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SOME OSTROWSKI TYPE INEQUALITIES FOR AN INTEGRAL OPERATOR AND n-TIME DIFFERENTIABLE FUNCTIONS

S. S. DRAGOMIR[1,2]

Abstract. In this paper we establish some Ostrowski type inequalities for the operator

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt + \frac{1}{b-x}\int_x^b f(t)\,dt\right],\ x \in (a,b)$$

in the case of functions $f:[a,b]\to\mathbb{C}$ whose n-derivatives $f^{(n)}$ are absolutely continuous on $[a,b]$. Several Hermite-Hadamard type inequalities are also provided.

1. Introduction

In 1938, A. Ostrowski [20], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1 (Ostrowski, 1938 [20])**.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)}|f'(t)| < \infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty (b-a), \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [12], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [19, p. 565],

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt = \frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt,\quad x\in[a,b],$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t) := \begin{cases} t-a & \text{if } t\in[a,x], \\ t-b & \text{if } t\in(x,b], \end{cases}$$

gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [12], Section 3).

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For other Ostrowski type inequalities for Lebesgue integral, see [11] and the recent survey [7].

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 2. *Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $z \in I$*

$$f(z) = T_n(f; c, z) + R_n(f; c, z), \tag{1.2}$$

where $T_n(f; c, z)$ is Taylor's polynomial, i.e.,

$$T_n(f; c, z) := \sum_{k=0}^{n} \frac{(z-c)^k}{k!} f^{(k)}(c). \tag{1.3}$$

Note that $f^{(0)} := f$ and $0! := 1$ and the remainder is given by

$$R_n(f; c, z) := \frac{1}{n!} \int_c^z (z-t)^n f^{(n+1)}(t)\, dt. \tag{1.4}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

Assume that the function $f : (a, b) \to \mathbb{C}$ is Lebesgue integrable on (a, b). We consider the following operator [8]

$$D_{a+,b-}f(x) := \frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\, dt + \frac{1}{b-x}\int_x^b f(t)\, dt\right],\ x \in (a, b). \tag{1.5}$$

We observe that if we take $x = \frac{a+b}{2}$, then we have

$$D_{a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{b-a}\int_a^b f(t)\, dt.$$

Moreover, if $f(a+) := \lim_{x\to a+} f(x)$ exists and is finite, then we have

$$\lim_{x\to a+} D_{a+,b-}f(x) = \frac{1}{2}\left[f(a+) + \frac{1}{b-a}\int_a^b f(t)\, dt\right]$$

and if $f(b-) := \lim_{x\to b-} f(x)$ exists and is finite, then we have

$$\lim_{x\to b-} D_{a+,b-}f(x) = \frac{1}{2}\left[f(b-) + \frac{1}{b-a}\int_a^b f(t)\, dt\right].$$

So, if $f : [a, b] \to \mathbb{C}$ is Lebesgue integrable on $[a, b]$ and continuous at right in a and at left in b, then we can extend the operator on the whole interval by putting

$$D_{a+,b-}f(a) := \frac{1}{2}\left[f(a) + \frac{1}{b-a}\int_a^b f(t)\, dt\right]$$

and

$$D_{a+,b-}f(b) := \frac{1}{2}\left[f(b) + \frac{1}{b-a}\int_a^b f(t)\, dt\right].$$

We say that the function $f : [a, b] \to \mathbb{C}$ is of *H-r-Hölder type* if

$$|f(t) - f(s)| \le H|t-s|^r$$

for any $t,\ s\in[a,b]$, where $H>0$ and $r\in(0,1]$. If $r=1$ and we put $H=L$, then we call the function of *L-Lipschitz type*.

In the recent paper [8] we obtained amongst other the following Ostrowski and midpoint type inequalities for $D_{a+,b-}f$:

Theorem 3. *If f is of H-r-Hölder type on $[a,b]$ with $H>0$ and $r\in(0,1]$, then for any $x\in(a,b)$ we have*

$$|D_{a+,b-}f(x)-f(x)|\le\frac{1}{2(r+1)}H\left[(x-a)^r+(b-x)^r\right]. \tag{1.6}$$

In particular, if f is of L-Lipschitz type, then

$$|D_{a+,b-}f(x)-f(x)|\le\frac{1}{4}L(b-a) \tag{1.7}$$

for any $x\in(a,b)$.

