

HERMITE-HADAMARD TYPE INEQUALITIES FOR A CLASS OF HARMONICALLY CONVEX FUNCTIONS

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ABSTRACT. In this paper, we introduce a new class of harmonically convex functions to establish some interesting Hermite-Hadamard type inequalities. Results obtained are extensions and generalizations of known results in literature. Applications to special means of real numbers are also given.

1. INTRODUCTION AND PRELIMINARIES

The role played by inequalities in mathematics cannot be undermined. Infact, most mathematical inequalities are basic tools for constructing analytic proofs of many important theorems. Over the years, the study of convex inequalities has steadily gained the attention of many researchers. Also, many classes of convex functions have been introduced to extend several known inequalities in literature (see [1], [11], [4] and the references therein). An important extension of convex function, the class of h -convex functions, was introduced by Varosanec in [12]. This was further generalized in [8] when the ϕ_{h-s} convex function was introduced by the authors. In [1], the class of harmonically convex functions was introduced and was significantly extended in [1] by the class of harmonically h -convex functions. In this paper, we further extend the class of harmonically h convex functions and establish some Hermite-Hadamard type inequalities.

Theorem 1.1 (Hermite-Hadamard inequality, [2]). *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}.$$

Definition 1.2. [4] *A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically convex if for every $x, y \in I$, $t \in [0, 1]$,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)f(x) + tf(y).$$

Definition 1.3. [1] *A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically Breckner s -convex where $0 < s \leq 1$, if for every $x, y \in I$, $t \in [0, 1]$,*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)^s f(x) + t^s f(y).$$

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Definition 1.4. [1] A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically s -Godunova-Levin of the second kind if for every $x, y \in I$, $t \in (0, 1)$ and $s \in [0, 1]$,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{(1-t)^s}f(x) + \frac{1}{t^s}f(y).$$

Definition 1.5. [1] A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically Godunova-Levin of the second kind if for every $x, y \in I$, $t \in (0, 1)$,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{(1-t)}f(x) + \frac{1}{t}f(y).$$

Definition 1.6. [1] A function $f : I \rightarrow \mathbb{R}$ is said to be harmonically P -function if for every $x, y \in I$, $t \in [0, 1]$,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x) + f(y).$$

Definition 1.7. [1] Let $h : [0, 1] \subseteq J \rightarrow \mathbb{R}$ be a non-negative function. A function $f : I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}$ is said to be harmonically h -convex if for all $x, y \in I$ and $t \in (0, 1)$,

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y).$$

2. THE CLASS OF HARMONICALLY ϕ_{h-s} CONVEX FUNCTIONS

We introduce the following definition in order to unify the classes of harmonically convex functions given in the previous section.

Definition 2.1. Let $h : [0, 1] \subseteq J \rightarrow (0, \infty)$, $s \in [0, 1]$, $t \in (0, 1)$ and ϕ be a given real valued function. Let $I \subseteq \mathbb{R} \setminus \{0\}$, then $f : I \rightarrow \mathbb{R}$ is an harmonically ϕ_{h-s} convex if for all $x, y \in I$,

$$(1) \quad f\left(\frac{\phi(x)\phi(y)}{t\phi(x) + (1-t)\phi(y)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(x)).$$

Remark. We discuss the special cases of the harmonically ϕ_{h-s} convex function. Denote by $\mathcal{HSX}(\phi_{h-s}, I)$, $\mathcal{HSX}(h, I)$, $\mathcal{HSX}(I)$, $\mathcal{HQ}_s(I)$, $\mathcal{HQ}(I)$ and $\mathcal{HP}(I)$ the class of all harmonically ϕ_{h-s} convex functions, harmonically h -convex functions, harmonically convex functions, harmonically s -Godunova-Levin functions, harmonically Godunova-Levin functions and harmonically P -functions.

