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° the interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) = (b-a)\int_0^{1/2} t f'(ta+(1-t)b)\,dt + \int_{1/2}^1 (t-1)\, f'(ta+(1-t)b)\,dt. \tag{1.3}$$

Definition 1. [7, page 12]. *A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 1. [7, page 12] *$f : (a, b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.*

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$ (see [5, page 69] *and* [9, page 4]*).*

In [8], Sarıkaya et al. proved the following Hermite-Hadamard type fractional integral inequality:

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Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha} f(b)+J_{b-}^{\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{1.4}$$

with $\alpha > 0$.

Remark 1. *In Theorem 2, it is not necessary supposing that f be a positive function and a, b are positive real numbers. From the Definition 2, it is clear that a, b are any real numbers such as $a < b$.*

In literature, there are hundreds studies for Hermite-Hadamard type inequality by using the left and right fractional integrals (such as Riemann-Liouville fractional integrals, Hadamard fractional integrals, Conformable fractional integrals etc.). In all of them, the left and right fractional integrals are used together. As much as we know, there is not any study for Hermite-Hadamard type inequality by using only the right fractional integrals or the left fractional integrals.

In this paper, our aim is obtaining new fractional Hermite-Hadamard type inequality by using only the right Riemann-Liouville fractional integral for convex functions. Also we desire proving new equalities to have new fractional trapezoid and midpoint type inequalities for the differentiable convex functions. This study will be fundamental to the new fractional integration studies for Hermite-Hadamard type inequalities.

2. The Right Fractional Hermite Hadamard Inequality

Theorem 3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{a+\alpha b}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) \leq \frac{f(a)+\alpha f(b)}{\alpha+1} \tag{2.1}$$

with $\alpha > 0$.

Proof. Let $\alpha > 0$. Since f is convex on $[a, b]$, using Theorem 1, there is at least one line of support

$$A(x)=f\left(\frac{a+\alpha b}{\alpha+1}\right)+m\left(x-\frac{a+\alpha b}{\alpha+1}\right) \leq f(x) \tag{2.2}$$

for all $x \in [a, b]$. From (2.2), we have

$$A(tb+(1-t)a)=f\left(\frac{a+\alpha b}{\alpha+1}\right)+m\left(tb+(1-t)a-\frac{a+\alpha b}{\alpha+1}\right) \leq f(tb+(1-t)a) \tag{2.3}$$

for all $t \in [0, 1]$. Multiplying both sides of (2.3) with $\alpha t^{\alpha-1}$ and integrating over $[0, 1]$ respect to t, we have

$$\begin{aligned}
&\alpha \int_0^1 t^{\alpha-1} A(tb+(1-t)a)\,dt=\alpha \int_0^1 t^{\alpha-1}\left[f\left(\frac{a+\alpha b}{\alpha+1}\right)+m\left(tb+(1-t)a-\frac{a+\alpha b}{\alpha+1}\right)\right] dt \\
&=\alpha f\left(\frac{a+\alpha b}{\alpha+1}\right) \int_0^1 t^{\alpha-1} dt+m\left[\int_0^1 \alpha\left[t^{\alpha} b+\left(t^{\alpha-1}-t^{\alpha}\right) a\right] dt-\frac{a+\alpha b}{\alpha+1} \alpha \int_0^1 t^{\alpha-1} dt\right] \\
&=f\left(\frac{a+\alpha b}{\alpha+1}\right)+m\left[\frac{a+\alpha b}{\alpha+1}-\frac{a+\alpha b}{\alpha+1}\right]=f\left(\frac{a+\alpha b}{\alpha+1}\right) \\
&\leq \alpha \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\,dt=\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} f(t)dt=\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a).
\end{aligned} \tag{2.4}$$

On the other hand, using the convexity of f on $[a, b]$, we have

$$f(tb+(1-t)a) \leq t f(b)+(1-t) f(a) \tag{2.5}$$

for all $t \in [0, 1]$. Multiplying both sides of (2.5) with $\alpha t^{\alpha-1}$ and integrating over $[0, 1]$ respect to t, we have

$$\begin{aligned}
&\alpha \int_0^1 t^{\alpha-1} f(tb+(1-t)a)\,dt=\frac{\alpha}{(b-a)^{\alpha}} \int_a^b (t-a)^{\alpha-1} f(t)dt=\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) \\
&\leq \alpha f(b) \int_0^1 t^{\alpha} dt+\alpha f(a) \int_0^1\left(t^{\alpha-1}-t^{\alpha}\right) dt=\frac{f(a)+\alpha f(b)}{\alpha+1}.
\end{aligned} \tag{2.6}$$

By using (2.4) and (2.6), we have (2.1). This completes the proof. □

Remark 2. *In Theorem 3, if one takes $\alpha = 1$, one has (1.1) (Hermite-Hadamard inequality).*

3. Lemmas

In this section we will prove the main equalities related to Lemma 1 and Lemma 2.

