Received 05/07/17

THE RIGHT CONFORMABLE FRACTIONAL HERMITE-HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

SERCAN TURHAN, İMDAT İŞCAN, MEHMET KUNT

Abstract. In this paper, a new fractional Hermite-Hadamard type inequality for convex functions is obtained by using only the right conformable fractional integral. Also, to have new fractional trapezoid and midpoint type inequalities for the differentiable convex functions, two new equalities are proved. Our results generalise the studies [2, 5, 6, 8].

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [3, 4].

In [2, 8], the authors used the following equality to obtain trapezoid type inequalities and some applications:

Lemma 1. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° is the interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\,du = \frac{b-a}{2}\int_0^1 (1-2t) f'(ta + (1-t)b)\,dt. \tag{1.2}$$

In [5], Kırmacı used the following equality to obtain midpoint type inequalities and some applications:

Lemma 2. *Let $a, b \in I$ with $a < b$ and $f : I^\circ \to \mathbb{R}$ is a differentiable mapping (I° the interior of I). If $f' \in L[a, b]$, then we have*

$$\frac{1}{b-a}\int_a^b f(u)\,du - f\left(\frac{a+b}{2}\right) = (b-a)\int_0^{1/2} t f'(ta + (1-t)b)\,dt + \int_{1/2}^1 (t-1) f'(ta + (1-t)b)\,dt. \tag{1.3}$$

Definition 1. [9, page 12]. *A function f defined on I has a support at $x_0 \in I$ if there exists an affine functions $A(x) = f(x_0) + m(x - x_0)$ such that $A(x) \leq f(x)$ for all $x \in I$. The graph of the support function A is called a line of support for f at x_0.*

Theorem 1. [9, page 12] *$f : (a, b) \to \mathbb{R}$ is a convex function if and only if there is at least one line of support for f at each $x_0 \in (a, b)$.*

Following definitions of the left and right side Riemann-Liouville fractional integrals are well known in the literature.

Definition 2. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f \in L[a, b]$. The left and right Riemann-Liouville fractional integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^\infty e^{-t} t^{\alpha-1} dt$ (see [7, page 69] *and* [11, page 4]*).*

2000 *Mathematics Subject Classification.* 26A51, 26A33, 26D10.

Key words and phrases. Convex functions, Hermite-Hadamard inequality, Left conformable fractional integral, Trapezoid type inequalities, Midpoint type inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 105, 10 pp

The beta function and incomplete beta function defined as follows:

$$B(u,v)=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}=\int_0^1 t^{u-1}(1-t)^{v-1}dt,\qquad u,v>0,$$

$$B_w(u,v)=\int_0^w t^{u-1}(1-t)^{v-1}dt\qquad u,v>0 \text{ and } 0\leq w\leq 1.$$

In [6], Kunt et al. proved the following fractional Hermite-Hadamard type inequality via the left Riemann-Liouville fractional integral and next equalities:

Theorem 2. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a convex function. If $f\in L[a,b]$, then the following inequality for the right Riemann-Liouville fractional integral holds:*

$$f\left(\frac{a+\alpha b}{\alpha+1}\right)\leq\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)\leq\frac{f(a)+\alpha f(b)}{\alpha+1}\tag{1.4}$$

with $\alpha>0$.

Lemma 3. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\frac{f(a)+\alpha f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)=\frac{b-a}{\alpha+1}\int_0^1\left[(\alpha+1)(1-t)^{\alpha}-1\right]f'(ta+(1-t)b)\,dt\tag{1.5}$$

with $\alpha>0$.

Lemma 4. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $f'\in L[a,b]$, then the following equality for the right Riemann-Liouville fractional integrals holds:*

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}J_{b-}^{\alpha}f(a)-f\left(\frac{a+\alpha b}{\alpha+1}\right)\\&=(b-a)\left[\int_0^{\frac{\alpha}{\alpha+1}}-t^{\alpha}f'(tb+(1-t)a)\,dt+\int_{\frac{\alpha}{\alpha+1}}^1(1-t^{\alpha})f'(tb+(1-t)a)\,dt\right]\end{aligned}\tag{1.6}$$

with $\alpha>0$.