(i). Then for $\phi(x) = x$, $h(t) \leq t$, we have

$$\begin{aligned} \mathcal{HP}(I) = \mathcal{HQ}_0(I) &= \mathcal{HSX}(\phi_{h-0}, I) \subseteq \mathcal{HSX}(\phi_{h-s_1}, I) \\ &\subseteq \mathcal{HSX}(\phi_{h-s_2}, I) \subseteq \mathcal{HSX}(\phi_{h-1}, I). \end{aligned}$$

(ii). Let ϕ be the identity function. Observe that

- (i). if $s = 0$, then $f \in \mathcal{HP}(I)$
- (ii). if $h(t) = 1$ and $s = 1$, then $f \in \mathcal{HSX}(I)$
- (iii). if $h(t) = t^{\frac{s}{s+1}}$, then $f \in \mathcal{HSX}(h, I)$
- (iv). if $h(t) = 1$, then f is harmonically Breckner s -convex
- (v). if $h(t) = t^2$, then $f \in \mathcal{HQ}_s(I)$
- (vi). if $h(t) = t^2$ and $s = 1$, then $f \in \mathcal{HQ}(I)$.

(iii). For $t = \frac{1}{2}$, we obtain the Jensen's type harmonically ϕ_{h-s} convex function or the harmonically-arithmetically (HA) ϕ_{h-s} convex function

$$(2) \quad f\left(\frac{2\phi(x)\phi(y)}{\phi(x)+\phi(y)}\right) \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s} (f(\phi(x)) + f(\phi(y))).$$

Example 2.2. (i). For $h(t) \leq t^{1-\frac{1}{s}}$ and $\phi(x) = x$, all known examples of harmonically convex functions are ϕ_{h-s} harmonically convex.

(ii). Let $I = [a, b] \subseteq \mathbb{R} \setminus \{0\}$. Consider the function $g : [\frac{1}{b}, \frac{1}{a}] \rightarrow \mathbb{R}$ defined by $g(x) = f(\frac{1}{x})$, then $f \in \mathcal{HSX}(\phi_{h-s}, I)$ if and only if $g \in \mathcal{SX}(\phi_{h-s}, J)$ where $\phi(x) = x$, $I = [a, b]$ and $J = [\frac{1}{b}, \frac{1}{a}]$.

(iii). Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval and $f : I \rightarrow \mathbb{R}$ be a function,

- (a) if $I \subset (0, \infty)$ and $f \in \mathcal{SX}(\phi_{h-s}, I)$ where f is nondecreasing on I then $f \in \mathcal{HSX}(\phi_{h-s}, I)$.
- (b) if $I \subset (0, \infty)$, $f \in \mathcal{HSX}(\phi_{h-s}, I)$ where f is nonincreasing on I , then $f \in \mathcal{SX}(I)$.
- (c) if $I \subset (-\infty, 0)$, $f \in \mathcal{HSX}(\phi_{h-s}, I)$ where f is nondecreasing on I then $f \in \mathcal{SX}(I)$.
- (d) if $I \subset (-\infty, 0)$, $f \in \mathcal{SX}(I)$ where f is nonincreasing on I , then $f \in \mathcal{HSX}(\phi_{h-s}, I)$.

Definition 2.3. The functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are said to be similarly ordered if for every $x, y \in \mathbb{R}$,

$$(f(x) - f(y))(g(x) - g(y)) \geq 0.$$

Proposition 2.4. Let $f, g \in \mathcal{HSX}(\phi_{h-s}, I)$. If f and g are similarly ordered and $H(t, s) = \left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s} \leq 1$, then the product $fg \in \mathcal{HSX}(\phi_{h-s}, I)$.

Proof. Since $f, g \in \mathcal{HSX}(\phi_{h-s}, I)$, then

$$\begin{aligned} & f\left(\frac{\phi(x)\phi(y)}{t\phi(x)+(1-t)\phi(y)}\right)g\left(\frac{\phi(x)\phi(y)}{t\phi(x)+(1-t)\phi(y)}\right) \\ & \leq \left(\frac{h(t)}{t}\right)^{-2s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-2s} f(\phi(x))g(\phi(x)) \\ & \quad + \left(\frac{h(t)}{t}\frac{h(1-t)}{1-t}\right)^{-s} \left(f(\phi(y))g(\phi(x)) + f(\phi(x))g(\phi(y))\right) \\ & \leq \left(\frac{h(t)}{t}\right)^{-2s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-2s} f(\phi(x))g(\phi(x)) \\ & \quad + \left(\frac{h(t)}{t}\frac{h(1-t)}{1-t}\right)^{-s} \left(f(\phi(x))g(\phi(x)) + f(\phi(y))g(\phi(y))\right) \\ & = H(t, s) \left(f(\phi(y))g(\phi(y))\left(\frac{h(t)}{t}\right)^{-s} + f(\phi(x))g(\phi(x))\left(\frac{h(1-t)}{1-t}\right)^{-s}\right) \\ & \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(y))g(\phi(y)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(x))g(\phi(x)). \end{aligned}$$