Lemma 3. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\frac{f(a) + \alpha f(b)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{b-}^{\alpha} f(a) = \frac{b - a}{\alpha + 1} \int_0^1 \left[(\alpha + 1)(1 - t)^{\alpha} - 1\right] f'(ta + (1 - t) b)\, dt \tag{3.1}$$

with $\alpha > 0$.

Proof. If we apply the partial integration to the right hand side of the equation (3.1), we have

$$\begin{aligned}
&\frac{b - a}{\alpha + 1} \int_0^1 \left[(\alpha + 1)(1 - t)^{\alpha} - 1\right] f'(ta + (1 - t) b)\, dt \\
&= (b - a) \left[\int_0^1 (1 - t)^{\alpha} f'(ta + (1 - t) b)\, dt - \frac{1}{\alpha + 1} \int_0^1 f'(ta + (1 - t) b)\, dt\right] \\
&= (b - a) \left[\int_0^1 t^{\alpha} f'(tb + (1 - t) a)\, dt - \frac{1}{\alpha + 1} \int_0^1 f'(tb + (1 - t) a)\, dt\right] \\
&= (b - a) \left[t^{\alpha} \frac{f(tb + (1 - t) a)}{b - a}\Big|_0^1 - \alpha \int_0^1 t^{\alpha - 1} \frac{f(tb + (1 - t) a)}{b - a} dt - \frac{1}{\alpha + 1} \frac{f(tb + (1 - t) a)}{b - a}\Big|_0^1\right] \\
&= \left[f(b) - \alpha \int_0^1 t^{\alpha - 1} f(tb + (1 - t) a)\, dt - \frac{f(b) - f(a)}{\alpha + 1}\right] \\
&= \frac{f(a) + \alpha f(b)}{\alpha + 1} - \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{b-}^{\alpha} f(a).
\end{aligned}$$

This completes the proof. □

Remark 3. *In Lemma 3, if one takes $\alpha = 1$, one has the Lemma 1.*

Lemma 4. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}
&\frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a + \alpha b}{\alpha + 1}\right) \\
&= (b - a) \left[\int_0^{\frac{\alpha}{\alpha + 1}} -t^{\alpha} f'(tb + (1 - t) a)\, dt + \int_{\frac{\alpha}{\alpha + 1}}^1 (1 - t^{\alpha}) f'(tb + (1 - t) a)\, dt\right]
\end{aligned} \tag{3.2}$$

with $\alpha > 0$.

Proof. If we apply the partial integration to the right hand side of the equation (3.2), we have

$$\begin{aligned}
&(b - a) \left[\int_0^{\frac{\alpha}{\alpha + 1}} -t^{\alpha} f'(tb + (1 - t) a)\, dt + \int_{\frac{\alpha}{\alpha + 1}}^1 (1 - t^{\alpha}) f'(tb + (1 - t) a)\, dt\right] \\
&= (b - a) \left[\int_0^1 -t^{\alpha} f'(tb + (1 - t) a)\, dt + \int_{\frac{\alpha}{\alpha + 1}}^1 f'(tb + (1 - t) a)\, dt\right] \\
&= (b - a) \left[\left(-t^{\alpha} \frac{f(tb + (1 - t) a)}{b - a}\Big|_0^1 + \alpha \int_0^1 t^{\alpha - 1} \frac{f(tb + (1 - t) a)}{b - a}\right) + \left(\frac{f(tb + (1 - t) a)}{b - a}\Big|_{\frac{\alpha}{\alpha + 1}}^1\right)\right] \\
&= \left[-f(b) + \alpha \int_0^1 t^{\alpha - 1} f(tb + (1 - t) a)\, dt + \left(f(b) - f\left(\frac{a + \alpha b}{\alpha + 1}\right)\right)\right] \\
&= \left[\alpha \int_0^1 t^{\alpha - 1} f(tb + (1 - t) a)\, dt - f\left(\frac{a + \alpha b}{\alpha + 1}\right)\right] \\
&= \frac{\Gamma(\alpha + 1)}{(b - a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a + \alpha b}{\alpha + 1}\right).
\end{aligned}$$