Following definitions of the left and right side conformable fractional integrals given in [1] (see also [10]):

Definition 3. *Let $\alpha\in(n,n+1]$, $n=0,1,2,...,$ $\beta=\alpha-n$, $a,b\in\mathbb{R}$ with $a<b$ and $f\in L[a,b]$. The left and right conformable fractional integrals $I_{\alpha}^{a}f$ and ${}^{b}I_{\alpha}f$ of order $\alpha>0$ are defined by*

$$I_{\alpha}^{a}f(t)=\frac{1}{n!}\int_a^t(t-x)^n(x-a)^{\beta-1}f(x)dx,\ t>a$$

and

$${}^{b}I_{\alpha}f(t)=\frac{1}{n!}\int_t^b(x-t)^n(b-x)^{\beta-1}f(x)dx,\ t<b$$

respectively.

It is easily seen that if one takes $\alpha=n+1$ in the Definition 3 (for the left and right conformable foractional integrals), one has the Definition 2 (the left and right Riemann-Liouville fractional integrals) for $\alpha\in\mathbb{N}$.

In [10], Set et al. proved following Hermite-Hadamrd type inequality via conformable fractional integrals:

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be a function with $0\leq a<b$ and $f\in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}\left[\ I_{\alpha}^{a}f(b)\ +\ {}^{b}I_{\alpha}f(a)\ \right]\leq\frac{f(a)+f(b)}{2}\tag{1.7}$$

with $\alpha\in(n,n+1]$.

In literature, there are hundreds studies for Hermite-Hadamard type inequality by using the left and right fractional integrals (such as Riemann-Liouville fractional integrals, Hadamard fractional integrals, Conformable fractional integrals etc.). In all of them, the left and right fractional integrals are used together. As much as we know, the first study for Hermite-Hadamard type inequality by using only the right Riemann-Liouville fractional integral is given in [6] by Kunt et al.

In this paper, our aim is obtaining new fractional Hermite-Hadamard type inequality by using only the righ conformable fractional integral for convex functions. Also we desire proving new equalities to have new conformable fractional trapezoid and midpoint type inequalities for the differentiable convex functions. This study generalise the studies [2, 5, 6, 8].

2. The Right Fractional Hermite Hadamard Inequality

Theorem 4. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a convex function. If $f \in L[a, b]$, then the following inequality for the right conformable fractional integral holds:*

$$f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \;{}^{b}I_{\alpha}f(a) \;\leq \frac{(\alpha-n)\,f(a)+(n+1)\,f(b)}{\alpha+1} \tag{2.1}$$

with $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$.

Proof. Let $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$. Since f is convex on $[a, b]$, using Theorem 1, there is at least one line of support

$$A(x) = f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) + m\left(x - \frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) \leq f(x) \tag{2.2}$$

for all $x \in [a, b]$. From (2.2), we have

$$\begin{aligned} A(tb+(1-t)\,a) &= f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) + m\left(tb+(1-t)\,a - \frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) \\ &\leq f(tb+(1-t)\,a) \end{aligned} \tag{2.3}$$

for all $t \in [0, 1]$. Multiplying both sides of (2.3) with $\frac{1}{n!}t^{n}(1-t)^{\alpha-n-1}$ and integrating over $[0, 1]$ respect to t, we have

$$\begin{aligned} &\frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n-1} A(tb+(1-t)\,a)\,dt \\ &= \frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n-1}\left[f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) + m\left(tb+(1-t)\,a - \frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\right]dt \\ &= f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n-1}\,dt \\ &\quad + \frac{m}{n!}\left[\int_0^1 t^{n}(1-t)^{\alpha-n-1}[tb+(1-t)\,a]\,dt - \frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\int_0^1 t^{n}(1-t)^{\alpha-n-1}\,dt\right] \\ &= f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\frac{B(n+1,\alpha-n)}{n!} \\ &\quad + \frac{m}{n!}\left[\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}B(n+1,\alpha-n) - \frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}B(n+1,\alpha-n)\right] \\ &= f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\frac{B(n+1,\alpha-n)}{n!} \leq \frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n-1} f(tb+(1-t)\,a)\,dt \\ &= \frac{1}{(b-a)^{\alpha}}\frac{1}{n!}\int_a^b (t-a)^{n}(b-t)^{\alpha-n-1} f(t)dt = \frac{1}{(b-a)^{\alpha}} \;{}^{b}I_{\alpha}f(a)\;. \end{aligned}$$