□

3. HERMITE-HADAMARD TYPE INEQUALITIES FOR $\mathcal{HSX}(\phi_{h-s}, I)$

Theorem 3.1. *Let $f \in \mathcal{HSX}(\phi_{h-s}, I)$. Suppose that $f \in L[a, b]$ where $a, b \in I$ with $a < b$ and ϕ is the identity function, then*

$$(3) \quad \frac{1}{2^{1-s} \left(h\left(\frac{1}{2}\right)\right)^{-s}} f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq (f(a) + f(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt.$$

Proof. Since $f \in \mathcal{HSX}(\phi_{h-s}, I)$, then by setting $t = \frac{1}{2}$ we obtain (2). Set $x = \frac{ab}{ta+(1-t)b}$ and $y = \frac{ab}{(1-t)a+tb}$, then

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \left(2h\left(\frac{1}{2}\right)\right)^{-s} \left(f\left(\frac{ab}{ta+(1-t)b}\right) + f\left(\frac{ab}{(1-t)a+tb}\right)\right) \\ &= \left(2h\left(\frac{1}{2}\right)\right)^{-s} \left(\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) dt + \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right) dt\right) \\ &\leq \left(2h\left(\frac{1}{2}\right)\right)^{-s} \int_0^1 \left(\left(\frac{h(t)}{t}\right)^{-s} f(b) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(a)\right. \\ &\quad \left.+ \left(\frac{h(t)}{t}\right)^{-s} f(a) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(b)\right) dt \\ &= 2 \left(2h\left(\frac{1}{2}\right)\right)^{-s} (f(a) + f(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt. \end{aligned}$$

But,

$$\int_0^1 f\left(\frac{ab}{ta+(1-t)b}\right) dt = \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right) dt = \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx.$$

Thus, we have that

$$f\left(\frac{2ab}{a+b}\right) \leq 2 \left(2h\left(\frac{1}{2}\right)\right)^{-s} \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx$$

and

$$2 \left(2h\left(\frac{1}{2}\right)\right)^{-s} \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq 2 \left(2h\left(\frac{1}{2}\right)\right)^{-s} (f(a) + f(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt.$$

This gives (3), hence the proof. \square

Theorem 3.2. *Let $f, g \in \mathcal{HSX}(\phi_{h-s}, I)$ be two non-negative functions where $a, b \in I$ with $a < b$. If $fg \in L[a, b]$, then*

$$\begin{aligned} \frac{\phi(a)\phi(b)}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} \frac{f(x)g(x)}{x^2} dx &\leq M(\phi(a), \phi(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-2s} dt \\ &\quad + N(\phi(a), \phi(b)) \int_0^1 \left(\frac{h(t)h(1-t)}{t(1-t)}\right)^{-s} dt \end{aligned}$$

where

$$\begin{aligned} M(\phi(a), \phi(b)) &= f(\phi(a))g(\phi(a)) + f(\phi(b))g(\phi(b)) \\ N(\phi(a), \phi(b)) &= f(\phi(b))g(\phi(a)) + f(\phi(a))g(\phi(b)). \end{aligned}$$

Proof. Since $f, g \in \mathcal{HSX}(\phi_{h-s}, I)$, then

$$(4) \quad f\left(\frac{\phi(a)\phi(b)}{t\phi(a) + (1-t)\phi(b)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} f(\phi(b)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} f(\phi(a))$$

$$(5) \quad g\left(\frac{\phi(a)\phi(b)}{(1-t)\phi(b) + t\phi(a)}\right) \leq \left(\frac{h(t)}{t}\right)^{-s} g(\phi(b)) + \left(\frac{h(1-t)}{1-t}\right)^{-s} g(\phi(a))$$

