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SOME INEQUALITIES FOR AN OPERATOR ASSOCIATED TO HERMITE-HADAMARD INEQUALITY FOR FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. In this paper we establish some inequalities for the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

in the case of functions $f : [a,b] \to \mathbb{C}$ of bounded variation. Several weighted and unweighted Hermite-Hadamard type inequalities are also provided.

1. INTRODUCTION

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : [a,b] \to \mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [9], the recent survey paper [5], the research papers [1]-[2], [10]-[18] and the references therein.

Assume that the function $f : (a,b) \to \mathbb{C}$ is Lebesgue integrable on (a,b). We introduce the following operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b). \tag{1.2}$$

We observe that if we take $x = \frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\,dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

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and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

So, if $f : [a,b] \to \mathbb{C}$ is Lebesgue integrable on $[a,b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\,dt\right]$$

and

$$D_{a+,b-}f(b) := \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\,dt\right].$$

If we change the variable $t = (1-s)a + sx$ for $x \in (a,b)$ then we have

$$\frac{1}{x-a}\int_a^x f(t)\,dt = \int_0^1 f((1-s)a + sx)\,ds$$

and if we change the variable $t = (1-s)x + sb$ for $x \in (a,b)$, then we also have

$$\frac{1}{b-x}\int_x^b f(t)\,dt = \int_0^1 f((1-s)x + sb)\,ds,$$

which gives the representation

$$D_{a+,b-}f(x) = \frac{1}{2}\int_0^1 [f((1-s)a + sx) + f((1-s)x + sb)]\,ds,\ x \in (a,b). \tag{1.3}$$

Using the representation (1.3), we observe that the operator $D_{a+,b-}$ is *linear, nonnegative* and *preserves the constant* functions, namely

$$D_{a+,b-}(\alpha f + \beta g) = \alpha D_{a+,b-}(f) + \beta D_{a+,b-}(g)$$

for any complex numbers α, β and integrable functions f, g. If $f \geq 0$ almost everywhere on $[a,b]$ and f is integrable, then $D_{a+,b-}f(x) \geq 0$ for any $x \in (a,b)$. Also, if $f = k$, a constant, then $D_{a+,b-}k(x) = k$ for any $x \in (a,b)$. If we define the function $\mathbf{1}(t) = 1$, $t \in [a,b]$, then, obviously, $D_{a+,b-}\mathbf{1} = \mathbf{1}$.

We say that the function $f : [a,b] \to \mathbb{C}$ is of *H-r-Hölder type* if

$$|f(t) - f(s)| \leq H|t-s|^r$$

for any t, $s \in [a,b]$, where $H > 0$ and $r \in (0,1]$. If $r = 1$ and we put $H = L$, then we call the function of *L-Lipschitz type*.

In the recent paper [7] we obtained the following results:

Theorem 1. *If f is of H-r-Hölder type on $[a,b]$ with $H > 0$ and $r \in (0,1]$, then for any $x \in (a,b)$ we have*

$$|D_{a+,b-}f(x) - f(x)| \leq \frac{1}{2(r+1)}H\left[(x-a)^r + (b-x)^r\right] \tag{1.4}$$

and

$$\left|D_{a+,b-}f(x) - \frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2(r+1)}H\left[(x-a)^r + (b-x)^r\right]. \tag{1.5}$$

In particular, if f is of L-Lipschitz type, then

$$|D_{a+,b-}f(x) - f(x)| \leq \frac{1}{4}L(b-a) \tag{1.6}$$

and

$$\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{4}L(b-a) \tag{1.7}$$

for any $x\in(a,b)$.

If we take in Theorem 1 $x=\frac{a+b}{2}$, then we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{2^r(r+1)}H(b-a)^r \tag{1.8}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2^r(r+1)}H(b-a)^r. \tag{1.9}$$

In particular, if f is of L-*Lipschitz type,* then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{4}L(b-a) \tag{1.10}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{4}L(b-a). \tag{1.11}$$

We also have:

Theorem 2. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then for any $x\in(a,b)$ we have*

$$\begin{aligned}&\left|D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right| \\ &\leq\frac{1}{2^{r+1}(r+1)}H\left[(x-a)^r+(b-x)^r\right].\end{aligned} \tag{1.12}$$

In particular, if f is of L-Lipschitz type, then

$$\left|D_{a+,b-}f(x)-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right|\leq\frac{1}{8}L(b-a) \tag{1.13}$$

for any $x\in(a,b)$.