This completes the proof. □

Remark 4. *In Lemma 4, if one takes $\alpha = 1$, one has the Lemma 2.*

4. The Right Fractional Trapezoid and Midpoint Type Inequalities

In this section we will obtain some new right Riemann-Liouville fractional trapezoid and midpoint type inequalities by using Lemma 3 and Lemma 4.

Theorem 4. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| \leq \frac{b-a}{\alpha+1}\left[\begin{array}{c} R_1(\alpha)|f'(b)|+R_2(\alpha)|f'(a)| \\ +R_3(\alpha)|f'(b)|+R_4(\alpha)|f'(a)| \end{array}\right] \tag{4.1}$$

where

$$\begin{aligned}
R_1(\alpha) &= \int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^{\alpha})\,tdt=\frac{\alpha}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}, \\
R_2(\alpha) &= \int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^{\alpha})(1-t)\,dt=\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-1\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}, \\
R_3(\alpha) &= \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^{\alpha}-1)\,tdt=\frac{\alpha\left(1+(\alpha+1)^{\frac{2}{\alpha}}\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}, \\
R_4(\alpha) &= \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^{\alpha}-1)(1-t)\,dt=\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-1-(\alpha+1)^{\frac{2}{\alpha}}\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}},
\end{aligned}$$

with $\alpha > 0$.

Proof. Using Lemma 3 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| \\
&\leq \frac{b-a}{\alpha+1}\int_0^1|(\alpha+1)(1-t)^{\alpha}-1|\,|f'(ta+(1-t)b)|\,dt \\
&= \frac{b-a}{\alpha+1}\int_0^1|(\alpha+1)t^{\alpha}-1|\,|f'(tb+(1-t)a)|\,dt \\
&\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c} \int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^{\alpha})[t|f'(b)|+(1-t)|f'(a)|]\,dt \\ +\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^{\alpha}-1)[t|f'(b)|+(1-t)|f'(a)|]\,dt \end{array}\right] \\
&\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c} \frac{\alpha}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|f'(b)|+\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-1\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|f'(a)| \\ +\frac{\alpha\left(1+(\alpha+1)^{\frac{2}{\alpha}}\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|f'(b)|+\frac{\alpha\left(2(\alpha+2)(\alpha+1)^{\frac{1}{\alpha}-1}-1-(\alpha+1)^{\frac{2}{\alpha}}\right)}{2(\alpha+2)(\alpha+1)^{\frac{2}{\alpha}}}|f'(a)| \end{array}\right].
\end{aligned}$$

This completes the proof. □

Remark 5. *In Theorem 4, if one takes $\alpha = 1$, one has the inequality proved in* [1, Theorem 2.2].

Theorem 5. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned}
&\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| \\
&\leq \frac{b-a}{\alpha+1}\left(\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}\right)^{1-\frac{1}{q}}\left(\begin{array}{c} R_1(\alpha)|f'(b)|^q+R_2(\alpha)|f'(a)|^q \\ +R_3(\alpha)|f'(b)|^q+R_4(\alpha)|f'(a)|^q \end{array}\right)^{\frac{1}{q}}
\end{aligned} \tag{4.2}$$

where $R_1(\alpha)$-$R_4(\alpha)$ are the same as in Theorem 4 and $\alpha > 0$.

Proof. Using Lemma 3, power mean inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\right| \\
&\leq \frac{b-a}{\alpha+1}\int_0^1|(\alpha+1)(1-t)^{\alpha}-1|\,|f'(ta+(1-t)b)|\,dt \\
&= \frac{b-a}{\alpha+1}\int_0^1|(\alpha+1)t^{\alpha}-1|\,|f'(tb+(1-t)a)|\,dt
\end{aligned}$$

$$\leq \frac{b-a}{\alpha+1}\left[\left(\int_0^1 |(\alpha+1)t^\alpha - 1|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |(\alpha+1)t^\alpha - 1|\,|f'(tb+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c}\left(\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^\alpha)\,dt+\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^\alpha-1)\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^\alpha)\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\\ +\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^\alpha-1)\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\end{array}\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{\alpha+1}\left(\frac{2\alpha}{(\alpha+1)^{1+\frac{1}{\alpha}}}\right)^{1-\frac{1}{q}}\left(\begin{array}{c}R_1(\alpha)|f'(b)|^q+R_2(\alpha)|f'(a)|^q\\ +R_3(\alpha)|f'(b)|^q+T_4(\alpha)|f'(a)|^q\end{array}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 6. *In Theorem 5, if one takes $\alpha = 1$, one has the inequality proved in* [6, Theorem 1].