Multiplying (4) by (5) and integrating with respect to t over $(0, 1)$, we obtain

$$\begin{aligned} & \int_0^1 f\left(\frac{\phi(a)\phi(b)}{t\phi(a) + (1-t)\phi(b)}\right) g\left(\frac{\phi(a)\phi(b)}{t\phi(a) + (1-t)\phi(b)}\right) dt \\ & \leq (f(\phi(b))g(\phi(b)) + f(\phi(a))g(\phi(a))) \int_0^1 \left(\frac{h(t)}{t}\right)^{-2s} dt \\ & \quad + (f(\phi(b))g(\phi(a)) + f(\phi(a))g(\phi(b))) \int_0^1 \left(\frac{h(t)h(1-t)}{t(1-t)}\right)^{-s} dt \end{aligned}$$

By using the substitution $x = \frac{\phi(a)\phi(b)}{t\phi(a) + (1-t)\phi(b)}$, we have

$$\begin{aligned} \frac{\phi(a)\phi(b)}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} \frac{f(x)g(x)}{x^2} dx & \leq M(\phi(a), \phi(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-2s} \\ & \quad + N(\phi(a), \phi(b)) \int_0^1 \left(\frac{h(t)h(1-t)}{t(1-t)}\right)^{-s} dt. \end{aligned}$$

□

Corollary 3.3. *Under the conditions of Theorem 3.2, suppose that f and g are similarly ordered and $\left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s} \leq 1$, then*

$$\frac{\phi(a)\phi(b)}{\phi(b) - \phi(a)} \int_{\phi(a)}^{\phi(b)} \frac{f(x)g(x)}{x^2} dx \leq 2M(\phi(a), \phi(b)) \int_0^1 \left(\frac{h(t)}{t}\right)^{-s} dt.$$

Remark. *Theorem 3.1 reduces to Theorem 2.4 in [4] when $h(t) = s = 1$. By setting $\phi(x) = x$, $h(t) = t^{\frac{1}{2}}$, $s = 1$ in Theorem 3.2, we obtain Theorem 3.6 in [1]. By setting $\phi(x) = x$, $h(t) = t^{\frac{s}{s+1}}$ in Theorem 3.1, we obtain Theorem 3.2 in [1] and by applying Remark 2(ii) accordingly, we obtain Corollaries 3.3 – 3.5 in [1]. Corollary 3.3 reduces to Theorem 3.7 in [1] when $\phi(x) = x$.*

4. INEQUALITIES FOR $\mathcal{HSX}(\phi_{h-s}, I)$ VIA FRACTIONAL INTEGRATION

Let $f \in L[a, b]$, the Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and J_{b-}^{α} of order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a \text{ and}$$

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$. Fractional integral reduces to the classical integral for $\alpha = 1$.

Hermite-Hadamard type inequalities have been proved for fractional integrals which

naturally extends the classical integrals (see for example, [3], [5], [9], [10], [13], [14]). Infact, M. Z. Sarikaya et. al.[9] proved the following Hermite-Hadamard inequalities for convex functions via fractional integrals.

Theorem 4.1. *Let $f : I \rightarrow \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L[a, b]$. If f is a convex function on $[a, b]$, then the following inequality for fractional integrals holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} (J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)) \leq \frac{f(a) + f(b)}{2}.$$

In this section, we establish Hermite-Hadamard type inequalities for harmonically ϕ_{h-s} via fractional integrals.

Theorem 4.2. *Let $f : I \subseteq \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$ where $a, b \in I$ with $a < b$. If $f, (f \circ \phi) \in \mathcal{HSX}(\phi_{h-s}, I)$, then the following holds.*

$$\begin{aligned} & \left(2h\left(\frac{1}{2}\right)\right)^s f\left(\frac{2ab}{a+b}\right) \\ & \leq \left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha+1) \left(J_{1/a-}^\alpha (f \circ \phi)(1/b) + J_{1/b+}^\alpha (f \circ \phi)(1/a)\right) \\ & \leq 2((f \circ \phi)^2(x) + (f \circ \phi^2)(y)) \int_0^1 t^{\alpha-1} \left(\left(\frac{h(t)}{t}\right)^{-s} + \left(\frac{h(1-t)}{1-t}\right)^{-s}\right) dt \end{aligned}$$

Proof. Set $x = \frac{ab}{tb+(1-t)a}$ and $y = \frac{ab}{ta+(1-t)b}$ in (2), then

$$(6) \quad f\left(\frac{2ab}{a+b}\right) \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s} \left((f \circ \phi)\left(\frac{ab}{tb+(1-t)a}\right) + (f \circ \phi)\left(\frac{ab}{ta+(1-t)b}\right)\right)$$

Multiplying both sides of (6) by $t^{\alpha-1}$ and integrating the result with respect to t over $(0, 1)$

$$\begin{aligned} & f\left(\frac{2ab}{a+b}\right) \\ & \leq \left(2h\left(\frac{1}{2}\right)\right)^{-s} \alpha \Gamma(\alpha) \left(\int_0^1 t^{\alpha-1} (f \circ \phi)\left(\frac{ab}{tb+(1-t)a}\right) dt\right. \\ & \quad \left.+ t^{\alpha-1} (f \circ \phi)\left(\frac{ab}{ta+(1-t)b}\right) dt\right) \\ & = \left(2h\left(\frac{1}{2}\right)\right)^{-s} \left(\frac{ab}{b-a}\right)^\alpha \alpha \Gamma(\alpha) \left(\int_{\frac{1}{b}}^{\frac{1}{a}} \left(u - \frac{1}{b}\right)^{\alpha-1} (f \circ \phi)\left(\frac{1}{u}\right) du\right. \\ & \quad \left.+ \int_{\frac{1}{b}}^{\frac{1}{a}} \left(\frac{1}{a} - v\right)^{\alpha-1} (f \circ \phi)\left(\frac{1}{v}\right) dv\right) \\ & = \left(2h\left(\frac{1}{2}\right)\right)^{-s} \left(\frac{ab}{b-a}\right)^\alpha \Gamma(\alpha+1) \left(J_{1/a-}^\alpha (f \circ \phi)(1/b) + J_{1/b+}^\alpha (f \circ \phi)(1/a)\right) \end{aligned}$$

This proves the first inequality. Since $(f \circ \phi) \in \mathcal{HSX}(\phi_{h-s}, I)$, then for $a, b \in I$, we have

$$\begin{aligned} (f \circ \phi) \left(\frac{ab}{tb + (1-t)a} \right) + (f \circ \phi) \left(\frac{ab}{ta + (1-t)b} \right) \\ \leq \left(\frac{h(t)}{t} \right)^{-s} (f \circ \phi^2)(y) + \left(\frac{h(1-t)}{1-t} \right)^{-s} (f \circ \phi^2)(x) \\ + \left(\frac{h(1-t)}{1-t} \right)^{-s} (f \circ \phi^2)(y) + \left(\frac{h(t)}{t} \right)^{-s} (f \circ \phi^2)(x) \end{aligned}$$

The remaining part of the proof follows by multiplying both sides by $t^{\alpha-1}$ and integrating the result with respect to t over $(0, 1)$. □

Corollary 4.3. [6] *Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a, b]$, then the following inequalities for fractional integral holds*

$$\begin{aligned} f \left(\frac{2ab}{a+b} \right) &\leq \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^\alpha \left(J_{1/a-}^\alpha (f \circ g)(1/b) + J_{1/b+}^\alpha (f \circ g)(1/a) \right) \\ &\leq \frac{f(a) + f(b)}{2} \quad \text{with } \alpha > 0. \end{aligned}$$

Proof. The result follows by setting $h(t) = 1$, $s = 1$ and $\phi(x) = x$ in Theorem 4.2. □

5. APPLICATION TO SPECIAL MEANS OF REAL NUMBERS

We recall the following definitions of some special means of two non-negative real numbers which are quite important for numerical approximations and computations.