If we take in Theorem 3 $x=\frac{a+b}{2}$, then we get the following *midpoint type inequality*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{2^r(r+1)}H(b-a)^r. \tag{1.8}$$

In particular, if f is of *L-Lipschitz type,* then we get the result from [9]:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{4}L(b-a). \tag{1.9}$$

Motivated by the above results, by the use of Taylor's formula with integral remainder (1.2), in this paper we establish an Ostrowski type representation for the operator $D_{a+,b-}f(x)$, $x\in(a,b)$ in the case of functions $f:[a,b]\to\mathbb{C}$ whose n-derivatives $f^{(n)}$ are absolutely continuous on $[a,b]$. As applications, several midpoint type inequalities are also provided.

2. Some Ostrowski Type Identities

We have the following representation:

Theorem 4. *Let $I\subset\mathbb{R}$ be an interval, $[a,b]\subset I$ and $f:I\longrightarrow\mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on $[a,b]$. Then for any $x\in(a,b)$ we have the representation*

$$\begin{aligned}D_{a+,b-}f(x)&=\sum_{k=0}^{n}\frac{(b-x)^k+(-1)^k(x-a)^k}{2(k+1)!}f^{(k)}(x)\\&\quad+\frac{1}{2n!}(b-x)^{n+1}\\&\quad\times\int_0^1\int_0^1 u^{n+1}s^nf^{(n+1)}(sx+(1-s)[(1-u)x+ub])\,dsdu\\&\quad+\frac{(-1)^{n+1}}{2n!}(x-a)^{n+1}\\&\quad\times\int_0^1\int_0^1 u^{n+1}s^nf^{(n+1)}(sx+(1-s)[ua+(1-u)x])\,dsdu.\end{aligned} \tag{2.1}$$

In particular,

$$\frac{1}{b-a}\int_{a}^{b}f\left(t\right)dt=\sum_{k=0}^{n}\frac{1+\left(-1\right)^{k}}{2^{k+1}\left(k+1\right)!}f^{(k)}\left(\frac{a+b}{2}\right)\left(b-a\right)^{k} \tag{2.2}$$
$$+\frac{\left(b-a\right)^{n+1}}{2^{n+2}n!}$$
$$\times\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}f^{(n+1)}\left(s\frac{a+b}{2}+\left(1-s\right)\left[\left(1-u\right)\frac{a+b}{2}+ub\right]\right)dsdu$$
$$+\frac{\left(-1\right)^{n+1}\left(b-a\right)^{n+1}}{2^{n+2}n!}$$
$$\times\int_{0}^{1}\int_{0}^{1}u^{n+1}s^{n}f^{(n+1)}\left(s\frac{a+b}{2}+\left(1-s\right)\left[ua+\left(1-u\right)\frac{a+b}{2}\right]\right)dsdu.$$

Proof. Using Taylor's representation with the integral remainder (1.2) we can write the following identity for $x\in(a,b)$

$$f\left(y\right)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(x\right)\left(y-x\right)^{k}+\frac{1}{n!}\int_{x}^{y}f^{(n+1)}\left(t\right)\left(y-t\right)^{n}dt, \tag{2.3}$$

where $y\in[a,b]$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t=(1-s)c+sd$, $s\in[0,1]$ that

$$\int_{c}^{d}h\left(t\right)dt=\left(d-c\right)\int_{0}^{1}h\left(\left(1-s\right)c+sd\right)ds.$$

Therefore,

$$\begin{aligned}
&\int_{x}^{y}f^{(n+1)}\left(t\right)\left(y-t\right)^{n}dt\\
&=\left(y-x\right)\int_{0}^{1}f^{(n+1)}\left(\left(1-s\right)x+sy\right)\left(y-\left(1-s\right)x-sy\right)^{n}ds\\
&=\left(y-x\right)^{n+1}\int_{0}^{1}f^{(n+1)}\left(\left(1-s\right)x+sy\right)\left(1-s\right)^{n}ds\\
&=\left(y-x\right)^{n+1}\int_{0}^{1}f^{(n+1)}\left(sx+\left(1-s\right)y\right)s^{n}ds,
\end{aligned}$$

where for the last equality we replaced s by $1-s$.