Theorem 6. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^q$ is convex on $[a, b]$ for $q > 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$(4.3)\qquad \left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a)\right|$$

$$\leq \frac{b-a}{\alpha+1}\left(R_5(\alpha,p)+R_6(\alpha,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}$$

where

$$R_5(\alpha,p) = \int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^\alpha)^p\,dt,$$
$$R_6(\alpha,p) = \int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^\alpha-1)^p\,dt,$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha > 0$.

Proof. Using Lemma 3, Hölder inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}J_{b-}^\alpha f(a)\right|$$

$$\leq \frac{b-a}{\alpha+1}\int_0^1 |(\alpha+1)(1-t)^\alpha-1|\,|f'(ta+(1-t)b)|\,dt$$

$$= \frac{b-a}{\alpha+1}\int_0^1 |(\alpha+1)t^\alpha-1|\,|f'(tb+(1-t)a)|\,dt$$

$$= \frac{b-a}{\alpha+1}\int_0^1 |1-(\alpha+1)t^\alpha|\,|f'(tb+(1-t)a)|\,dt$$

$$\leq \frac{b-a}{\alpha+1}\left[\left(\int_0^1 |1-(\alpha+1)t^\alpha|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 |f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\right]$$

$$\leq \frac{b-a}{\alpha+1}\left[\begin{array}{c}\left(\int_0^{\frac{1}{\sqrt[\alpha]{\alpha+1}}}(1-(\alpha+1)t^\alpha)^p\,dt+\int_{\frac{1}{\sqrt[\alpha]{\alpha+1}}}^1((\alpha+1)t^\alpha-1)^p\,dt\right)^{\frac{1}{p}}\\ \times\left(\int_0^1\left[t|f'(b)|^q+(1-t)|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{\alpha+1}\left[\left(R_5(\alpha,p)+R_6(\alpha,p)\right)^{\frac{1}{p}}\left(\frac{|f'(b)|^q+|f'(a)|^q}{2}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Remark 7. *In Theorem 6, if one takes $\alpha = 1$, one has the inequality proved in* [1, Theorem 2.3].

Theorem 7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \leq (b-a)\left[\begin{array}{c} R_7(\alpha)|f'(b)| + R_8(\alpha)|f'(a)| \\ +R_9(\alpha)|f'(b)| + R_{10}(\alpha)|f'(a)| \end{array}\right] \tag{4.4}$$

where

$$\begin{aligned} R_7(\alpha) &= \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha+1} dt = \frac{\alpha\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}, \\ R_8(\alpha) &= \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha} - t^{\alpha+1} dt = \frac{2\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}, \\ R_9(\alpha) &= \int_{\frac{\alpha}{\alpha+1}}^1 t - t^{\alpha+1} dt = \frac{\alpha\left(2\alpha^{\alpha} + (\alpha+1)^{\alpha}\right)}{2(\alpha+1)^{\alpha+2}(\alpha+2)}, \\ R_{10}(\alpha) &= \int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})(1-t)\, dt = \frac{4\alpha^{\alpha+1} - \alpha(\alpha+1)^{\alpha}}{2(\alpha+1)^{\alpha+2}(\alpha+2)}, \end{aligned}$$

with $\alpha > 0$.

Proof. Using Lemma 4 and the convexity of $|f'|$, we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \\ &\leq (b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}|f'(tb+(1-t)a)|\, dt + \int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})|f'(tb+(1-t)a)|\, dt\right] \\ &\leq (b-a)\left[\begin{array}{c} \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}[t|f'(b)| + (1-t)|f'(a)|]\, dt \\ +\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})[t|f'(b)| + (1-t)|f'(a)|]\, dt \end{array}\right] \\ &\leq (b-a)\left[\begin{array}{c} \frac{\alpha\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}|f'(b)| + \frac{2\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}|f'(a)| \\ +\frac{\alpha(2\alpha^{\alpha}+(\alpha+1)^{\alpha})}{2(\alpha+1)^{\alpha+2}(\alpha+2)}|f'(b)| + \frac{4\alpha^{\alpha+1}-\alpha(\alpha+1)^{\alpha}}{2(\alpha+1)^{\alpha+2}(\alpha+2)}|f'(a)| \end{array}\right]. \end{aligned}$$

This completes the proof. □

Remark 8. *In Theorem 7, if one takes $\alpha = 1$, one has the inequality proved in* [4, Theorem 2.2].

Theorem 8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \\ &\leq (b-a)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} \left(R_7(\alpha)|f'(b)|^q + R_8(\alpha)|f'(a)|^q\right)^{\frac{1}{q}} \\ +\left(R_9(\alpha)|f'(b)|^q + R_{10}(\alpha)|f'(a)|^q\right)^{\frac{1}{q}} \end{array}\right]. \end{aligned} \tag{4.5}$$

where $R_7(\alpha)$-$R_{10}(\alpha)$ are the same as in Theorem 7 and $\alpha > 0$.

Proof. Using Lemma 4, power mean inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} J_{b-}^{\alpha} f(a) - f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \\ &\leq (b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}|f'(tb+(1-t)a)|\, dt + \int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})|f'(tb+(1-t)a)|\, dt\right] \\ &\leq (b-a)\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha} dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\, dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})|f'(tb+(1-t)a)|^q dt\right)^{\frac{1}{q}} \end{array}\right] \\ &\leq (b-a)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[t|f'(b)|^q + (1-t)|f'(a)|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[t|f'(b)|^q + (1-t)|f'(a)|^q\right] dt\right)^{\frac{1}{q}} \end{array}\right] \end{aligned}$$

$$\leq (b-a)\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(\frac{\alpha\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}\left|f'(b)\right|^{q}+\frac{2\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}}{(\alpha+1)(\alpha+2)}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\\ +\left(\frac{\alpha(2\alpha^{\alpha}+(\alpha+1)^{\alpha})}{2(\alpha+1)^{\alpha+2}(\alpha+2)}\left|f'(b)\right|^{q}+\frac{4\alpha^{\alpha+1}-\alpha(\alpha+1)^{\alpha}}{2(\alpha+1)^{\alpha+2}(\alpha+2)}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\end{array}\right].$$

This completes the proof. □

Remark 9. *In Theorem 8, if one takes $\alpha=1$, one has the following midpoint type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left[\left(\frac{\left|f'(b)\right|^{q}+2\left|f'(a)\right|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{2\left|f'(b)\right|^{q}+\left|f'(a)\right|^{q}}{3}\right)^{\frac{1}{q}}\right] \tag{4.6}$$

Theorem 9. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\right| \qquad (4.7)\\ &\leq (b-a)\left[\begin{array}{c} R_{11}^{\frac{1}{p}}(\alpha,p)\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(b)\right|^{q}+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\\ +R_{12}^{\frac{1}{p}}(\alpha,p)\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(b)\right|^{q}+\frac{1}{2(\alpha+1)^2}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\end{array}\right]\end{aligned}$$

where

$$\begin{aligned} R_{11}(\alpha,p) &= \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt,\\ R_{12}(\alpha,p) &= \int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})^p\,dt,\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. Using Lemma 4, Hölder inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\right|\\ &\leq (b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})\left|f'(tb+(1-t)a)\right|dt\right]\\ &\leq (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}}\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}\end{array}\right]\\ &\leq (b-a)\left[\begin{array}{c}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}}\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})^p\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\end{array}\right]\\ &\leq (b-a)\left[\begin{array}{c} R_{11}^{\frac{1}{p}}(\alpha,p)\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(b)\right|^{q}+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\\ +R_{12}^{\frac{1}{p}}(\alpha,p)\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(b)\right|^{q}+\frac{1}{2(\alpha+1)^2}\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\end{array}\right].\end{aligned}$$

This completes the proof. □

Remark 10. *In Theorem 9, if one takes $\alpha=1$, one has the inequality proved in* [4, Theorem 2.3].

5. Competing Interests

The authors declare that they have no competing interests.

6. Authors' Contributions

The majority of this article has made by the first author. All authors read and approved final form of the article.

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