Definition 5.1. (1) *The Arithmetic mean $A = A(a, b) := \frac{a+b}{2}$.*

(2) *The Geometric mean $G = G(a, b) := \sqrt{ab}$.*

(3) *The Harmonic mean $H = H(a, b) := \frac{2ab}{a+b}$.*

(4) *The Logarithmic mean $L = L(a, b) := \frac{b-a}{\ln b - \ln a}$.*

(5) *The p -logarithmic mean $L_p = L_p(a, b) := \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}$, $p \neq -1, 0$.*

Proposition 5.2. *Let $h(t) = t^{1-\frac{1}{2s}}$, $s \in (0, 1)$, $\phi(x) = x$ and $0 < a < b$, then*

- (i). $\frac{1}{\sqrt{2}} H(a, b) \leq \frac{G^2(a, b)}{L(a, b)} \leq \frac{4}{3} A(a, b)$.
- (ii). $\frac{1}{\sqrt{2}} H^2(a, b) \leq G^2(a, b) \leq \frac{4}{3} A(a^2, b^2)$.
- (iii). $\frac{1}{\sqrt{2}} H^n(a, b) \leq \frac{p+3}{p+1} G^2 L_p^p(a, b) \leq \frac{4}{3} A(a^n, b^n)$
where $p = n - 2$, $p \neq -1, 0$

Proof. Define $f : (0, \infty) \rightarrow \mathbb{R}$ by $f(x) = x$, clearly $f \in \mathcal{HSX}(I)$ and so by Example 2.2, $f \in \mathcal{HSX}(\phi_{h-s}, I)$ since $h(t) \leq t^{1-\frac{1}{s}}$. The remaining part of the proof then follows from Theorem 3.1. The proofs of (ii) and (iii) similarly follow by setting $f(x) = x^2$ and $f(x) = x^n$ respectively. □

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REFERENCES

- [1] Aslam M. A., Noor K. I., Awan M. U., Costache S., *Some integral inequalities for harmonically h -convex functions*, U.P.B. Sci. Bull. Series A, **77**(1), (2015), 5-16.
- [2] Dragomir, S.S. and Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.
- [3] Iscan I., *New general integral inequalities for quasi-geometrically convex functions via fractional integrals*, J. Inequal. Appl. **491** (2013), 1-15.
- [4] ———, *Hermite-Hadamard type inequalities for harmonically convex functions*, Hacettepe Journal of Mathematics and Statistics, **43**(6), (2014), 925-942.
- [5] ———, *Generalization of different type integral inequalities for s -convex functions via fractional integrals*, Appl. Anal., **93**(9), (2014), 1846-1862.
- [6] Iscan I., Wu S., *Hermite-Hadamard type inequalities for harmonically convex functions via fractional integrals*, Appl. Maths. Comp. **238**(2014), 237-244.
- [7] Iscan I., *Ostrowski type inequalities for harmonically s -convex functions*, Konuralp J. Math., **3**(1), (2015), 63-74.
- [8] Olanipekun, P. O., Mogbademu, A. A., Omotoyinbo, O., *Jensen-type inequalities for a class of convex functions*, Ser. Math. Inform. Facta Universitatis, Vol. 31, No. 3 (2016), 655-666.
- [9] Sarikaya M. Z., Set E., Yaldiz H., Basak N., *HermiteHadamards inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model. **57** (2013), 24032407.
- [10] Set E., *New inequalities of Ostrowski type for mappings whose derivatives are s -convex in the second sense via fractional integrals*, Comput. Math. Appl. **63**(2012), 11471154.
- [11] Set, E., Ozdemir, M.E. and Dragomir, S.S., *On Hadamard-type inequalities involving several kinds of convexity*, Journal of Inequalities and Applications, 2010, Article ID 286845, 12 pages, 2010. doi:10.1155/2010/286845.
- [12] Varosanec S., *On h -convexity*, J. Math. Anal. Appl. **326**, (2007), 303-311.
- [13] Wang J., Li X., Feckan M., Zhou Y., *HermiteHadamard-type inequalities for RiemannLiouville fractional integrals via two kinds of convexity*, Appl. Anal. **92**(11), (2012) 2241-2253, <http://dx.doi.org/10.1080/00036811.2012.727986>.
- [14] Wang J., Zhu C., Zhou Y., *New generalized HermiteHadamard type inequalities and applications to special means*, J. Inequal. Appl. **325**(2013), 1-15

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