We can then write the equality (2.3) as

$$\begin{aligned}
f\left(y\right)&=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}\left(x\right)\left(y-x\right)^{k}\\
&\quad+\frac{1}{n!}\left(y-x\right)^{n+1}\int_{0}^{1}f^{(n+1)}\left(sx+\left(1-s\right)y\right)s^{n}ds,
\end{aligned} \tag{2.4}$$

for any $y\in[a,b]$.

Now, for $x \in (a,b)$, if we integrate (2.4) on $[a,x]$ over y, then we get

$$\begin{aligned}\int_a^x f(y)\,dy &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \int_a^x (y-x)^k\,dy \\ &+ \frac{1}{n!}\int_a^x (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy, \\ &= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} f^{(k)}(x)(x-a)^{k+1} \\ &+ \frac{1}{n!}\int_a^x (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy,\end{aligned}$$

which gives

$$\begin{aligned}\frac{1}{x-a}\int_a^x f(y)\,dy &= \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} f^{(k)}(x)(x-a)^{k} \\ &+ \frac{1}{n!}\frac{1}{x-a}\int_a^x (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy.\end{aligned} \tag{2.5}$$

Also, if we integrate (2.4) on $[x,b]$ over y, then we get

$$\begin{aligned}\int_x^b f(y)\,dy &= \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x) \int_x^b (y-x)^k\,dy \\ &+ \frac{1}{n!}\int_x^b (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy \\ &= \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(x)(b-x)^{k+1} \\ &+ \frac{1}{n!}\int_x^b (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy,\end{aligned}$$

which gives

$$\begin{aligned}\frac{1}{b-x}\int_x^b f(y)\,dy &= \sum_{k=0}^{n} \frac{1}{(k+1)!} f^{(k)}(x)(b-x)^{k} \\ &+ \frac{1}{n!}\frac{1}{b-x}\int_x^b (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy.\end{aligned} \tag{2.6}$$

Now, if we make the change of variable $y=(1-u)a+ux$, $u \in [0,1]$, then

$$\begin{aligned}&\int_a^x (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)\,s^n ds\right)dy. \\ &= (-1)^{n+1}(x-a)^{n+2} \\ &\times \int_0^1 (1-u)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)[(1-u)a+ux])\,s^n ds\right)du \\ &= (-1)^{n+1}(x-a)^{n+2}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)[ua+(1-u)x])\,s^n ds\right)du\end{aligned}$$

and by (2.5) we get

$$\frac{1}{x-a}\int_a^x f(y)\,dy=\sum_{k=0}^n \frac{(-1)^k}{(k+1)!}f^{(k)}(x)(x-a)^k \tag{2.7}$$
$$+\frac{1}{n!}(-1)^{n+1}(x-a)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[ua+(1-u)x])\,dsdu,$$

for $x\in(a,b)$.

Also, if we make the change of variable $y=(1-u)x+ub$, $u\in[0,1]$ then

$$\int_x^b (y-x)^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)y)s^n ds\right)dy$$
$$=(b-x)^{n+2}\int_0^1 u^{n+1}\left(\int_0^1 f^{(n+1)}(sx+(1-s)[(1-u)x+ub])s^n ds\right)du$$

and by (2.6) we get

$$\frac{1}{b-x}\int_x^b f(y)\,dy=\sum_{k=0}^n \frac{1}{(k+1)!}f^{(k)}(x)(b-x)^k \tag{2.8}$$
$$+\frac{1}{n!}(b-x)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[(1-u)x+ub])\,dsdu$$

for $x\in(a,b)$.