It means that

$$f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right) \leq \frac{n!}{B(n+1,\alpha-n)}\frac{1}{(b-a)^{\alpha}} \;{}^{b}I_{\alpha}f(a) \;= \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \;{}^{b}I_{\alpha}f(a) \tag{2.4}$$

On the other hand, using the convexity of f on $[a, b]$, we have

$$f(tb+(1-t)\,a) \leq tf(b) + (1-t)\,f(a) \tag{2.5}$$

for all $t \in [0, 1]$. Multiplying both sides of (2.5) with $\frac{1}{n!}t^{n}(1-t)^{\alpha-n-1}$ and integrating over $[0, 1]$ respect to t, we have

$$\begin{aligned} &\frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n-1} f(tb+(1-t)\,a)\,dt = \frac{1}{(b-a)^{\alpha}}\frac{1}{n!}\int_a^b (t-a)^{n}(b-t)^{\alpha-n-1} f(t)dt \\ &= \frac{1}{(b-a)^{\alpha}} \;{}^{b}I_{\alpha}f(a) \;\leq f(b)\frac{1}{n!}\int_0^1 t^{n+1}(1-t)^{\alpha-n-1}\,dt + f(a)\frac{1}{n!}\int_0^1 t^{n}(1-t)^{\alpha-n}\,dt \\ &= \frac{1}{n!}\frac{(\alpha-n)\,f(a)+(n+1)\,f(b)}{\alpha+1}B(n+1,\alpha-n)\,. \end{aligned}$$

It means that

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)} \;{}^{b}I_{\alpha}f(a) \;= \frac{n!}{B(n+1,\alpha-n)}\frac{1}{(b-a)^{\alpha}} \;{}^{b}I_{\alpha}f(a) \;\leq \frac{(\alpha-n)\,f(a)+(n+1)\,f(b)}{\alpha+1}. \tag{2.6}$$

By using (2.4) and (2.6), we have (2.1). This completes the proof. $\square$

Remark 1. *In Theorem 4,*

(1) *If one takes* $\alpha = n+1$, *one has* (1.4),
(2) *If one takes* $\alpha = n+1$, *after that if one takes* $\alpha = 1$, *one has* (1.1) *(Hermite-Hadamard inequality).*

3. Lemmas

In this section we will prove the main equalities related to Lemma 1, Lemma 2, Lemma 3 and Lemma 4.

Lemma 5. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be a differentiable function on* (a, b). *If* $f' \in L[a, b]$, *then the following equality for the right conformable fractional integrals holds:*

$$\frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha} f(a) \tag{3.1}$$

$$= \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \int_0^1 \left[\begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right] f'(ta+(1-t)b)\, dt$$

with $n = 0, 1, 2\ldots$ *and* $\alpha \in (n, n+1]$.

Proof. If we apply the partial integration to the right hand side of the equation (3.1), we have

$$\frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \int_0^1 \left[\begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right] f'(ta+(1-t)b)\, dt$$

$$= \frac{b-a}{(\alpha+1)} \int_0^1 \left[(\alpha+1) \frac{B_{1-t}(n+1,\alpha-n)}{B(n+1,\alpha-n)} - (\alpha-n) \right] f'(ta+(1-t)b)\, dt$$

$$= (b-a) \left[\frac{1}{B(n+1,\alpha-n)} \int_0^1 B_{1-t}(n+1,\alpha-n) f'(ta+(1-t)b)\, dt - \frac{\alpha-n}{\alpha+1} \int_0^1 f'(ta+(1-t)b)\, dt \right]$$

$$= (b-a) \left[\begin{array}{c} \frac{1}{B(n+1,\alpha-n)} \int_0^1 \left(\int_0^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) f'(ta+(1-t)b)\, dt \\ -\frac{\alpha-n}{\alpha+1} \left. \frac{f(ta+(1-t)b)}{a-b} \right|_0^1 \end{array} \right]$$