If we take in Theorem 2 $x=\frac{a+b}{2}$, then we get

$$\begin{aligned}0&\leq\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right| \\ &\leq\frac{1}{4^r(r+1)}H(b-a)^r\end{aligned} \tag{1.14}$$

and

$$0\leq\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\right|\leq\frac{1}{8}L(b-a). \tag{1.15}$$

Motivated by the above results, in this paper we establish some inequalities for the operator $D_{a+,b-}f(x)$, $x\in(a,b)$ in the case of functions $f:[a,b]\to\mathbb{C}$ of bounded variation. Several weighted and unweighted Hermite-Hadamard type inequalities are also provided.

2. Some Inequalities for $D_{a+,b-}f$

We have:

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $x\in(a,b)$ we have*

$$\begin{aligned}
&|D_{a+,b-}f(x)-f(x)| \\
&\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_t^x(f)\right)dt+\frac{1}{b-x}\int_x^b\left(\bigvee_x^t(f)\right)dt\right]\leq\frac{1}{2}\bigvee_a^b(f)
\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right| \\
&\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt+\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right]\leq\frac{1}{2}\bigvee_a^b(f).
\end{aligned}\tag{2.2}$$

Proof. If $f:[a,b]\to\mathbb{C}$ is Lebesgue integrable on $[a,b]$ and $\lambda,\ \mu\in[a,b]$, then we have the following simple equality

$$\begin{aligned}
&D_{a+,b-}f(x)-\frac{\lambda+\mu}{2} \\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x[f(t)-\lambda]\,dt+\frac{1}{b-x}\int_x^b[f(t)-\mu]\,dt\right].
\end{aligned}\tag{2.3}$$

If we take $\lambda=\mu=f(x)$, then we get the equality

$$D_{a+,b-}f(x)-f(x)=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x[f(t)-f(x)]\,dt+\frac{1}{b-x}\int_x^b[f(t)-f(x)]\,dt\right].$$

while for $\lambda=f(a)$ and $\mu=f(b)$ we get

$$\begin{aligned}
&D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2} \\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x[f(t)-f(a)]\,dt+\frac{1}{b-x}\int_x^b[f(t)-f(b)]\,dt\right].
\end{aligned}$$

Now, if $f:[a,b]\to\mathbb{C}$ is a function of bounded variation on $[a,b]$, then by the properties of the modulus we have

$$
\begin{aligned}
&\left|D_{a+,b-}f(x)-f(x)\right| \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\left|\int_a^x [f(t)-f(x)]\,dt\right| + \frac{1}{b-x}\left|\int_x^b [f(t)-f(x)]\,dt\right|\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\int_a^x |f(t)-f(x)|\,dt + \frac{1}{b-x}\int_x^b |f(t)-f(x)|\,dt\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\int_a^x \left(\bigvee_t^x (f)\right) dt + \frac{1}{b-x}\int_x^b \left(\bigvee_x^t (f)\right) dt\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\bigvee_a^x (f)\int_a^x dt + \frac{1}{b-x}\bigvee_x^b (f)\int_x^b dt\right] \\
&= \frac{1}{2}\left(\bigvee_a^x (f) + \bigvee_x^b (f)\right) = \frac{1}{2}\bigvee_a^b (f),
\end{aligned}
$$

which proves the inequality (2.1).

Similarly, we have

$$
\begin{aligned}
&\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right| \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\left|\int_a^x [f(t)-f(a)]\,dt\right| + \frac{1}{b-x}\left|\int_x^b [f(t)-f(b)]\,dt\right|\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\int_a^x |f(t)-f(a)|\,dt + \frac{1}{b-x}\int_x^b |f(t)-f(b)|\,dt\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\int_a^x \left(\bigvee_a^t (f)\right) dt + \frac{1}{b-x}\int_x^b \left(\bigvee_t^b (f)\right) dt\right] \\
&\le \frac{1}{2}\left[\frac{1}{x-a}\bigvee_a^x (f)\int_a^x dt + \frac{1}{b-x}\bigvee_x^b (f)\int_x^b dt\right] \\
&= \frac{1}{2}\left(\bigvee_a^x (f) + \bigvee_x^b (f)\right) = \frac{1}{2}\bigvee_a^b (f),
\end{aligned}
$$

which proves the inequality (2.2). $\square$

Remark 1. *Observe that*

$$\begin{aligned}
&\frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_t^x(f)\right)dt+\frac{1}{b-x}\int_x^b\left(\bigvee_x^t(f)\right)dt\right]\\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_a^x(f)-\bigvee_a^t(f)\right)dt+\frac{1}{b-x}\int_x^b\left(\bigvee_x^b(f)-\bigvee_t^b(f)\right)dt\right]\\
&=\frac{1}{2}\left[\bigvee_a^x(f)-\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt+\bigvee_x^b(f)-\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right]\\
&=\frac{1}{2}\left[\bigvee_a^b(f)-\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt-\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right]
\end{aligned}$$

and then the inequality (2.1) can be written as

$$\begin{aligned}
&|D_{a+,b-}f(x)-f(x)| \\
&\leq\frac{1}{2}\left[\bigvee_a^b(f)-\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt-\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right]\leq\frac{1}{2}\bigvee_a^b(f)
\end{aligned}\tag{2.4}$$

for any $x\in(a,b)$.

If we take in (2.1) and (2.2) $x=\frac{a+b}{2}$, then we get, see also [5]

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\\
&\leq\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(\bigvee_t^{\frac{a+b}{2}}(f)\right)dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\left(\bigvee_{\frac{a+b}{2}}^t(f)\right)dt\leq\frac{1}{2}\bigvee_a^b(f)
\end{aligned}\tag{2.5}$$

and

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{f(a)+f(b)}{2}\right|\\
&\leq\frac{1}{b-a}\int_a^{\frac{a+b}{2}}\left(\bigvee_a^t(f)\right)dt+\frac{1}{b-a}\int_{\frac{a+b}{2}}^b\left(\bigvee_t^b(f)\right)dt\leq\frac{1}{2}\bigvee_a^b(f).
\end{aligned}\tag{2.6}$$

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then for any $x\in(a,b)$ we have*

$$\left|D_{a+,b-}f(x)-\frac{1}{2}\left(f(x)+\frac{f(a)+f(b)}{2}\right)\right|\leq\frac{1}{4}\bigvee_a^b(f).\tag{2.7}$$

Proof. If we take in (2.3) $\lambda=\frac{1}{2}[f(a)+f(x)]$ and $\mu=\frac{1}{2}[f(x)+f(b)]$ then we get

$$\begin{aligned}
&D_{a+,b-}f(x)-\frac{1}{2}\left(f(x)+\frac{f(a)+f(b)}{2}\right)\\
&=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left[f(t)-\frac{1}{2}[f(a)+f(x)]\right]dt\right.\\
&\left.+\frac{1}{b-x}\int_x^b\left[f(t)-\frac{1}{2}[f(x)+f(b)]\right]dt\right],
\end{aligned}\tag{2.8}$$

for any $x \in (a,b)$.

Since $f : [a,b] \to \mathbb{C}$ is a function of bounded variation on $[a,b]$, then

$$\begin{aligned}\left|f(t)-\frac{1}{2}[f(a)+f(x)]\right| &= \frac{|f(t)-f(a)+f(t)-f(x)|}{2} \\ &\leq \frac{1}{2}[|f(t)-f(a)|+|f(x)-f(t)|] \leq \frac{1}{2}\bigvee_a^x(f)\end{aligned}$$

for any $t \in [a,x]$, and, similarly

$$\left|f(t)-\frac{1}{2}[f(x)+f(b)]\right| \leq \frac{1}{2}\bigvee_x^b(f)$$

for any $t \in [x,b]$.

By taking the modulus in (2.8) we get

$$\begin{aligned}&\left|D_{a+,b-}f(x)-\frac{1}{2}\left(f(x)+\frac{f(a)+f(b)}{2}\right)\right| \\ &\leq \frac{1}{2}\left[\frac{1}{x-a}\left|\int_a^x\left[f(t)-\frac{1}{2}[f(a)+f(x)]\right]dt\right|\right. \\ &\left.+\frac{1}{b-x}\left|\int_x^b\left[f(t)-\frac{1}{2}[f(x)+f(b)]\right]dt\right|\right] \\ &\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left|f(t)-\frac{1}{2}[f(a)+f(x)]\right|dt\right. \\ &\left.+\frac{1}{b-x}\int_x^b\left|\left[f(t)-\frac{1}{2}[f(x)+f(b)]\right]\right|dt\right] \\ &\leq \frac{1}{2}\left[\frac{1}{2}\frac{1}{x-a}\int_a^x\left(\bigvee_a^x(f)\right)dt+\frac{1}{2}\frac{1}{b-x}\int_x^b\left(\bigvee_x^b(f)\right)dt\right] \\ &= \frac{1}{4}\left(\bigvee_a^x(f)+\bigvee_x^b(f)\right)=\frac{1}{4}\bigvee_a^b(f),\end{aligned}$$

which proves the inequality (2.7). □

Remark 2. *If we add the inequalities (2.2) and (2.4) we get*

$$\begin{aligned}&\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|+|D_{a+,b-}f(x)-f(x)| \\ &\leq \frac{1}{2}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt+\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right] \\ &+\frac{1}{2}\left[\bigvee_a^b(f)-\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt-\frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right] \\ &= \frac{1}{2}\bigvee_a^b(f)\end{aligned} \tag{2.9}$$

for any $x \in (a,b)$.

By the triangle inequality we also have

$$\begin{aligned}&\left|2D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}-f(x)\right|\\&\leq\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|+\left|D_{a+,b-}f(x)-f(x)\right|\end{aligned}$$

namely

$$\begin{aligned}&\left|D_{a+,b-}f(x)-\frac{1}{2}\left(\frac{f(a)+f(b)}{2}+f(x)\right)\right|\\&\leq\frac{1}{2}\left(\left|D_{a+,b-}f(x)-\frac{f(a)+f(b)}{2}\right|+\left|D_{a+,b-}f(x)-f(x)\right|\right)\end{aligned}\tag{2.10}$$

for any $x\in(a,b)$.

Therefore by (2.9) and (2.10) we also get (2.7).

If we take $x=\frac{a+b}{2}$ in (2.7), then we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right)\right|\leq\frac{1}{4}\bigvee_a^b(f).\tag{2.11}$$

3. Some Integral Inequalities

The following lemma is of interest in itself, see also [6]:

Lemma 1. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b) and $f(a+)$, $f(b-)$ exists and are finite. Then we have*

$$\int_a^b D_{a+,b-}f(x)\,dx=\int_a^b\ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx.\tag{3.1}$$

Proof. We have

$$\int_a^b D_{a+,b-}f(x)\,dx:=\frac{1}{2}\left[\int_a^b\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx+\int_a^b\left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx\right].$$

Observe that, integrating by parts, we have

$$\begin{aligned}\int_a^b\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx&=\int_a^b\left(\int_a^x f(t)\,dt\right)d\left(\ln(x-a)\right)\\&=\ln(x-a)\left(\int_a^x f(t)\,dt\right)\Big|_{a+}^b-\int_a^b\ln(x-a)f(x)\,dx\end{aligned}$$

$$=\ln(b-a)\left(\int_a^b f(t)\,dt\right)-\lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\,dt\right)\right]-\int_a^b\ln(x-a)f(x)\,dx.$$

Since

$$\begin{aligned}\lim_{x\to a+}\left[\ln(x-a)\left(\int_a^x f(t)\,dt\right)\right]&=\lim_{x\to a+}\left[(x-a)\ln(x-a)\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)\right]\\&=\lim_{x\to a+}\left[(x-a)\ln(x-a)\right]\lim_{x\to a+}\left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)=0f(a+)=0,\end{aligned}$$

hence

$$\int_a^b \left(\frac{1}{x-a}\int_a^x f(t)\,dt\right)dx = \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(x-a)\,f(x)\,dx$$
$$= \int_a^b [\ln(b-a) - \ln(x-a)]\,f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{x-a}\right)f(x)\,dx$$

Also, integrating by parts, we have

$$\int_a^b \left(\frac{1}{b-x}\int_x^b f(t)\,dt\right)dx = -\int_a^b \left(\int_x^b f(t)\,dt\right)d(\ln(b-x))$$
$$= -\ln(b-x)\left(\int_x^b f(t)\,dt\right)\Bigg|_a^{b-} + \int_a^b \ln(b-x)\,d\left(\int_x^b f(t)\,dt\right)$$
$$= -\lim_{x\to b-}\left[\ln(b-x)\left(\int_x^b f(t)\,dt\right)\right] + \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x)\,f(x)\,dx$$
$$= \ln(b-a)\left(\int_a^b f(t)\,dt\right) - \int_a^b \ln(b-x)\,f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{b-x}\right)f(x)\,dx.$$

Therefore

$$\int_a^b D_{a+,b-}f(x)\,dx = \frac{1}{2}\left[\int_a^b \ln\left(\frac{b-a}{x-a}\right)f(x)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right)f(x)\,dx\right]$$
$$= \frac{1}{2}\int_a^b \ln\left[\frac{(b-a)^2}{(x-a)(b-x)}\right]f(x)\,dx = \int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx$$

and the equality (3.1) is obtained. □

The following result holds:

By the use of Theorem 1 we have:

Theorem 5. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then we have*

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{(r+1)^2}H(b-a)^r \tag{3.2}$$

and

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{(r+1)^2}H(b-a)^r. \tag{3.3}$$

Proof. If we integrate the inequality the inequality (1.4), we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{1}{b-a}\int_a^b \left|D_{a+,b-}f(x)-f(x)\right|dx \\
&\le \frac{1}{2(r+1)}H\frac{1}{b-a}\int_a^b \left[(x-a)^r+(b-x)^r\right]dx \\
&= \frac{1}{2(r+1)}H\frac{1}{b-a}\left[\frac{(b-a)^{r+1}}{r+1}+\frac{(b-a)^{r+1}}{r+1}\right]=\frac{1}{(r+1)^2}H(b-a)^r
\end{aligned}$$

and by (3.1) we get (3.2).

The inequality (3.3) follows in a similar way from (1.5). □

The inequality (3.2) can be actually improved as follows:

Theorem 6. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
&\le \frac{1}{2^r(r+1)^2}H(b-a)^r.
\end{aligned}\tag{3.4}$$

Proof. If we take the integral mean in (1.12) we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx-\frac{1}{2}\frac{1}{b-a}\int_a^b\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx\right| \\
&\le \frac{1}{b-a}\int_a^b\left|D_{a+,b-}f(x)\,dx-\frac{1}{2}\left[f\left(\frac{x+a}{2}\right)+f\left(\frac{x+b}{2}\right)\right]\right|dx \\
&\le \frac{1}{2^{r+1}(r+1)}H\frac{1}{b-a}\int_a^b\left[(x-a)^r+(b-x)^r\right] \\
&= \frac{1}{2^{r+1}(r+1)}H\frac{1}{b-a}\frac{2(b-a)^{r+1}}{r+1}=\frac{1}{2^r(r+1)^2}H(b-a)^r.
\end{aligned}\tag{3.5}$$

Using the change of variable we have

$$\int_a^b f\left(\frac{a+x}{2}\right)dx=2\int_a^{\frac{a+b}{2}}f(y)\,dy \text{ and } \int_a^b f\left(\frac{x+b}{2}\right)dx=2\int_{\frac{a+b}{2}}^b f(y)\,dy$$

and then

$$\frac{1}{2}\int_a^b\left[f\left(\frac{a+x}{2}\right)+f\left(\frac{x+b}{2}\right)\right]dx=\int_a^{\frac{a+b}{2}}f(y)\,dy+\int_{\frac{a+b}{2}}^b f(y)\,dy=\int_a^b f(y)\,dy.$$

By utilising (3.5) we then get (3.4) □

The case of Lipschitzian functions is as follows:

Corollary 1. *If f is of L-Lipschitz type with $L > 0$, then*

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{1}{4}L(b-a) \tag{3.6}$$

and

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{1}{8}L(b-a)^r. \tag{3.7}$$

In the case of functions of bounded variation, we have:

Theorem 7. *Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and continuous at the extremities. Then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\le \frac{1}{2}\left[\bigvee_a^b(f) - \frac{1}{b-a}\left(\int_a^b \ln\left(\frac{b-a}{x-a}\right)\bigvee_a^x(f)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right)\bigvee_x^b(f)\,dx\right)\right] \\ &\le \frac{1}{2}\bigvee_a^b(f)\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \\ &\le \frac{1}{2}\frac{1}{b-a}\left(\int_a^b \ln\left(\frac{b-a}{x-a}\right)\bigvee_a^x(f)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right)\bigvee_x^b(f)\,dx\right) \le \frac{1}{2}\bigvee_a^b(f).\end{aligned} \tag{3.9}$$

Proof. Taking the integral in (2.4) we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\le \frac{1}{b-a}\int_a^b \left|D_{a+,b-}f(x) - f(x)\right| dx \\ &\le \frac{1}{2}\bigvee_a^b(f) \\ &- \frac{1}{2}\frac{1}{b-a}\int_a^b\left[\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt - \frac{1}{b-x}\int_x^b\left(\bigvee_t^b(f)\right)dt\right]dx \\ &\le \frac{1}{2}\bigvee_a^b(f).\end{aligned} \tag{3.10}$$

Observe that

$$\begin{aligned}
&\int_a^b \frac{1}{x-a}\left(\int_a^x\left(\bigvee_a^t(f)\right)dt\right)dx \\
&= \int_a^b\left(\int_a^x\left(\bigvee_a^t(f)\right)dt\right)d\left(\ln(x-a)\right) \\
&= \ln(x-a)\int_a^x\left(\bigvee_a^t(f)\right)dt\Bigg|_{a+}^{b} - \int_a^b \ln(x-a)\bigvee_a^x(f)\,dx \\
&= \ln(b-a)\int_a^b\left(\bigvee_a^t(f)\right)dt - \int_a^b \ln(x-a)\bigvee_a^x(f)\,dx
\end{aligned}$$

since

$$\begin{aligned}
&\lim_{x\to a+}\left[\ln(x-a)\int_a^x\left(\bigvee_a^t(f)\right)dt\right] \\
&= \lim_{x\to a+}\left[(x-a)\ln(x-a)\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt\right] \\
&= \lim_{x\to a+}\left[(x-a)\ln(x-a)\right]\lim_{x\to a+}\left[\frac{1}{x-a}\int_a^x\left(\bigvee_a^t(f)\right)dt\right] \\
&= 0\lim_{x\to a+}\bigvee_a^t(f) = 0.
\end{aligned}$$

Also, in a similar way

$$\begin{aligned}
&\int_a^b \frac{1}{b-x}\left(\int_x^b\left(\bigvee_t^b(f)\right)dt\right)dx \\
&= -\int_a^b\left(\int_x^b\left(\bigvee_t^b(f)\right)dt\right)d\left(\ln(b-x)\right) \\
&= -\left[\ln(b-x)\int_x^b\left(\bigvee_t^b(f)\right)dt\Bigg|_a^{b-} - \int_a^b \ln(b-x)\,d\left(\int_x^b\left(\bigvee_t^b(f)\right)dt\right)\right] \\
&= -\left[\ln(b-x)\int_x^b\left(\bigvee_t^b(f)\right)dt\Bigg|_a^{b-} + \int_a^b \ln(b-x)\bigvee_x^b(f)\,dx\right] \\
&= -\left[0-\ln(b-a)\int_a^b\left(\bigvee_t^b(f)\right)dt + \int_a^b \ln(b-x)\bigvee_x^b(f)\,dx\right] \\
&= \ln(b-a)\int_a^b\left(\bigvee_t^b(f)\right)dt - \int_a^b \ln(b-x)\bigvee_x^b(f)\,dx.
\end{aligned}$$

If we add these equalities we get

$$\int_a^b \frac{1}{x-a}\left(\int_a^x \left(\bigvee_a^t (f)\right) dt\right) dx + \int_a^b \frac{1}{b-x}\left(\int_x^b \left(\bigvee_t^b (f)\right) dt\right) dx$$

$$= \ln(b-a)\int_a^b \left(\bigvee_a^t (f)\right) dt - \int_a^b \ln(x-a)\bigvee_a^x (f)\,dx$$

$$+ \ln(b-a)\int_a^b \left(\bigvee_t^b (f)\right) dt - \int_a^b \ln(b-x)\bigvee_x^b (f)\,dx$$

$$= (b-a)\bigvee_a^b (f)\ln(b-a) - \int_a^b \ln(x-a)\bigvee_a^x (f)\,dx - \int_a^b \ln(b-x)\bigvee_x^b (f)\,dx$$

$$= (b-a)\bigvee_a^x (f)\ln(b-a) + (b-a)\bigvee_x^b (f)\ln(b-a)$$

$$- \int_a^b \ln(x-a)\bigvee_a^x (f)\,dx - \int_a^b \ln(b-x)\bigvee_x^b (f)\,dx$$

$$= \int_a^b \ln\left(\frac{b-a}{x-a}\right)\bigvee_a^x (f)\,dx + \int_a^b \ln\left(\frac{b-a}{b-x}\right)\bigvee_x^b (f)\,dx$$

and by (3.10) we get (3.1).

The inequality (3.9) follows by(2.2). □

If we add the inequalities (3.8) with (3.9), use the triangle inequality and divide by 2 we get

Corollary 2. *With the assumptions of Theorem 7 we have*

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{1}{2}\left(\frac{1}{b-a}\int_a^b f(x)\,dx + \frac{f(a)+f(b)}{2}\right)\right| \leq \frac{1}{4}\bigvee_a^b (f). \tag{3.11}$$

4. Some Weighted Integral Inequalities

We need the following equality as well, see also [6]:

Lemma 2. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b). Then we have*

$$\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx = \frac{1}{4}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx. \tag{4.1}$$

Proof. We have

$$\int_a^b (x-a)(b-x)\,D_{a+,b-}f(x)\,dx = \frac{1}{2}\left[\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right) dx + \int_a^b (x-a)\left(\int_x^b f(t)\,dt\right) dx\right]. \tag{4.2}$$

Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^b (b-x)\left(\int_a^x f(t)\,dt\right)dx \\
&= -\int_a^b \left(\int_a^x f(t)\,dt\right) d\left(\frac{(b-x)^2}{2}\right) \\
&= -\left(\int_a^x f(t)\,dt\right)\frac{(b-x)^2}{2}\Bigg|_{a+}^{b} + \int_a^b \frac{(b-x)^2}{2} d\left(\int_a^x f(t)\,dt\right) \\
&= \frac{1}{2}\int_a^b (b-x)^2 f(x)\,dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b (x-a)\left(\int_x^b f(t)\,dt\right)dx \\
&= \int_a^b \left(\int_x^b f(t)\,dt\right) d\left(\frac{(x-a)^2}{2}\right) \\
&= \frac{(x-a)^2}{2}\int_x^b f(t)\,dt\Bigg|_a^{b-} - \int_a^b \frac{(x-a)^2}{2} d\left(\int_x^b f(t)\,dt\right) \\
&= \frac{1}{2}\int_a^b (x-a)^2 f(x)\,dx,
\end{aligned}$$

which, by (4.2) produces the desired result (4.1). $\square$

Theorem 8. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then we have*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx \right.\\
&\qquad\left. -\frac{2}{b-a}\int_a^b (x-a)(b-x) f(x)\,dx\right| \\
&\qquad\qquad \le \frac{2H}{(r+1)(r+2)(r+3)}(b-a)^{r+3}
\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx - (b-a)^2\frac{f(a)+f(b)}{6}\right| \\
&\le \frac{2H}{(r+1)(r+2)(r+3)}(b-a)^{r+3}.
\end{aligned} \tag{4.4}$$

Proof. From (1.4) and (1.5) we have

$$\begin{aligned}
&|D_{a+,b-}f(x)(x-a)(b-x) - f(x)(x-a)(b-x)| \\
&\le \frac{1}{2(r+1)}H\left[(x-a)^{r+1}(b-x)+(x-a)(b-x)^{r+1}\right]
\end{aligned} \tag{4.5}$$

and

$$\left|D_{a+,b-}f(x)(x-a)(b-x)-\frac{f(a)+f(b)}{2}(x-a)(b-x)\right| \le \frac{1}{2(r+1)}H\left[(x-a)^{r+1}(b-x)+(x-a)(b-x)^{r+1}\right] \tag{4.6}$$

for any $x\in[a,b]$.

Taking the integral mean in (4.5), we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx\right.\\&\left.-\frac{1}{b-a}\int_a^b f(x)(x-a)(b-x)\,dx\right|\\&\le\frac{1}{b-a}\int_a^b\left|D_{a+,b-}f(x)(x-a)(b-x)-f(x)(x-a)(b-x)\right|dx\\&\le\frac{1}{2(r+1)}\\&\times H\left[\frac{1}{b-a}\int_a^b(x-a)^{r+1}(b-x)\,dx+\frac{1}{b-a}\int_a^b(x-a)(b-x)^{r+1}\,dx\right].\end{aligned} \tag{4.7}$$

Since

$$\int_a^b(x-a)^{r+1}(b-x)\,dx=\frac{1}{(r+2)(r+3)}(b-a)^{r+3}$$

and

$$\int_a^b(x-a)(b-x)^{r+1}\,dx=\frac{1}{(r+2)(r+3)}(b-a)^{r+3},$$

then by (4.7) we have

$$\begin{aligned}&\left|\frac{1}{4}\frac{1}{b-a}\int_a^b\left[(x-a)^2+(b-x)^2\right]f(x)\,dx-\frac{1}{b-a}\int_a^b(x-a)(b-x)f(x)\,dx\right|\\&\le\frac{H}{(r+1)(r+2)(r+3)}(b-a)^{r+3},\end{aligned}$$

which, by multiplying by 2 proves the inequality (4.3).

Taking the integral mean in (4.6), we have

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx\right.\\&\left.-\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b(x-a)(b-x)\,dx\right|\\&\le\frac{1}{b-a}\int_a^b\left|D_{a+,b-}f(x)(x-a)(b-x)-\frac{f(a)+f(b)}{2}(x-a)(b-x)\right|dx\\&\le\frac{1}{2(r+1)}H\left[\frac{1}{b-a}\int_a^b(x-a)^{r+1}(b-x)\,dx+\frac{1}{b-a}\int_a^b(x-a)(b-x)^{r+1}\,dx\right].\end{aligned} \tag{4.8}$$

Since

$$\frac{1}{b-a}\int_a^b (x-a)(b-x)\,dx = \frac{1}{6}(b-a)^2,$$

then by (4.8) we get

$$\left|\frac{1}{4}\frac{1}{b-a}\int_a^b \left[(x-a)^2+(b-x)^2\right] f(x)\,dx - \frac{1}{6}(b-a)^2\frac{f(a)+f(b)}{2}\right|$$
$$\leq \frac{H}{(r+1)(r+2)(r+3)}(b-a)^{r+3}$$

and by multiplying with 2 we get (4.4). □

Corollary 3. *If f is of L-Lipschitz type with $L>0$, then*

$$\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx - \frac{2}{b-a}\int_a^b (x-a)(b-x) f(x)\,dx\right| \leq \frac{1}{12}L(b-a)^{r+3} \tag{4.9}$$

and

$$\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx - (b-a)^2\frac{f(a)+f(b)}{6}\right| \leq \frac{1}{12}L(b-a)^{r+3}. \tag{4.10}$$

Theorem 9. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx - \frac{2}{b-a}\int_a^b (x-a)(b-x) f(x)\,dx\right| \\ &\leq \frac{1}{2}\left(\frac{1}{3}\bigvee_a^b(f)(b-a)^2 - \int_a^b (b-x)^2\bigvee_a^x(f)\,dx - \int_a^b (x-a)^2\bigvee_x^b(f)\,dx\right) \\ &\leq \frac{1}{6}\bigvee_a^b(f)(b-a)^2\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \left[\frac{(x-a)^2+(b-x)^2}{2}\right] f(x)\,dx - (b-a)^2\frac{f(a)+f(b)}{6}\right| \\ &\leq \frac{1}{2}\left(\int_a^b (b-x)^2\bigvee_a^x(f)\,dx + \int_a^b (x-a)^2\bigvee_x^b(f)\,dx\right) \leq \frac{1}{6}\bigvee_a^b(f)(b-a)^2.\end{aligned} \tag{4.12}$$

Proof. From (2.4) we have

$$\begin{aligned}&\left|D_{a+,b-}f(x)(x-a)(b-x)-f(x)(x-a)(b-x)\right|\\&\leq\frac{1}{2}\left[\bigvee_a^b(f)(x-a)(b-x)-(b-x)\int_a^x\left(\bigvee_a^t(f)\right)dt-(x-a)\int_x^b\left(\bigvee_t^b(f)\right)dt\right]\end{aligned}$$

while from (2.2) we have

$$\begin{aligned}&\left|D_{a+,b-}f(x)(x-a)(b-x)-\frac{f(a)+f(b)}{2}(x-a)(b-x)\right|\\&\leq\frac{1}{2}\left[(b-x)\int_a^x\left(\bigvee_a^t(f)\right)dt+(x-a)\int_x^b\left(\bigvee_t^b(f)\right)dt\right]\end{aligned}$$

for any $x\in(a,b)$.

Taking the integral mean we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx\right.\\&\qquad\left.-\frac{1}{b-a}\int_a^b f(x)(x-a)(b-x)\,dx\right|\\&\leq\frac{1}{2}\left[\bigvee_a^b(f)\frac{1}{b-a}\int_a^b(x-a)(b-x)\,dx\right.\\&\quad-\frac{1}{b-a}\int_a^b(b-x)\left(\int_a^x\left(\bigvee_a^t(f)\right)dt\right)dx\\&\qquad\left.-\frac{1}{b-a}\int_a^b(x-a)\left(\int_x^b\left(\bigvee_t^b(f)\right)dt\right)dx\right]\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b D_{a+,b-}f(x)(x-a)(b-x)\,dx\right.\\&\qquad\left.-\frac{f(a)+f(b)}{2}\frac{1}{b-a}\int_a^b(x-a)(b-x)\,dx\right|\\&\leq\frac{1}{2}\left[\frac{1}{b-a}\int_a^b(b-x)\left(\int_a^x\left(\bigvee_a^t(f)\right)dt\right)dx\right.\\&\qquad\left.+\frac{1}{b-a}\int_a^b(x-a)\left(\int_x^b\left(\bigvee_t^b(f)\right)dt\right)dx\right].\end{aligned}\tag{4.14}$$

Using the integration by parts, we have

$$\begin{aligned}
&\int_a^b (b-x)\left(\int_a^x \left(\bigvee_a^t (f)\right) dt\right) dx \\
&= -\frac{1}{2}\int_a^b \left(\int_a^x \left(\bigvee_a^t (f)\right) dt\right) d\left((b-x)^2\right) \\
&= -\frac{1}{2}\left[(b-x)^2 \int_a^x \left(\bigvee_a^t (f)\right) dt\Big|_a^b - \int_a^b (b-x)^2 \bigvee_a^x (f)\, dx\right] \\
&= \frac{1}{2}\int_a^b (b-x)^2 \bigvee_a^x (f)\, dx
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b (x-a)\left(\int_x^b \left(\bigvee_t^b (f)\right) dt\right) dx \\
&= \frac{1}{2}\int_a^b \left(\int_x^b \left(\bigvee_t^b (f)\right) dt\right) d\left((x-a)^2\right) \\
&= \frac{1}{2}\left[(x-a)^2 \int_x^b \left(\bigvee_t^b (f)\right) dt\Big|_a^b + \int_a^b (x-a)^2 \bigvee_x^b (f)\, dx\right] \\
&= \frac{1}{2}\int_a^b (x-a)^2 \bigvee_x^b (f)\, dx
\end{aligned}$$

and by (4.13) and (4.14) we get the desired results (4.11) and (4.120. □

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