Therefore, by (2.7) and (2.8) we get

$$D_{a+,b-}f(x)=\frac{1}{2}\left[\frac{1}{x-a}\int_a^x f(t)\,dt+\frac{1}{b-x}\int_x^b f(t)\,dt\right]$$
$$=\sum_{k=0}^n \frac{1}{(k+1)!}f^{(k)}(x)\frac{(b-x)^k+(-1)^k(x-a)^k}{2}$$
$$+\frac{1}{2n!}(b-x)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[(1-u)x+ub])\,dsdu$$
$$+\frac{1}{2n!}(-1)^{n+1}(x-a)^{n+1}\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[ua+(1-u)x])\,dsdu,$$

which proves the desired result (2.1). □

Remark 1. *For $n=0$ we get*

$$D_{a+,b-}f(x) \tag{2.9}$$
$$=f(x)+\frac{1}{2}(b-x)\int_0^1\int_0^1 uf'(sx+(1-s)[(1-u)x+ub])\,dsdu$$
$$-\frac{1}{2}(x-a)\int_0^1\int_0^1 uf'(sx+(1-s)[ua+(1-u)x])\,dsdu,$$

where $x\in(a,b)$.

In particular,

$$\begin{aligned}(2.10)\quad & \frac{1}{b-a}\int_a^b f(t)\,dt \\ &= f\left(\frac{a+b}{2}\right)+\frac{b-a}{4}\int_0^1\int_0^1 u\left[f'\left(s\frac{a+b}{2}+(1-s)\left[(1-u)\frac{a+b}{2}+ub\right]\right)\right.\\ &\qquad \left.-f'\left(s\frac{a+b}{2}+(1-s)\left[ua+(1-u)\frac{a+b}{2}\right]\right)\right]dsdu.\end{aligned}$$

For $n=1$ we get

$$\begin{aligned}(2.11)\quad D_{a+,b-}f(x) &= f(x)+\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\\ &+\frac{1}{2}(b-x)^2\int_0^1\int_0^1 u^2 s f''(sx+(1-s)[(1-u)x+ub])\,dsdu\\ &+\frac{1}{2}(x-a)^2\int_0^1\int_0^1 u^2 s f''(sx+(1-s)[ua+(1-u)x])\,dsdu\end{aligned}$$

where $x\in(a,b)$.

In particular,

$$\begin{aligned}(2.12)\quad & \frac{1}{b-a}\int_a^b f(t)\,dt = f\left(\frac{a+b}{2}\right)\\ &+\frac{1}{8}(b-a)^2\int_0^1\int_0^1 u^2 s\left[f''\left(s\frac{a+b}{2}+(1-s)\left[(1-u)\frac{a+b}{2}+ub\right]\right)\right.\\ &\qquad \left.+f''\left(s\frac{a+b}{2}+(1-s)\left[ua+(1-u)\frac{a+b}{2}\right]\right)\right]dsdu.\end{aligned}$$

In [8] the first author obtained the following equality:

Lemma 1. *Assume that the function $f:(a,b)\to\mathbb{C}$ is Lebesgue integrable on (a,b) and $f(a+)$, $f(b-)$ exists and are finite. Then we have*

$$(2.13)\qquad \int_a^b D_{a+,b-}f(x)\,dx=\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right)f(x)\,dx.$$

By taking the integral mean in the equality (2.1) we can state the following corollary as well:

Corollary 1. *With the assumptions of Theorem 4 we have*

$$\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \tag{2.14}$$
$$= \sum_{k=0}^{n}\frac{1}{2(k+1)!}\frac{1}{b-a}\int_a^b \left[(b-x)^k+(-1)^k(x-a)^k\right] f^{(k)}(x)\,dx$$
$$+\frac{1}{2n!}\frac{1}{b-a}\int_a^b (b-x)^{n+1}$$
$$\times\left(\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[(1-u)x+ub])\,dsdu\right)dx$$
$$+\frac{(-1)^{n+1}}{2n!}\frac{1}{b-a}\int_a^b (x-a)^{n+1}$$
$$\times\left(\int_0^1\int_0^1 u^{n+1}s^n f^{(n+1)}(sx+(1-s)[ua+(1-u)x])\,dsdu\right)dx.$$