$$= (b-a) \left[\begin{array}{c} \frac{1}{B(n+1,\alpha-n)} \left(\left. \left(\int_0^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{f(ta+(1-t)b)}{a-b} \right|_0^1 + \int_0^1 (1-t)^n t^{\alpha-n-1} \frac{f(ta+(1-t)b)}{a-b} dt \right) \\ +\frac{\alpha-n}{\alpha+1} \frac{f(a)-f(b)}{b-a} \end{array} \right]$$

$$= \left[\begin{array}{c} \frac{1}{B(n+1,\alpha-n)} \left(B(n+1,\alpha-n) f(b) - \int_0^1 t^n (1-t)^{\alpha-n-1} f(tb+(1-t)a)\, dt \right) \\ +\frac{\alpha-n}{\alpha+1} (f(a)-f(b)) \end{array} \right]$$

$$= \left[\frac{\alpha-n}{\alpha+1} (f(a)-f(b)) + f(b) - \frac{1}{B(n+1,\alpha-n)} \int_0^1 t^n (1-t)^{\alpha-n-1} f(tb+(1-t)a)\, dt \right]$$

$$= \left[\frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{1}{B(n+1,\alpha-n)} \int_0^1 t^n (1-t)^{\alpha-n-1} f(tb+(1-t)a)\, dt \right]$$

$$= \frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha} f(a) \ .$$

This completes the proof. □

Remark 2. *In Lemma 5,*

(1) *If one takes* $\alpha = n+1$, *one has the Lemma 3,*
(2) *If one takes* $\alpha = n+1$, *after that if one takes* $\alpha = 1$, *one has the Lemma 1.*

Lemma 6. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a, b] \to \mathbb{R}$ *be a differentiable function on* (a, b). *If* $f' \in L[a, b]$, *then the following equality for the right conformable fractional integrals holds:*

$$\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha} f(a) \ - f\left(\frac{(\alpha-n) a + (n+1) b}{\alpha+1} \right) \tag{3.2}$$

$$= \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} -B_t(n+1,\alpha-n) f'(tb+(1-t)a)\, dt \\ + \int_{\frac{n+1}{\alpha+1}}^1 \left(B(n+1,\alpha-n) - B_t(n+1,\alpha-n) \right) f'(tb+(1-t)a)\, dt \end{array} \right]$$

with $n = 0, 1, 2\ldots$ *and* $\alpha \in (n, n+1]$.

Proof. If we apply the partial integration to the right hand side of the equation (3.2), we have

$$\frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}-B_t(n+1,\alpha-n)\,f'(tb+(1-t)a)\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^1\left(B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right)f'(tb+(1-t)a)\,dt\end{array}\right]$$

$$=(b-a)\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}-\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}f'(tb+(1-t)a)\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^1\left(1-\frac{B_t(n+1,\alpha-n)}{B(n+1,\alpha-n)}\right)f'(tb+(1-t)a)\,dt\end{array}\right]$$

$$=(b-a)\left[\int_{\frac{n+1}{\alpha+1}}^1 f'(tb+(1-t)a)\,dt-\frac{1}{B(n+1,\alpha-n)}\int_0^1 B_t(n+1,\alpha-n)\,f'(tb+(1-t)a)\,dt\right]$$

$$=(b-a)\left[\begin{array}{c}\left(\left.\frac{f(tb+(1-t)a)}{b-a}\right|_{\frac{n+1}{\alpha+1}}^1\right)\\ -\frac{1}{B(n+1,\alpha-n)}\left(B_t(n+1,\alpha-n)\left.\frac{f(tb+(1-t)a)}{b-a}\right|_0^1-\int_0^1 t^n(1-t)^{\alpha-n+1}\frac{f(tb+(1-t)a)}{b-a}dt\right)\end{array}\right]$$

$$=\left[\begin{array}{c}\left(f(b)-f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\right)\\ -\left(f(b)-\frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n(1-t)^{\alpha-n+1}f(tb+(1-t)a)\,dt\right)\end{array}\right]$$

$$=\frac{1}{B(n+1,\alpha-n)}\int_0^1 t^n(1-t)^{\alpha-n+1}f(tb+(1-t)a)\,dt-f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)$$

$$=\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\ {}^bI_\alpha f(a)\ -f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right).$$

This completes the proof. □

Remark 3. *In Lemma 6,*

(1) *If one takes $\alpha=n+1$, one has the Lemma 4,*
(2) *If one takes $\alpha=n+1$, after that if one takes $\alpha=1$, one has the Lemma 2.*

4. The Right Conformable Fractional Trapezoid and Midpoint Type Inequalities

In this section we will obtain some new right conformable fractional trapezoid and midpoint type inequalities by using Lemma 5 and Lemma 6.

Theorem 5. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is convex on $[a,b]$, then the following right conformable fractional integral inequality holds:*

$$\left|\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\ {}^bI_\alpha f(a)\right| \tag{4.1}$$
$$\leq\frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\left[|f'(a)|R_1(\alpha,n)+|f'(b)|\right]R_2(\alpha,n)$$

where

$$\begin{aligned}R_1(\alpha,n)&=\int_0^1\left|(\alpha+1)B_{1-t}(n+1,\alpha-n)-(\alpha-n)B(n+1,\alpha-n)\right|td,\\ R_2(\alpha,n)&=\int_0^1\left|(\alpha+1)B_{1-t}(n+1,\alpha-n)-(\alpha-n)B(n+1,\alpha-n)\right|(1-t)\,dt,\end{aligned}$$

with $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Proof. Using Lemma 5 and the convexity of $|f'|$, we have

$$\left|\frac{(\alpha-n)f(a)+(n+1)f(b)}{\alpha+1}-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha\Gamma(\alpha-n)}\ {}^bI_\alpha f(a)\right|$$

$$\leq\frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\int_0^1\left|\begin{array}{c}(\alpha+1)B_{1-t}(n+1,\alpha-n)\\-(\alpha-n)B(n+1,\alpha-n)\end{array}\right||f'(ta+(1-t)b)|\,dt$$

$$\leq\frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\int_0^1\left|\begin{array}{c}(\alpha+1)B_{1-t}(n+1,\alpha-n)\\-(\alpha-n)B(n+1,\alpha-n)\end{array}\right|\left[t|f'(a)|+(1-t)|f'(b)|\right]dt$$

$$\leq\frac{b-a}{(\alpha+1)B(n+1,\alpha-n)}\left[\begin{array}{c}|f'(a)|\int_0^1|(\alpha+1)B_{1-t}(n+1,\alpha-n)-(\alpha-n)B(n+1,\alpha-n)|\,tdt\\+|f'(b)|\int_0^1|(\alpha+1)B_{1-t}(n+1,\alpha-n)-(\alpha-n)B(n+1,\alpha-n)|(1-t)\,dt\end{array}\right]$$

This completes the proof. □

Remark 4. *In Theorem 5,*

(1) *If one takes* $\alpha = n+1$*, one has the inequality proved in* [6, Theorem 4],
(2) *If one takes* $\alpha = n+1$*, after that if one takes* $\alpha = 1$*, one has the inequality proved in* [2, Theorem 2.2].

Theorem 6. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b)*. If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q \geq 1$*, then the following right conformable fractional integral inequality holds:*

$$\left|\frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ {}^{b}I_{\alpha}f(a)\right| \tag{4.2}$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} R_3^{1-\frac{1}{q}}(\alpha,n)\left(|f'(a)|^q R_1(\alpha,n) + |f'(b)|^q R_2(\alpha,n)\right)^{\frac{1}{q}}$$

where $R_1(\alpha,n)$ *and* $R_2(\alpha,n)$ *are same as in Theorem 5 and*

$$R_3(\alpha,n) = \int_0^1 |(\alpha+1) B_{1-t}(n+1,\alpha-n) - (\alpha-n) B(n+1,\alpha-n)|\, dt,$$

with $n = 0, 1, 2...$ *and* $\alpha \in (n, n+1]$.

Proof. Using Lemma 5, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ {}^{b}I_{\alpha}f(a)\right|$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| |f'(ta+(1-t)b)|\, dt$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| |f'(ta+(1-t)b)|^q\, dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}}$$
$$\left(\int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| \left[t|f'(a)|^q + (1-t)|f'(b)|^q\right] dt\right)^{\frac{1}{q}}$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left|\begin{array}{c}(\alpha+1) B_{1-t}(n+1,\alpha-n)\\ -(\alpha-n) B(n+1,\alpha-n)\end{array}\right| dt\right)^{1-\frac{1}{q}}$$
$$\left(\begin{array}{c}|f'(a)|^q \int_0^1 |(\alpha+1) B_{1-t}(n+1,\alpha-n) - (\alpha-n) B(n+1,\alpha-n)|\, t dt\\ + |f'(b)|^q \int_0^1 |(\alpha+1) B_{1-t}(n+1,\alpha-n) - (\alpha-n) B(n+1,\alpha-n)| (1-t)\, dt\end{array}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 5. *In Theorem 6,*

(1) *If one takes* $\alpha = n+1$*, one has the inequality proved in* [6, Theorem 5],
(2) *If one takes* $\alpha = n+1$*, after that if one takes* $\alpha = 1$*, one has the inequality proved in* [8, Theorem 1].

Theorem 7. *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b)*. If* $|f'|^q$ *is convex on* $[a,b]$ *for* $q > 1$*, then the following right conformable fractional integral inequality holds:*

$$\left|\frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ {}^{b}I_{\alpha}f(a)\right| \tag{4.3}$$
$$\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} R_4^{\frac{1}{p}}(\alpha,n)\left(\frac{|f'(a)|^q + |f'(b)|^q}{2}\right)^{\frac{1}{q}}$$

where

$$R_4(\alpha,n) = \int_0^1 |(\alpha+1) B_{1-t}(n+1,\alpha-n) - (\alpha-n) B(n+1,\alpha-n)|^p\, dt,$$

with $\frac{1}{p} + \frac{1}{q} = 1$, $n = 0, 1, 2...$ *and* $\alpha \in (n, n+1]$.

Proof. Using Lemma 5, Hölder inequality and the convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left| \frac{(\alpha-n) f(a) + (n+1) f(b)}{\alpha+1} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha}f(a) \ \right| \\
&\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \int_0^1 \left| \begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right| |f'(ta+(1-t)b)| \, dt \\
&\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left| \begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right|^p dt \right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 |f'(ta+(1-t)b)|^q \, dt \right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left| \begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right|^p dt \right)^{\frac{1}{p}} \\
&\quad \left(\int_0^1 \left[t|f'(a)|^q + (1-t)|f'(b)|^q \right] dt \right)^{\frac{1}{q}} \\
&\leq \frac{b-a}{(\alpha+1) B(n+1,\alpha-n)} \left(\int_0^1 \left| \begin{array}{c} (\alpha+1) B_{1-t}(n+1,\alpha-n) \\ -(\alpha-n) B(n+1,\alpha-n) \end{array} \right|^p dt \right)^{\frac{1}{p}} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.
\end{aligned}$$

This completes the proof. ☐

Remark 6. *In Theorem 7,*

(1) *If one takes $\alpha = n+1$, one has the inequality proved in* [6, Theorem 6],
(2) *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in* [2, Theorem 2.3].

Theorem 8. *Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b). If $|f'|$ is convex on $[a,b]$, then the following right conformable fractional integral inequality holds:*

$$\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha}f(a) \ - f\left(\frac{(\alpha-n)a + (n+1)b}{\alpha+1} \right) \right| \\
&\leq \frac{b-a}{B(n+1,\alpha-n)} \left[|f'(b)| R_5(\alpha,n) + |f'(a)| R_6(\alpha,n) \right]
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
R_5(\alpha,n) &= \left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, t dt \\ + \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| \, t dt \end{array} \right), \\
R_6(\alpha,n) &= \left(\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t) \, dt \\ + \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| (1-t) \, dt \end{array} \right),
\end{aligned}$$

with $n = 0, 1, 2...$ and $\alpha \in (n, n+1]$.

Proof. Using Lemma 6 and the convexity of $|f'|$, we have

$$\begin{aligned}
&\left| \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)} \ {}^{b}I_{\alpha}f(a) \ - f\left(\frac{(\alpha-n)a + (n+1)b}{\alpha+1} \right) \right| \\
&\leq \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| |f'(tb+(1-t)a)| \, dt \\ + \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| |f'(tb+(1-t)a)| \, dt \end{array} \right] \\
&\leq \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \left[t|f'(b)| + (1-t)|f'(a)| \right] dt \\ + \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| \left[t|f'(b)| + (1-t)|f'(a)| \right] dt \end{array} \right] \\
&\leq \frac{b-a}{B(n+1,\alpha-n)} \left[\begin{array}{c} |f'(b)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| \, t dt + |f'(a)| \int_0^{\frac{n+1}{\alpha+1}} |B_t(n+1,\alpha-n)| (1-t) \, dt \\ + |f'(b)| \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| \, t dt \\ + |f'(a)| \int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n) - B_t(n+1,\alpha-n)| (1-t) \, dt \end{array} \right]
\end{aligned}$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} |f'(b)|\left(\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,tdt \\ +\int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,tdt\end{array}\right) \\ +|f'(a)|\left(\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|(1-t)\,dt \\ +\int_{\frac{n+1}{\alpha+1}}^1 |B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|(1-t)\,dt\end{array}\right)\end{array}\right].$$

This completes the proof. □

Remark 7. *In Theorem 8,*

(1) *If one takes $\alpha = n+1$, one has the inequality proved in* [6, Theorem 7]*,*
(2) *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in* [5, Theorem 2.2]*.*

Theorem 9. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q\geq 1$, then the following right conformable fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ {}^bI_{\alpha}f(a)\ -f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\right| \tag{4.5}$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c} R_7^{1-\frac{1}{q}}(\alpha,n)\left(\begin{array}{c}|f'(b)|^q R_8(\alpha,n) \\ +|f'(a)|^q R_9(\alpha,n)\end{array}\right)^{\frac{1}{q}} \\ +R_{10}^{1-\frac{1}{q}}(\alpha,n)\left(\begin{array}{c}|f'(b)|^q R_{11}(\alpha,n) \\ +|f'(a)|^q R_{12}(\alpha,n)\end{array}\right)^{\frac{1}{q}}\end{array}\right]$$

where

$$\begin{aligned}
R_7(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt,\\
R_8(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,tdt,\\
R_9(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|(1-t)\,dt,\\
R_{10}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,dt,\\
R_{11}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,tdt,\\
R_{12}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|(1-t)\,dt,
\end{aligned}$$

with $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Proof. Using Lemma 6, power mean inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\Gamma(\alpha-n)}\ {}^bI_{\alpha}f(a)\ -f\left(\frac{(\alpha-n)a+(n+1)b}{\alpha+1}\right)\right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)a)|\,dt \\ +\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)a)|\,dt\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)a)|^q\,dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\left[t\,|f'(b)|^q+(1-t)\,|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\left[t\,|f'(b)|^q+(1-t)\,|f'(a)|^q\right]dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}|f'(b)|^q\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,tdt\\ +|f'(a)|^q\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,(1-t)\,dt\end{array}\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,dt\right)^{1-\frac{1}{q}}\\ \times\left(\begin{array}{c}|f'(b)|^q\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,tdt\\ +|f'(a)|^q\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,(1-t)\,dt\end{array}\right)^{\frac{1}{q}}\end{array}\right].$$

This completes the proof. □

Remark 8. *In Theorem 9,*

(1) *If one takes $\alpha=n+1$, one has the inequality proved in* [6, Theorem 8]*,*
(2) *If one takes $\alpha=n+1$, after that if one takes $\alpha=1$, one has the inequality proved in* [6, Remark 9].

Theorem 10. *Let $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b). If $|f'|^q$ is convex on $[a,b]$ for $q>1$, then the following right Riemann-Liouville fractional integral inequality holds:*

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\ {}^{b}I_{\alpha}f(a)\ -f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\right| \tag{4.6}$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}R_{13}^{\frac{1}{p}}(\alpha,n)\left(\frac{(n+1)^2}{2(\alpha+1)^2}\,|f'(b)|^q+\frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}\,|f'(a)|^q\right)^{\frac{1}{q}}\\ +R_{14}^{\frac{1}{p}}(\alpha,n)\left(\frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}\,|f'(b)|^q+\frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}\,|f'(a)|^q\right)^{\frac{1}{q}}\end{array}\right]$$

where

$$\begin{aligned}R_{13}(\alpha,n) &= \int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|^p\,dt,\\ R_{14}(\alpha,n) &= \int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|^p\,dt,\end{aligned}$$

with$\frac{1}{p}+\frac{1}{q}=1$, $n=0,1,2...$ and $\alpha\in(n,n+1]$.

Proof. Using Lemma 6, Hölder inequality and the convexity of $|f'|^q$, we have

$$\left|\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}\,\Gamma(\alpha-n)}\ {}^{b}I_{\alpha}f(a)\ -f\left(\frac{(\alpha-n)\,a+(n+1)\,b}{\alpha+1}\right)\right|$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)\,a)|\,dt\\ +\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|\,|f'(tb+(1-t)\,a)|\,dt\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}|B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\\ \times\left(\int_0^{\frac{n+1}{\alpha+1}}|f'(tb+(1-t)\,a)|^q\,dt\right)^{\frac{1}{q}}\\ +\left(\int_{\frac{n+1}{\alpha+1}}^1|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)|^p\,dt\right)^{\frac{1}{p}}\\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1|f'(tb+(1-t)\,a)|^q\,dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}\left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{n+1}{\alpha+1}}\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1\left|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{n+1}{\alpha+1}}^1\left[t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q\right] dt\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b-a}{B(n+1,\alpha-n)}\left[\begin{array}{c}\left(\int_0^{\frac{n+1}{\alpha+1}}\left|B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\frac{(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{2(\alpha+1)(n+1)-(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{n+1}{\alpha+1}}^1\left|B(n+1,\alpha-n)-B_t(n+1,\alpha-n)\right|^p dt\right)^{\frac{1}{p}} \\ \times\left(\frac{(\alpha+1)^2-(n+1)^2}{2(\alpha+1)^2}\left|f'(b)\right|^q+\frac{(\alpha+1)^2-2(\alpha+1)(n+1)+(n+1)^2}{2(\alpha+1)^2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\end{array}\right].$$

This completes the proof. □

Remark 9. *In Theorem 10,*

(1) *If one takes $\alpha = n+1$, one has the inequality proved in* [6, Theorem 9],
(2) *If one takes $\alpha = n+1$, after that if one takes $\alpha = 1$, one has the inequality proved in* [5, Theorem 2.3].

5. Competing Interests

The authors declare that they have no competing interests.

6. Authors' Contributions

The majority of this article has made by the first author. All authors read and approved final form of the article.

References

[1] T. Abdeljawad, On conformable fractional calculus, J. Comp. Appl. Math., 279 (2015), 57-66.
[2] S. S. Dragomir, R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula, Appl. Math. Lett., 11(5) (1998), 91-95.
[3] J. Hadamard, Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann, J. Math. Pures Appl., 58 (1893), 171-215.
[4] Ch. Hermite, Sur deux limites d'une intégrale définie, Mathesis, 3 (1883), 82–83.
[5] U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp. 147 (2004) 137-146.
[6] M. Kunt, D. Karapınar, S. Turhan, İ. İşcan, The right Rieaman-Liouville fractional Hermite-Hadamard type inequalities for convex functions, RGMIA Research Report Collection, 20 (2017), Article 102, 8 pp.
[7] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations. Elsevier, Amsterdam (2006).
[8] C. E. M. Pearce, J. Pecaric, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13 (2000), 51-55.
[9] A. W. Roberts, D. E. Varberg, Convex functions, Academic Press, New York (1973.)
[10] E. Set., A. O. Akdemir, İ. Mumcu, Hermite-Hadamard's inequality and its extensions for conformable fractional integrals of any order $\alpha > 0$, Available online from: https://www.researchgate.net/publication/303382221.
[11] Y. Zhou, Basic theory of fractional differential equations, World Scientific, New Jersey (2014).

Department of Mathematics, Faculty of Sciences and Arts, Giresun University, 28200, Giresun, Turkey
E-mail address: sercan.turhan@giresun.edu.tr; imdat.iscan@giresun.edu.tr

Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey
E-mail address: mkunt@ktu.edu.tr