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IMPROVEMENT OF FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITY AND SOME NEW FRACTIONAL MIDPOINT TYPE INEQUALITIES FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, it is proved that fractional Hermite-Hadamard inequality and fractional Hermite-Hadamard-Fejér inequality are just results of Hermite-Hadamard-Fejér inequality. After this, a new fractional Hermite-Hadamard inequality which is not a result of Hermite-Hadamard-Fejér inequality and better than given in [9] by Sarıkaya et al. is obtained. Also, a new equality is proved and some new fractional midpoint type inequalities are given. Our results have some relations with the results given in [5] by Kırmacı.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [2, 3].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequality or its weighted versions, the so-called Hermite-Hadamard-Fejér inequality.

In [1], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then, the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (i.e. $g(x) = g(a + b - x)$ for all $x \in [a, b]$).

In [5], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 1. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° tne interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) = (b-a)\int_0^{1/2} tf'(ta+(1-t)b)\,dt + \int_{1/2}^1 (t-1)f'(ta+(1-t)b)\,dt. \tag{1.3}$$

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 1. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t}t^{\alpha-1}dt$ (see [8, page 69] *and* [10, page 4]*).*

In [9], Sarıkaya et al. proved the following fractional Hermite-Hadamard type inequality:

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Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}\tag{1.4}$$

with $\alpha>0$.

Remark 1. *In Theorem 2, it is not necessary supposing that f be a positive function and a,b are positive real numbers. From the Definition 1, it is clear that a,b are any real numbers such as $a<b$.*

In [4], İşcan proved the following fractional Hermite-Hadamard-Fejér type inequality:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a convex function with $a<b$ and $f\in L[a,b]$. If $g:[a,b]\to\mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a+b}{2}$, then the following inequality for fractional integrals holds:*

$$f\left(\frac{a+b}{2}\right)\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\leq\left[J_{a+}^{\alpha}(fg)(b)+J_{b-}^{\alpha}(fg)(a)\right]\leq\frac{f(a)+f(b)}{2}\left[J_{a+}^{\alpha}g(b)+J_{b-}^{\alpha}g(a)\right]\tag{1.5}$$

with $\alpha>0$.

In [6], Kunt et al. proved the following left Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

Theorem 4. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the left Riemann-Liouville fractional integral holds:*

$$f\left(\frac{\alpha a+b}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a+}^{\alpha}f(b)\leq\frac{\alpha f(a)+f(b)}{\alpha+1}\tag{1.6}$$

with $\alpha>0$.

Lemma 2. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the left Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{a+}^{\alpha}f(b)-f\left(\frac{\alpha a+b}{\alpha+1}\right)\\&=(b-a)\left[\int_{0}^{\frac{\alpha}{\alpha+1}}t^{\alpha}f'(ta+(1-t)b)\,dt+\int_{\frac{\alpha}{\alpha+1}}^{1}(t^{\alpha}-1)f'(ta+(1-t)b)\,dt\right]\end{aligned}\tag{1.7}$$

with $\alpha>0$.

In [7], Kunt et al. proved the following right Riemann-Liouville fractional Hermite-Hadamard type inequality and next equality:

Theorem 5. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{a+\alpha b}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\leq\frac{f(a)+\alpha f(b)}{\alpha+1}\tag{1.8}$$

with $\alpha>0$.

Lemma 3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\\&=(b-a)\left[\int_{0}^{\frac{\alpha}{\alpha+1}}-t^{\alpha}f'(tb+(1-t)a)\,dt+\int_{\frac{\alpha}{\alpha+1}}^{1}(1-t^{\alpha})f'(tb+(1-t)a)\,dt\right]\end{aligned}\tag{1.9}$$

with $\alpha>0$.

In our studies we noticed that fractional Hermite-Hadamard type inequality given in Theorem 2 and fractional Hermite-Hadamard-Fejér type inequality given in Theorem 3 are just result of Hermite-Hadamard-Fejér inequality (given in Theorem 1), with a special selection of the weighted function. This show how strong the Hermite-Hadamard-Fejér inequality is. However, we will prove new fractional Hermite-Hadamard type inequality which is not a result of Theorem 1. Also, we will have new fractional midpoint type inequalities.

2. Results of Hermite-Hadamard-Fejér inequality

Proposition 1. *Theorem 2 is a result of Theorem 1.*

Proof. In Theorem 1, let we choose $g(x) = (x-a)^{\alpha-1} + (b-x)^{\alpha-1}$ for $\alpha > 0$, $a, b \in \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ (It is clear $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\int_a^b g(x)dx = \int_a^b (x-a)^{\alpha-1} + (b-x)^{\alpha-1} dx = \frac{2(b-a)^\alpha}{\alpha}, \tag{2.1}$$

$$\begin{aligned} \int_a^b f(x)g(x)dx &= \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)dx \\ &= \int_a^b (x-a)^{\alpha-1} f(x)dx + \int_a^b (b-x)^{\alpha-1} f(x)dx \\ &= \Gamma(\alpha)\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right]. \end{aligned} \tag{2.2}$$

Combining (1.2), (2.1) and (2.2) we have (1.4). This completes the proof. □

Proposition 2. *Theorem 3 is a result of Theorem 1.*

Proof. In Theorem 1, let we choose $w(x) = \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] g(x)$ for $\alpha > 0$, $a, b \in \mathbb{R}$, $g : [a,b] \to \mathbb{R}$ and $g(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$ (It is clear $w(x)$ nonnegative, integrable and symmetric to $\frac{a+b}{2}$). Computing the following integrals, we have

$$\begin{aligned} \int_a^b w(x)\,dx &= \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] g(x)\,dx \\ &= \int_a^b (x-a)^{\alpha-1} g(x)\,dx + \int_a^b (b-x)^{\alpha-1} g(x)\,dx \\ &= \Gamma(\alpha)\left[J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)\right], \end{aligned} \tag{2.3}$$

$$\begin{aligned} \int_a^b f(x)w(x)dx &= \int_a^b \left[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}\right] f(x)g(x)\,dx \\ &= \int_a^b (x-a)^{\alpha-1} f(x)g(x)\,dx + \int_a^b (b-x)^{\alpha-1} f(x)g(x)\,dx \\ &= \Gamma(\alpha)\left[J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)\right]. \end{aligned} \tag{2.4}$$

Combining (1.2), (2.3) and (2.4) we have (1.5). This completes the proof. □

Remark 2. *Theorem 4 and Theorem 5 are not results of Theorem 1.*

3. Improvement of Fractional Hermite-Hadamard Type Inequality

We will use Theorem 4 and Theorem 5 to have new fractional Hermite-Hadamard type inequality better than (1.4).

Theorem 6. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a convex function. If $f \in L[a,b]$, then the following inequality for fractional integral holds:*

$$\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right) + f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2} \tag{3.1}$$

with $\alpha > 0$.

Proof. If (1.6) and (1.8) gather side by side and dividing into 2, it is hold the desired result. □

Remark 3. *Since, f is a convex function on $[a,b]$, it is clear $f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}$ for $\alpha > 0$. It means that*

1. *Theorem 6 is better than Theorem 2,*
2. *In Theorem 6 if one takes $\alpha = 1$, one has (1.1),*
3. *Theorem 6 is not a result of Theorem 1.*

4. New Fractional Midpoint Type Inequalities

We will now prove an equality to have new fractional midpoint type inequalities.

Lemma 4. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $f' \in L[a, b]$, then the following equality for the fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2} \tag{4.1}$$

$$=\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha} f'(ta+(1-t)b)\,dt+\int_{\frac{\alpha}{\alpha+1}}^1 (t^{\alpha}-1) f'(ta+(1-t)b)\,dt \\ +\int_0^{\frac{\alpha}{\alpha+1}} -t^{\alpha} f'(tb+(1-t)a)\,dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha}) f'(tb+(1-t)a)\,dt\end{array}\right]$$

Proof. If (1.7) and (1.9) gather side by side and dividing into 2, it is hold the desired result. □

Corollary 1. *In Lemma 4, if one takes $\alpha = 1$, one has Lemma 1.*

Theorem 7. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|$ is convex on $[a, b]$, then the following fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right| \tag{4.2}$$

$$\leq (b-a)\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[|f'(a)|+|f'(b)|\right]$$

with $\alpha > 0$.

Proof. Using Lemma 4 and the convexity of $|f'|$, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right|$$

$$\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}|f'(ta+(1-t)b)|\,dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})|f'(ta+(1-t)b)|\,dt \\ +\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}|f'(tb+(1-t)a)|\,dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})|f'(tb+(1-t)a)|\,dt\end{array}\right]$$

$$\leq\frac{b-a}{2}\left[\begin{array}{c}\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[t|f'(a)|+(1-t)|f'(b)|\right]dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[t|f'(a)|+(1-t)|f'(b)|\right]dt \\ +\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[t|f'(b)|+(1-t)|f'(a)|\right]dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[t|f'(b)|+(1-t)|f'(a)|\right]dt\end{array}\right]$$

$$=\frac{b-a}{2}\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[|f'(a)|+|f'(b)|\right]dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[|f'(a)|+|f'(b)|\right]dt\right]$$

$$=\frac{b-a}{2}\left[\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\,dt\right]\left[|f'(a)|+|f'(b)|\right]$$

$$=(b-a)\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[|f'(a)|+|f'(b)|\right].$$

This completes the proof. □

Corollary 2. *In Theorem 7, if one takes $\alpha = 1$, one has* [5, Theorem 2.2].

Theorem 8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function on (a, b). If $|f'|^q$ is convex on $[a, b]$ for $q \geq 1$, then the following fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right| \tag{4.3}$$

$$\leq\frac{b-a}{2}\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[\begin{array}{c}\left(\frac{\alpha}{\alpha+2}|f'(a)|^q+\frac{2}{\alpha+2}|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\alpha+1)^{\alpha}+2\alpha^{\alpha+1}}{2\alpha^{\alpha}(\alpha+2)}|f'(a)|^q+\frac{4\alpha^{\alpha}-(\alpha+1)^{\alpha}}{2\alpha^{\alpha}(\alpha+2)}|f'(b)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{\alpha}{\alpha+2}|f'(b)|^q+\frac{2}{\alpha+2}|f'(a)|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\alpha+1)^{\alpha}+2\alpha^{\alpha+1}}{2\alpha^{\alpha}(\alpha+2)}|f'(b)|^q+\frac{4\alpha^{\alpha}-(\alpha+1)^{\alpha}}{2\alpha^{\alpha}(\alpha+2)}|f'(a)|^q\right)^{\frac{1}{q}}\end{array}\right]$$

with $\alpha > 0$.

Proof. Using Lemma 4, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right|$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(ta+(1-t)b)\right|dt \\ +\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(tb+(1-t)a)\right|dt \end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}dt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\left(\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\right)^{1-\frac{1}{q}}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right]dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+2}}\left[\begin{array}{c} \left(\frac{\alpha}{\alpha+2}\left|f'(a)\right|^q+\frac{2}{\alpha+2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\alpha+1)^{\alpha}+2\alpha^{\alpha+1}}{2\alpha^{\alpha}(\alpha+2)}\left|f'(a)\right|^q+\frac{4\alpha^{\alpha}-(\alpha+1)^{\alpha}}{2\alpha^{\alpha}(\alpha+2)}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{\alpha}{\alpha+2}\left|f'(b)\right|^q+\frac{2}{\alpha+2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{(\alpha+1)^{\alpha}+2\alpha^{\alpha+1}}{2\alpha^{\alpha}(\alpha+2)}\left|f'(b)\right|^q+\frac{4\alpha^{\alpha}-(\alpha+1)^{\alpha}}{2\alpha^{\alpha}(\alpha+2)}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. □

Corollary 3. *In Theorem 8, if one takes $\alpha=1$, one has the following midpoint type inequality,*

$$\left|\frac{1}{b-a}\int_a^b f(u)\,du-f\left(\frac{a+b}{2}\right)\right|\leq\frac{b-a}{8}\left[\begin{array}{c}\left(\frac{1}{3}\left|f'(a)\right|^q+\frac{2}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\frac{2}{3}\left|f'(a)\right|^q+\frac{1}{3}\left|f'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right]. \tag{4.4}$$

Theorem 9. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following fractional midpoint type inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right| \tag{4.5}$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(a)\right|^q+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(a)\right|^q+\frac{1}{2(\alpha+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})^p dt\right)^{\frac{1}{p}}\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{1}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \end{array}\right]$$

with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>0$.

Proof. Using Lemma 4, Holder inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]-\frac{f\left(\frac{\alpha a+b}{\alpha+1}\right)+f\left(\frac{a+\alpha b}{\alpha+1}\right)}{2}\right|$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(ta+(1-t)b)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(ta+(1-t)b)\right|dt \\ +\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha}\left|f'(tb+(1-t)a)\right|dt+\int_{\frac{\alpha}{\alpha+1}}^1 (1-t^{\alpha})\left|f'(tb+(1-t)a)\right|dt \end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}} \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left|f'\left(ta+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}} \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left|f'\left(tb+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}} \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\alpha}{\alpha+1}} \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \end{array}\right]$$

$$\leq \frac{b-a}{2}\left[\begin{array}{c} \left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(a)\right|^q+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(a)\right|^q+\frac{1}{2(\alpha+1)^2}\left|f'(b)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_0^{\frac{\alpha}{\alpha+1}} t^{\alpha p}dt\right)^{\frac{1}{p}}\left(\frac{\alpha^2}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{\alpha^2+2\alpha}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{\alpha}{\alpha+1}}^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left(\frac{2\alpha+1}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{1}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \end{array}\right].$$

This completes the proof. □

Corollary 4. *In Theorem 9, if one takes* $\alpha = 1$*, one has* [5, Theorem 2.3].

5. Competing Interests

The authors declare that they have no competing interests.

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