Remark 2. *By taking the integral mean in (2.9) we get*

$$\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx = \frac{1}{b-a}\int_a^b f(x)\,dx \tag{2.15}$$
$$+\frac{1}{2}\frac{1}{b-a}\int_a^b\left((b-x)\int_0^1\int_0^1 uf'(sx+(1-s)[(1-u)x+ub])\,dsdu\right)dx$$
$$-\frac{1}{2}\frac{1}{b-a}\int_a^b (x-a)\left(\int_0^1\int_0^1 uf'(sx+(1-s)[ua+(1-u)x])\,dsdu\right)dx,$$

where $f:[a,b]\to\mathbb{C}$ is absolutely continuous.

Also, by taking the integral mean in (2.11) we get

$$\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \tag{2.16}$$
$$=\frac{1}{2}\left(\frac{3}{b-a}\int_a^b f(x)-\frac{f(a)+f(b)}{2}\right)$$
$$+\frac{1}{2}\frac{1}{b-a}\int_a^b (b-x)^2\left(\int_0^1\int_0^1 u^2 sf''(sx+(1-s)[(1-u)x+ub])\,dsdu\right)dx$$
$$+\frac{1}{2}\frac{1}{b-a}\int_a^b (x-a)^2\left(\int_0^1\int_0^1 u^2 sf''(sx+(1-s)[ua+(1-u)x])\,dsdu\right)dx.$$

3. Some Ostrowski and HH-Type Inequalities

The following integral inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(t)\,dt\le\frac{f(a)+f(b)}{2}, \tag{3.1}$$

which holds for any convex function $f:[a,b]\to\mathbb{R}$, is well known in the literature as the *Hermite-Hadamard (HH) inequality.*

There is an extensive amount of literature devoted to this simple and nice result which has many applications in the Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [11], the recent survey paper [7], the research papers [1]-[2], [14]-[27] and the references therein.

The following result provides an inequality related to the first Hermite-Hadamard inequality in (3.1).

Theorem 5. *Let $I \subset \mathbb{R}$ be an interval, $[a,b] \subset I$ and $f : I \longrightarrow \mathbb{C}$ is such that the $2m+2$-derivative $f^{(2m+2)}$ is nonnegative on $[a,b]$, where $m \geq 0$, then for any $x \in (a,b)$ we have the trapezoid type inequality*

$$D_{a+,b-}f(x) \geq \sum_{k=0}^{2m+1} \frac{(b-x)^k + (-1)^k (x-a)^k}{2(k+1)!} f^{(k)}(x). \tag{3.2}$$

In particular, we have

$$\frac{1}{b-a} \int_a^b f(t)\,dt \geq \sum_{k=0}^{2m+1} \frac{1+(-1)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)(b-a)^k \tag{3.3}$$

that was obtained in [22].

Proof. By the representation (2.1) we have

$$\begin{aligned} D_{a+,b-}f(x) &= \sum_{k=0}^{2m+1} \frac{(b-x)^k + (-1)^k (x-a)^k}{2(k+1)!} f^{(k)}(x) \\ &\quad + \frac{1}{2(2m+1)!}(b-x)^{2m+2} \\ &\quad \times \int_0^1 \int_0^1 u^{2m+2} s^{2m+1} f^{(2m+2)}(sx + (1-s)[(1-u)x + ub])\,dsdu \\ &\quad + \frac{1}{2n!}(x-a)^{2m+2} \\ &\quad \times \int_0^1 \int_0^1 u^{2m+2} s^{2m+1} f^{(n+1)}(sx + (1-s)[ua + (1-u)x])\,dsdu \\ &\geq \sum_{k=0}^{2m+1} \frac{(b-x)^k + (-1)^k (x-a)^k}{2(k+1)!} f^{(k)}(x) \end{aligned}$$

since the last two integrals are nonnegative due to the fact that $f^{(2m+2)}$ is nonnegative on $[a,b]$. $\square$

Remark 3. *For $m=0$ we obtain from Theorem 5 that*

$$D_{a+,b-}f(x) \geq f(x) + \frac{1}{2}\left(\frac{a+b}{2} - x\right) f'(x) \tag{3.4}$$

for any $x \in (a,b)$, where f is differentiable and convex on $[a,b]$, which for $x = \frac{a+b}{2}$ reduces to the first Hermite-Hadamard inequality in (3.1).

We use the ∞-norm of an essentially bounded function f on the interval $[c,d]$ defined by

$$\|f\|_{[c,d],\infty} := \operatorname*{essup}_{t\in[c,d]} |f(t)| < \infty,\ f \in L_\infty[c,d].$$

Theorem 6. *Let $I \subset \mathbb{R}$ be an interval, $[a,b] \subset I$ and $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on $[a,b]$ and $f^{(n+1)} \in L_{\infty}[a,b]$. Then for any $x \in (a,b)$ we have the inequality*

$$\left| D_{a+,b-}f(x) - \sum_{k=0}^{n} \frac{(b-x)^k + (-1)^k (x-a)^k}{2(k+1)!} f^{(k)}(x) \right| \tag{3.5}$$
$$\leq \frac{1}{2(n+2)!} \left[(x-a)^{n+1} \left\| f^{(n+1)} \right\|_{[a,x],\infty} + (b-x)^{n+1} \left\| f^{(n+1)} \right\|_{[x,b],\infty} \right]$$
$$\leq \frac{1}{2(n+2)!} \left[(x-a)^{n+1} + (b-x)^{n+1} \right] \left\| f^{(n+1)} \right\|_{[a,b],\infty}.$$

In particular,

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - \sum_{k=0}^{2m+1} \frac{1+(-1)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)(b-a)^k \right| \tag{3.6}$$
$$\leq \frac{1}{2^{n+2}(n+2)!} \left[\left\| f^{(n+1)} \right\|_{\left[a,\frac{a+b}{2}\right],\infty} + \left\| f^{(n+1)} \right\|_{\left[\frac{a+b}{2},b\right],\infty} \right] (b-a)^{n+1}$$
$$\leq \frac{1}{2^{n+1}(n+2)!} \left\| f^{(n+1)} \right\|_{[a,b],\infty} (b-a)^{n+1}.$$

Proof. By taking the modulus in the equality (2.1) we get

$$\left| D_{a+,b-}f(x) - \sum_{k=0}^{n} \frac{(b-x)^k + (-1)^k (x-a)^k}{2(k+1)!} f^{(k)}(x) \right| \tag{3.7}$$
$$+ \frac{1}{2n!}(x-a)^{n+1} \int_0^1 \int_0^1 u^{n+1}s^n \left| f^{(n+1)}(sx + (1-s)[ua + (1-u)x]) \right| dsdu$$
$$\frac{1}{2n!}(b-x)^{n+1} \int_0^1 \int_0^1 u^{n+1}s^n \left| f^{(n+1)}(sx + (1-s)[(1-u)x + ub]) \right| dsdu$$
$$=: C(x,n).$$

Observe that $sx + (1-s)[ua + (1-u)x] \in [a,x]$ for any u, $s \in [0,1]$. Therefore

$$\sup_{(s,u)\in[0,1]^2} \left| f^{(n+1)}(sx + (1-s)[ua + (1-u)x]) \right| \leq \left\| f^{(n+1)} \right\|_{[a,x],\infty}$$

and

$$\int_0^1 \int_0^1 u^{n+1}s^n \left| f^{(n+1)}(sx + (1-s)[ua + (1-u)x]) \right| dsdu$$
$$\leq \left\| f^{(n+1)} \right\|_{[a,x],\infty} \int_0^1 \int_0^1 u^{n+1}s^n dsdu$$
$$= \frac{1}{(n+1)(n+2)} \left\| f^{(n+1)} \right\|_{[a,x],\infty}.$$

Similarly, we have

$$\int_0^1 \int_0^1 u^{n+1}s^n \left| f^{(n+1)}(sx + (1-s)[(1-u)x + ub]) \right| dsdu$$
$$\leq \frac{1}{(n+1)(n+2)} \left\| f^{(n+1)} \right\|_{[x,b],\infty}.$$

Therefore

$$\begin{aligned}
&C(x,n)\\
&\leq \frac{1}{2n!}(x-a)^{n+1}\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[a,x],\infty}\\
&+\frac{1}{2n!}(b-x)^{n+1}\frac{1}{(n+1)(n+2)}\left\|f^{(n+1)}\right\|_{[x,b],\infty}\\
&=\frac{1}{2(n+2)!}\left[(x-a)^{n+1}\left\|f^{(n+1)}\right\|_{[a,x],\infty}+(b-x)^{n+1}\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right]\\
&\leq \frac{1}{2(n+2)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\max\left\{\left\|f^{(n+1)}\right\|_{[a,x],\infty},\left\|f^{(n+1)}\right\|_{[x,b],\infty}\right\}\\
&=\frac{1}{2(n+2)!}\left[(x-a)^{n+1}+(b-x)^{n+1}\right]\left\|f^{(n+1)}\right\|_{[a,b],\infty}
\end{aligned}$$

and the inequality (3.5) is thus proved. □

We note that the inequality between the first and last term in (3.6) was obtained in a different way in [3].

Remark 4. *If we take in (3.5) $n=0$, then we get*

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-f(x)\right|\\
&\leq \frac{1}{4}\left[(x-a)\left\|f'\right\|_{[a,x],\infty}+(b-x)\left\|f'\right\|_{[x,b],\infty}\right]\leq \frac{1}{4}(b-a)\left\|f'\right\|_{[a,b],\infty}
\end{aligned}\tag{3.8}$$

for any $x\in(a,b)$, and in particular [6]

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{1}{8}\left[\left\|f'\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\left\|f'\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a)\leq \frac{1}{4}(b-a)\left\|f'\right\|_{[a,b],\infty}.
\end{aligned}\tag{3.9}$$

If we take in (3.5) $n=1$, then we get

$$\begin{aligned}
&\left|D_{a+,b-}f(x)-f(x)-\frac{1}{2}\left(\frac{a+b}{2}-x\right)f'(x)\right|\\
&\leq \frac{1}{12}\left[(x-a)^2\left\|f''\right\|_{[a,x],\infty}+(b-x)^2\left\|f''\right\|_{[x,b],\infty}\right]\\
&\leq \frac{1}{6}\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\left\|f''\right\|_{[a,b],\infty}
\end{aligned}\tag{3.10}$$

for any $x\in(a,b)$, and in particular

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{1}{48}\left[\left\|f''\right\|_{\left[a,\frac{a+b}{2}\right],\infty}+\left\|f''\right\|_{\left[\frac{a+b}{2},b\right],\infty}\right](b-a)^2\leq \frac{1}{24}(b-a)^2\left\|f''\right\|_{[a,b],\infty}.
\end{aligned}\tag{3.11}$$

If we take the integral mean in (3.8), we get

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \tag{3.12}$$
$$\le \frac{1}{4}\left[\frac{1}{b-a}\int_a^b (x-a)\,\|f'\|_{[a,x],\infty}\,dx + \frac{1}{b-a}\int_a^b (b-x)\,\|f'\|_{[x,b],\infty}\,dx\right]$$
$$\le \frac{1}{4}(b-a)\,\|f'\|_{[a,b],\infty}\,.$$

Also, if we take the integral mean in (3.10) we get

$$\left|\frac{1}{b-a}\int_a^b \ln\left(\frac{b-a}{\sqrt{(x-a)(b-x)}}\right) f(x)\,dx \right. \tag{3.13}$$
$$\left. -\frac{1}{2}\left(\frac{3}{b-a}\int_a^b f(x) - \frac{f(a)+f(b)}{2}\right)\right|$$
$$\le \frac{1}{12}\left[\frac{1}{b-a}\int_a^b (x-a)^2\,\|f''\|_{[a,x],\infty}\,dx + \frac{1}{b-a}\int_a^b (b-x)^2\,\|f''\|_{[x,b],\infty}\,dx\right]$$
$$\le \frac{1}{18}\,\|f''\|_{[a,b],\infty}\,(b-a)^2\,.$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa