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Magic Squares with Perfect Square Number Sums

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Abstract

*This short paper shows how to create magic squares in such a way that total sum of their numbers becomes a **perfect square.** This has been done in two ways: Firstly, Take the sum of **odd numbers**, and secondly, take the numbers in a **sequential way**. In the first case, for all orders of magic squares, one can always have a perfect square sum. In the second case, magic squares with perfect square magic sums exist, but only for odd order magic squares. For the even order magic squares, such as 4, 6, 8, etc. it is not possible to write sequential number magic squares with perfect square sums. For the odd order magic squares, at least three examples of sequential numbers are given in each case. This is done from the 3rd to the 25th orders of magic squares. For the prime number orders, examples are considered up to order 11. Further results for order 13, 17, etc. are presented along similar lines. **Finally, we reach a conclusion that we can always create a magic square, so that, if the order of magic square is k, then the number of elements are k^2, the magic sum is k^3, and the sum of all numbers on the square is k^4, for all k=3,4,5,…***

An equation means nothing to me unless it expresses a thought of God.
- S. Ramanujan.

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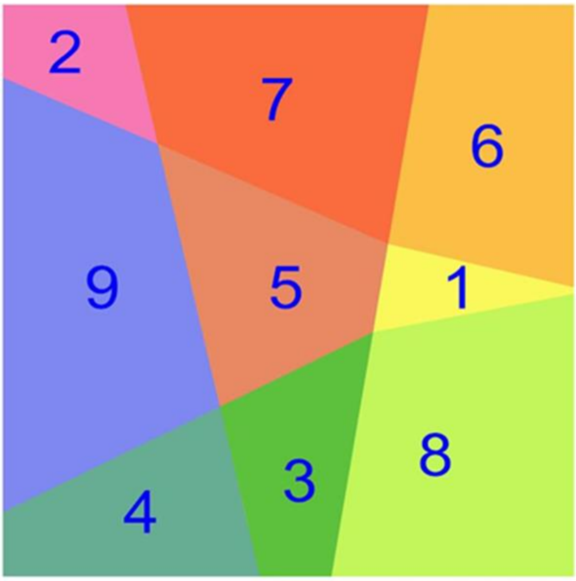
1. Introduction

Recently, William Walkington started an interesting discussion as to how to create magic squares with cells that had the same areas as their numbers. Below is a graphic design for a 2017 seasonal greetings card, showing a magic square with approximate areas that was constructed by William Walkington (2016):



William Walkington - 2016

Figure 1

Lee Sallows (2017) also constructed another magic square representing the areas as rectangles. See below:

Lee Sallows - 2017

Figure 2

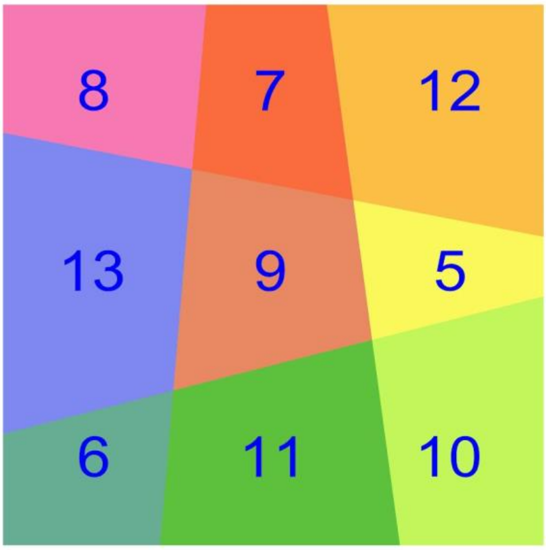
The sum of all the numbers is given by

$$1+2+3+4+5+6+7+8+9=45. \qquad (1)$$

The number 45 is not a perfect square. If we make a slight change, then we can transform the sum into a perfect square:

$$5+6+7+8+9+10+11+12+13=81=9^2 \qquad (2)$$

Using this number sequence, Walter Trump (2017) was able to construct the following area magic square:

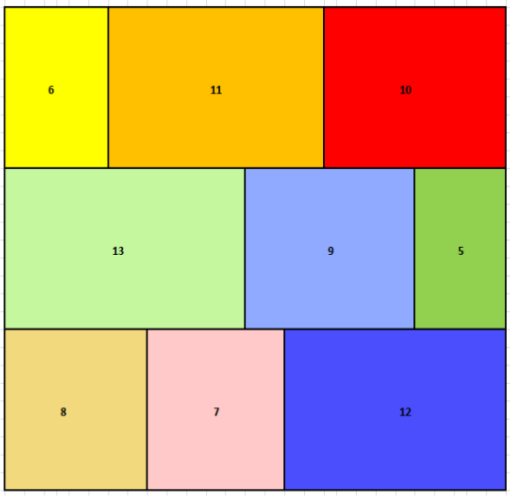


Walter Trump, 2017-01-06, based on ideas of William Walkington and Inder Taneja

Figure 3

Adding 4 to each number in (1), we obtain the numbers in (3). Observing area-wise the figures 1 and 3, there is a considerable difference: For example, from numbers 1 to 2, the cell area is doubled, while from numbers 5 to 6, there is proportionally less increase between the cell areas.

According to Lee Sallows's approach, the area-wise magic square for equation (2) is as follows:



Lee Sallows-Style

Figure 4

In order to construct a magic square with cell areas that are in proportion to their numbers it is not necessary that the numbers always sum to a perfect square. Below is another example constructed by William Walkington (2017) with sequential numbers from 3 to 11:

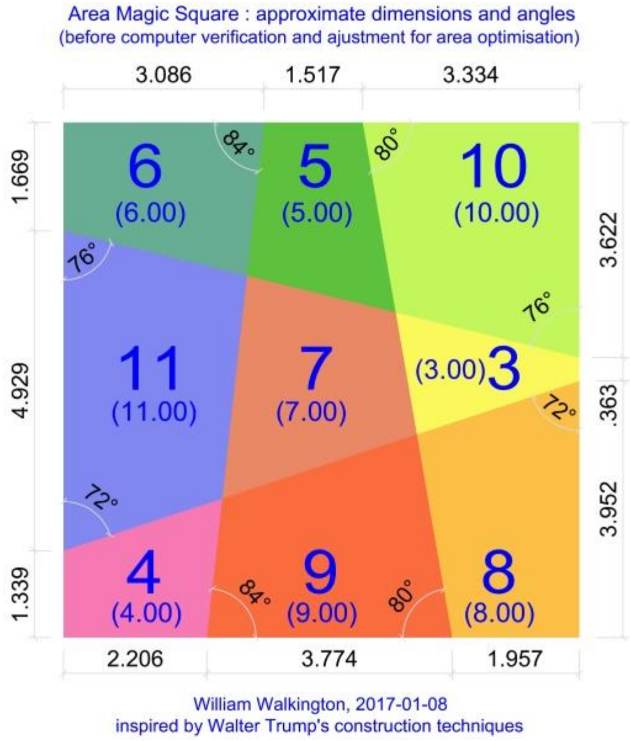


Figure 5

We observe that in Figures 1, 3 and 5, the number cell areas are proportional and aligned in both directions. In Figures 2 and 4, the proportionality of the areas is only present in one direction, which is horizontal. In figure 4 we cannot have proportionality in both directions, because the magic square sum is $S_{3\times 3} := 27$, whilst the sum of numbers is $81 = 27 \times 3$ (and not 27×27).

Below are two examples of classical order 4 magic squares with cell areas that are proportional to their numbers:

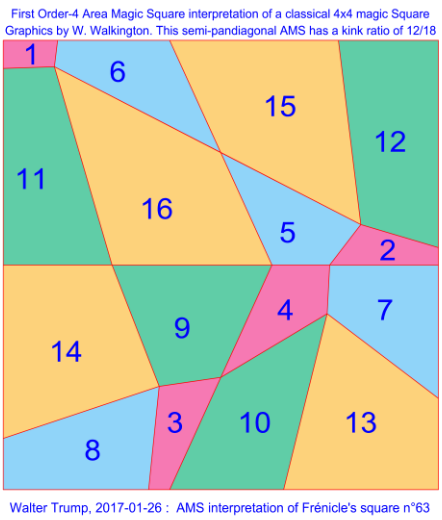


Figure 6

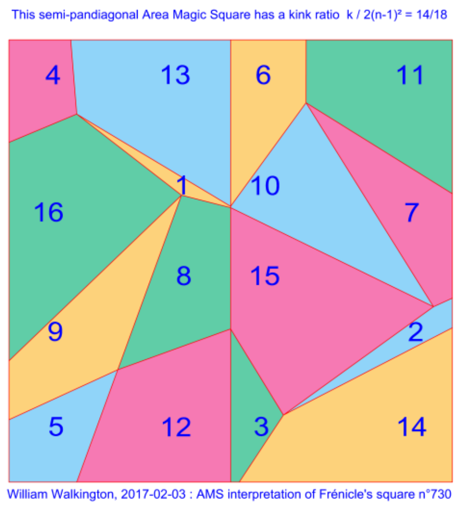


Figure 7

More examples of similar kinds of order 4 area magic squares, together with order 6 area magic squares, can be seen in William Walkington's pages [8].

From equation (2), the question arises, how to create higher order magic squares such that the sum of numbers are always a perfect square. ***Based on this idea, in this paper we establish some formulas, so that sum of the numbers in magic squares always becomes a perfect square.***

2. Series and Magic Square Sums

This section brings some basic idea of series and magic square sums.

2.1 Sum of Natural Numbers Series

It is well-known that the positive natural number series sum is given by

$$\mathbf{T(n) := 1+2+3+...+n = \frac{n(n+1)}{2}, \quad n \geq 1.} \tag{3}$$

The sequence $\mathbf{T(n)}$, is also famous **Pascal's triangle values**.

2.2 Sum of Odd Numbers Series

It is well-known that sum of **odd-number series** is given by

$$\mathbf{F_n := 1+3+5+...+(2n-1) = n^2, \quad n \geq 1.} \tag{4}$$

Since we are working with magic squares, let us write, $\mathbf{n = k^2}$, then we have

$$\mathbf{F_{k^2} := 1+3+5+...+(2k^2-1) = k^4, \quad k \geq 1.} \tag{5}$$

The number k is the order of a magic square. Thus for all $k \geq 3$, we can always have a odd numbers magic square with sum of all numbers a perfect square.

2.3 Sequential Perfect Square Sum

Below is a general formula to write sequential magic squares.

Let us consider

$$\mathbf{K := T(n) - T(n-k) = k\left(n - \frac{k-1}{2}\right), \quad n > k\,.} \tag{6}$$

a) For **even order magic squares**, $k = 2p$:

$$\mathbf{K := k\left(n - \frac{2p-1}{2}\right)}. \tag{7}$$

In this case, the expression $n - \frac{2p-1}{2}$ is never a natural number. Thus, **there is no sequential magic square with sum of numbers as a perfect square for even order magic squares**.

b) For odd order magic squares, $k = 2p+1$:

$$\mathbf{K := T(n) - T(n-k) := k\left(n - \frac{2p+1-1}{2}\right) = k(n-p)}. \tag{8}$$

In this case we can always find a natural number, such that $n-p$ a perfect square with $n-p \geq k$. The minimum value is $n-p=k$. This gives $\mathbf{K := k^2}$ and the sum of all numbers is $\mathbf{F_{k^2} := k^4}$.

2.4 Magic Square Sum and Sum of Numbers

It is well known that **magic sum** of a magic square of order k has total elements, $1, 2, 3, ..., k^2$ is given by

$$\mathbf{S_{k\times k} := \frac{k(k^2+1)}{2}}. \tag{9}$$

Sum of all members is given by

$$\mathbf{F_{k^2} := \frac{k^2(k^2+1)}{2}}. \tag{10}$$

Thus, from (5) and (8), we conclude that ***at least there are two ways of writing magic squares with sum of all members a perfect square*:**

(i) Magic squares formed by odd order numbers;

(ii) For magic square of odd orders, such as, 3, 5, 7,..., one can always find sequential numbers such that the sum of all numbers is a perfect square.

Based on this idea, sections below give examples magic squares with numbers sum a perfect square.

3. Magic Square of Order 3

According to (5) and (8), one can have a perfect square sum magic square of order 3 in two ways. One using odd order numbers and secondly having sequential values. Below are examples of both types.

3.1 First Approach – Odd Numbers

Take $k=3$ in (5), we get

$$1+3+5+...+(2\times 3^2-1)=3^4,$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+9+11+13+15+17=9^2=81.} \tag{11}$$

According to numbers given in (11), the magic square of order 3 is given by

			27
3	13	11	27
17	9	1	27
7	5	15	27
27	27	27	27

Example 1

3.2 Second Approach – Sequential

According to (8),

$$K:=\frac{n(n+1)}{2}-\frac{(n-9)(n-8)}{2}=9(n-4).$$

Take $n=13$, we get a perfect square, i.e.,

$$K:=T(13)-T(4)=\frac{13\times 14}{2}-\frac{4\times 5}{2}=9\times 9=81.$$

Simplifying, one get

$$\mathbf{5+6+7+8+9+10+11+12+13=81.} \tag{12}$$

Again, there is a perfect square for the magic square of order 3 Take the nine numbers in a sequence from 5 to 13. In this case the magic square of order 3 is given by

			27
6	11	10	27
13	9	5	27
8	7	12	27
27	27	27	27

Example 2

In both the examples given above the magic sum is $\mathbf{S_{3\times 3}:=27=3^3}$ and sum of all numbers is $\mathbf{F_9:=81=3^4}$.

We observe that the **minimum number** bigger than 9, and giving $n-4$ as a perfect square is $n=13$. Thus we can say that the sum of numbers starting from 5 to 13 is a perfect square sum with least possible value. Still, there are more sequences with the same property. See examples below:

$$\mathbf{T(20)-T(11)=12+13+...+20=144=12^2};$$
$$\mathbf{T(29)-T(20)=21+22+...+29=225=15^2};$$
$$\mathbf{T(40)-T(31)=32+33+...+40=324=18^2}.$$

4. Magic Square of Order 4

According to (8), we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 4. Take $k=4$ in (5), we get

$$1+3+5+...+(2\times 4^2-1)=4^4.$$

$$\Rightarrow \quad \mathbf{1+3+5+7+9+11+13+15+17+} \\ \mathbf{+19+21+23+25+27+29+31=256=16^2}. \tag{13}$$

According to values given in (13), the **pan diagonal magic square of order 4** is given by

		64	64	64	64
	13	23	1	27	64
64	3	25	15	21	64
64	31	5	19	9	64
64	17	11	29	7	64
64	64	64	64	64	64

Example 3

In this case, the magic sum is $\mathbf{S_{4\times 4}:=64=4^3}$, and sum of numbers is $\mathbf{F_{16}:=256=16^2=4^4}$.

5. Magic Square of Order 5

According to (5) and (8), one can have a perfect square sum magic square of order 5 in two ways. One using odd order numbers and secondly having sequential values. Below are examples of both type.

5.1 First Approach – Odd Numbers

Take $k=5$ in (5), we get

$$1+3+5+...+(2\times 5^2-1)=5^4.$$

$$\Rightarrow \quad \mathbf{1+3+5+7+9+11+13+15+17+19+21+23+25+} \\ \mathbf{+27+29+31+33+35+37+39+41+43+45+47+49=25^2=625.} \tag{14}$$

According to 25 values given in (14), the **pan diagonal magic square of order 5** is given by

		125	125	125	125	125
	1	17	23	39	45	125
125	33	49	5	11	27	125
125	15	21	37	43	9	125
125	47	3	19	25	31	125
125	29	35	41	7	13	125
	125	125	125	125	125	125

Example 4

5.2 Second Approach – Sequential

According to (8),

$$K := \frac{n(n+1)}{2} - \frac{(n-25)(n-24)}{2} = 25(n-12).$$

Take, $n = 37$, we get a perfect square, i.e.,

$$K := T(37) - T(12) = \frac{37\times 38}{2} - \frac{12\times 13}{2} = 37\times 19 - 6\times 13$$
$$= 703 - 78 = 625 = 25^2 = 5^4.$$

Simplifying, one get

$$\mathbf{13+14+...+37 = 5^4 = 25^2 = 625}. \tag{15}$$

This gives a perfect square sum for 25 numbers in a sequential way from 13 to 37. In this case the **pan diagonal magic square of order 5** is given by

		125	125	125	125	125
	13	21	24	32	35	125
125	29	37	15	18	26	125
125	20	23	31	34	17	125
125	36	14	22	25	28	125
125	27	30	33	16	19	125
	125	125	125	125	125	125

Example 5

In both the examples given above the magic sum is $\mathbf{S_{5\times 5} := 125 = 5^3}$ and sum of all numbers is $\mathbf{F_{25} := 625 = 25^2 = 5^4}$.

Above magic square is for minimal number of sequential values. Still, there are more possibilities of getting sequential numbers for the magic square of order 5, where magic sum is a perfect square. See below more examples,

$$\mathbf{T(48) - T(23) = 24 + 25 + ... + 48 = 900 = 30^2};$$
$$\mathbf{T(61) - T(36) = 37 + 38 + ... + 61 = 1225 = 35^2};$$
$$\mathbf{T(76) - T(51) = 52 + 53 + ... + 76 = 1600 = 40^2}$$

6. Magic Square of Order 6

According to (7) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 6. Take $k=6$ in (5), we get

$$1+3+5+...+(2\times 6^2-1)=6^4$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+...+67+69+71=36^2=1296.} \qquad (16)$$

Thus, there is a perfect square sum for the magic square of order 6 of odd numbers. According to values given in (16), the magic square of order 6 is given by

						216
1	45	55	67	33	15	216
57	13	69	27	41	9	216
23	11	25	53	61	43	216
63	31	7	47	19	49	216
37	65	21	5	59	29	216
35	51	39	17	3	71	216
216	216	216	216	216	216	216

Example 6

In this case, the magic sum is $\mathbf{S_{6\times 6} := 216 = 6^3}$, and sum of number is $\mathbf{F_{36} := 1296 = 36^2 = 6^4}$.

7. Magic Square of Order 7

According to (5) and (8), one can have a perfect square sum magic square of order 7 in two ways. One using odd order numbers and secondly having sequential values. Below are examples of both type.

7.1 First Approach – Odd Numbers

Take $k=7$ in (5), we get

$$1+3+5+...+(2\times 7^2-1)=7^4,$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+...+93+95+97=49^2=2401.} \qquad (17)$$

According to values given in (17), a **pan diagonal magic square of order 7** is given by

		343	343	343	343	343	343	343
	1	17	33	49	65	81	97	343
343	79	95	13	15	31	47	63	343
343	45	61	77	93	11	27	29	343
343	25	41	43	59	75	91	9	343
343	89	7	23	39	55	57	73	343
343	69	71	87	5	21	37	53	343
343	35	51	67	83	85	3	19	343
	343	343	343	343	343	343	343	343

Example 7

In this case, the magic sum is $\mathbf{S_{7\times 7} := 343 = 7^3}$, and sum of all number is $\mathbf{F_{49} := 2401 = 7^4}$.

7.2 Second Approach – Sequential

According to (8),

$$K := \frac{n(n+1)}{2} - \frac{(n-49)(n-48)}{2} = 49(n-24)$$

Take $n = 73$, we get a perfect square sum, i.e.,

$$K := T(73) - T(24) = \frac{73\times 74}{2} - \frac{24\times 25}{2} = 73\times 37 - 12\times 25$$
$$= 2701 - 300 = 2401 = 49^2 = 7^4.$$

Simplifying, one get

$$\mathbf{25 + 26 + ... + 72 + 73 = 7^4 = 49^2 = 2401}. \tag{18}$$

This gives a perfect square sum for 49 values put in a sequence starting from 25. According to values given in (18), a **pan diagonal magic square of order 7** is given by

		343	343	343	343	343	343	343
	25	33	41	49	57	65	73	343
343	64	72	31	32	40	48	56	343
343	47	55	63	71	30	38	39	343
343	37	45	46	54	62	70	29	343
343	69	28	36	44	52	53	61	343
343	59	60	68	27	35	43	51	343
343	42	50	58	66	67	26	34	343
	343	343	343	343	343	343	343	343

Example 8

Still, there are more possibilities of getting sequential numbers magic square of order 7 with perfect square sum. See below examples,

$$\mathbf{T(88) - T(39) = 40 + 41 + ... + 88 = 3136 = 56^2};$$
$$\mathbf{T(105) - T(56) = 57 + 88 + ... + 105 = 3969 = 63^2};$$
$$\mathbf{T(124) - T(79) = 80 + 81 + ... + 124 = 4900 = 70^2}.$$

In this case, also the magic sum is $\mathbf{S_{7\times 7} := 343}$, and sum of number is $\mathbf{F_{49} := 2401 = 49^2 = 7^4}$.

8. Magic Square of Order 8

According to (7) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 8. Take $k = 8$ in (5), we get

$$1 + 3 + 5 + ... + (2\times 8^2 - 1) = 8^4$$
$$\Rightarrow \quad \mathbf{1 + 3 + 5 + 7 + ... + 123 + 125 + 127 = 64^2 = 4096.} \tag{19}$$

According to values given in (19), the magic square of order 8 is given by

	512	512	512	512	512	512	512	512	512
	31	81	71	9	53	123	109	35	512
512	51	125	107	37	25	87	65	15	512
512	1	79	89	23	43	101	115	61	512
512	45	99	117	59	7	73	95	17	512
512	75	5	19	93	97	47	57	119	512
512	103	41	63	113	77	3	21	91	512
512	85	27	13	67	127	49	39	105	512
512	121	55	33	111	83	29	11	69	512
	512	512	512	512	512	512	512	512	512

Example 9

In this case the magic sum is $\mathbf{S_{8\times 8} := 512}$, and sum of number is $\mathbf{F_{64} := 4096 = 64^2} = \mathbf{8^4}$. The above magic square is also a bimagic with **bimagic sum** $\mathbf{Sb_{8\times 8} := 43688}$.

9. Magic Square of Order 9

According to (5) and (8), one can have a perfect square sum magic squares of order 9 in two ways. One, using odd order numbers, and second with sequential values.

9.1 First Approach – Odd Numbers

Take $k = 9$ in (5), we get

$$1+3+5+...+(2\times 9^2-1)=9^4,$$

$$\Rightarrow \quad \mathbf{1+3+5+7+...+157+159+161=81^2=6561.} \tag{20}$$

According to values given in (20), the magic square of order 9 is given by

									729
1	35	45	69	79	95	119	129	157	729
65	75	103	109	143	153	15	25	41	729
123	133	149	11	21	49	55	89	99	729
53	9	19	97	59	87	147	121	137	729
93	67	83	161	117	127	43	5	33	729
151	113	141	39	13	29	107	63	73	729
27	37	17	77	105	61	139	155	111	729
85	101	57	135	145	125	23	51	7	729
131	159	115	31	47	3	81	91	71	729
729	729	729	729	729	729	729	729	729	729

Example 10

In this case the magic sum is $\mathbf{S_{9\times 9} := 729 = 9^3}$, and sum of numbers is $\mathbf{F_{81} := 6561 = 81^2 = 9^4}$. It is also a bimagic square with bimagic sum is $\mathbf{Sb_{9\times 9} := 78729}$.

9.2 Second Approach – Sequential

According to (8),

$$K := T(n) - T(n-81), \quad n > 81,$$

i.e.,

$$K := \frac{n(n+1)}{2} - \frac{(n-81)(n-80)}{2} = 81n - 3240 = 81(n-40).$$

Take $n = 121$, we get a perfect square, i.e.,

$$K := T(121) - T(40) = \frac{121\times 122}{2} - \frac{40\times 41}{2}$$
$$= 121\times 61 - 20\times 41$$
$$= 7381 - 820 = 6561 = 81^2 = 9^4.$$

Simplifying, one get

$$\mathbf{41 + 42 + ... + 120 + 121 = 9^4 = 81^2 = 6561}. \tag{21}$$

Thus, we have a perfect square sum for the 81 numbers starting from 41. According to values given in (21), the magic square of order 9 is given by

									729
41	58	63	75	80	88	100	105	119	729
73	78	92	95	112	117	48	53	61	729
102	107	115	46	51	65	68	85	90	729
67	45	50	89	70	84	114	101	109	729
87	74	82	121	99	104	62	43	57	729
116	97	111	60	47	55	94	72	77	729
54	59	49	79	93	71	110	118	96	729
83	91	69	108	113	103	52	66	44	729
106	120	98	56	64	42	81	86	76	729
729	729	729	729	729	729	729	729	729	729

Example 11

In this case also the magic sum is $\mathbf{S_{9\times 9} := 729 = 9^3}$, and sum of number is $\mathbf{F_{81} := 6561 = 81^2 = 9^4}$. This magic square is also a bimagic with bimagic sum is $\mathbf{Sb_{9\times 9} := 63969}$.

Still, there are more possibilities of getting sequential numbers magic squares of order 9, where numbers sum is a perfect square. See below examples,

$$\mathbf{T(140) - T(59) = 60 + 61 + ... + 140 = 8100 = 90^2};$$
$$\mathbf{T(161) - T(81) = 82 + 88 + ... + 161 = 9801 = 99^2};$$
$$\mathbf{T(184) - T(103) = 104 + 105 + ... + 184 = 11644 = 108^2}.$$

10. Magic Square of Order 10

According to (8) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 10. Take $n=100$ in (5) or $k=10$ in (6), we get

$$1+3+5+...+(2\times 10^2-1)=10^4$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+...+197+198+199=100^2=10000.} \tag{22}$$

According to values given in (22), the magic square of order 10 is given by

										1000
1	97	31	183	175	47	149	73	119	125	1000
71	23	135	165	59	197	81	107	9	153	1000
123	115	45	99	191	13	77	161	147	29	1000
39	173	109	67	141	85	195	3	131	57	1000
187	41	17	113	89	143	171	139	25	75	1000
169	127	61	157	33	111	19	185	55	83	1000
95	5	159	51	103	69	133	37	181	167	1000
53	189	87	21	137	179	105	155	63	11	1000
145	79	193	129	7	35	43	91	177	101	1000
117	151	163	15	65	121	27	49	93	199	1000
1000	1000	1000	1000	1000	1000	1000	1000	1000	1000	1000

Example 12

In this case the magic sum is $\mathbf{S_{10\times 10}:=1000=10^3}$, and sum of numbers is $\mathbf{F_{100}:=10000=100^2=10^4}$.

11. Magic Square of Order 11

According to (5) and (8), one can have a perfect square sum magic square of order 11 in two ways. One, using odd order numbers, and second with sequential values.

11.1 First Approach – Odd Numbers

Take $k=11$ in (5), we get

$$1+3+5+...+(2\times 11^2-1)=11^4,$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+...+239+241=121^2=14641.} \tag{23}$$

According to values given in (23), the magic square of order 11 is given by

		1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331
	1	41	59	77	95	113	153	171	189	207	225	1331
1331	201	241	17	35	53	71	89	129	147	165	183	1331
1331	159	177	217	235	11	29	47	87	105	123	141	1331
1331	117	135	175	193	211	229	5	23	63	81	99	1331
1331	75	93	111	151	169	187	205	223	21	39	57	1331
1331	33	51	69	109	127	145	163	181	199	239	15	1331
1331	233	9	27	45	85	103	121	139	157	197	215	1331
1331	191	209	227	3	43	61	79	97	115	133	173	1331
1331	149	167	185	203	221	19	37	55	73	91	131	1331
1331	107	125	143	161	179	219	237	13	31	49	67	1331
1331	65	83	101	119	137	155	195	213	231	7	25	1331
	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331

Example 13

In this case the magic sum is $\mathbf{S_{11\times 11} := 1331 = 11^3}$, and sum of numbers is $\mathbf{F_{121} := 14641 = 121^2 = 11^4}$. In this case the magic square is **pan diagonal**.

11.2 Second Approach – Sequential

According to (8), we have

$$K := T(n) - T(n-121), \quad n > 121,$$

i.e.,

$$K := \frac{n(n+1)}{2} - \frac{(n-121)(n-120)}{2} = 121n - 7260 = 121(n-60).$$

Take $n = 181$, we get a perfect square sum, i.e.,

$$\begin{aligned} K := T(181) - T(60) &= \frac{181\times 182}{2} - \frac{60\times 61}{2} \\ &= 181\times 91 - 30\times 61 \\ &= 16471 - 1830 = 14641 = 121^2 = 11^4. \end{aligned}$$

Simplifying,

$$\mathbf{61 + 62 + \ldots + 180 + 181 = 11^4 = 121^2 = 14641}. \tag{24}$$

Thus, we have a perfect square sum for the 121 numbers starting from 61. According to values given in (24), the magic square of order 11 is given by

		1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331
	61	81	90	99	108	117	137	146	155	164	173	1331
1331	161	181	69	78	87	96	105	125	134	143	152	1331
1331	140	149	169	178	66	75	84	104	113	122	131	1331
1331	119	128	148	157	166	175	63	72	92	101	110	1331
1331	98	107	116	136	145	154	163	172	71	80	89	1331
1331	77	86	95	115	124	133	142	151	160	180	68	1331
1331	177	65	74	83	103	112	121	130	139	159	168	1331
1331	156	165	174	62	82	91	100	109	118	127	147	1331
1331	135	144	153	162	171	70	79	88	97	106	126	1331
1331	114	123	132	141	150	170	179	67	76	85	94	1331
1331	93	102	111	120	129	138	158	167	176	64	73	1331
	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331	1331

Example 14

In both the examples 13 and 14, the magic sum is $\mathbf{S_{11\times 11} := 1331 = 11^3}$, and the sum of numbers is $\mathbf{F_{121} := 14641 = 121^2 = 11^4}$. Still, there are more possibilities of getting sequential number magic squares for the magic square of order 11, where the numbers sum is a perfect square. See below examples,

$$\mathbf{T(204) - T(83) = 84 + 85 + ... + 204 = 17424 = 132^2};$$
$$\mathbf{T(229) - T(108) = 109 + 110 + ... + 229 = 20449 = 143^2};$$
$$\mathbf{T(256) - T(135) = 136 + 137 + ... + 256 = 23716 = 154^2}.$$

12. Magic Square of Order 12

According to (8) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 12. Take $k = 12$ in (5), we get

$$1 + 3 + 5 + ... + (2 \times 12^2 - 1) = 12^4$$

$$\Rightarrow \qquad \mathbf{1 + 3 + 5 + ... + 285 + 287 = 12^4 = 20736}. \tag{25}$$

According to values given in (25), the magic square of order 12 is given by

												1728
135	165	3	273	89	211	53	223	109	191	25	251	1728
9	267	141	159	55	221	91	209	35	241	119	181	1728
285	15	153	123	235	65	199	77	263	37	179	97	1728
147	129	279	21	197	79	233	67	169	107	253	47	1728
85	215	49	227	111	189	27	249	137	163	5	271	1728
59	217	95	205	33	243	117	183	7	269	139	161	1728
239	61	203	73	261	39	177	99	283	17	151	125	1728
193	83	229	71	171	105	255	45	149	127	281	19	1728
113	187	29	247	133	167	1	275	87	213	51	225	1728
31	245	115	185	11	265	143	157	57	219	93	207	1728
259	41	175	101	287	13	155	121	237	63	201	75	1728
173	103	257	43	145	131	277	23	195	81	231	69	1728
1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728	1728

Example 15

In this case the magic sum is $\mathbf{S_{12\times 12} := 1728 = 12^3}$, and sum of numbers is $\mathbf{F_{144} := 20736 = 144^2 = 12^4}$. Each block of order 4 is **pan diagonal magic square** with magic sum $\mathbf{S_{4\times 4} := 576}$.

13. Magic Square of Order 14

According to (7) we don't have sequential magic square with numbers sum a perfect square. We will obtain only odd numbers magic square of order 14. Take $k = 14$ in (5), we get

$$1+3+5+...+(2\times 14^2-1)=14^4$$

$$\Rightarrow \quad \mathbf{1+3+5+...+389+391 = 14^4 = 38416}. \tag{26}$$

According to values given in (26), the magic square of order 14 is given by

														2744
1	189	381	223	275	155	115	63	33	317	291	355	249	97	2744
293	31	219	351	375	253	173	233	7	105	139	165	75	325	2744
251	323	61	273	343	125	159	11	93	51	193	197	297	367	2744
379	221	299	91	309	229	9	153	67	339	45	191	279	133	2744
327	167	109	23	241	171	57	201	265	123	345	49	371	295	2744
207	19	177	73	41	**331**	**357**	**307**	**389**	147	87	267	117	225	2744
37	69	263	113	5	**305**	**391**	**329**	**359**	183	243	101	143	203	2744
89	129	141	179	71	**385**	**303**	**361**	**335**	237	259	3	205	47	2744
59	85	119	145	187	**363**	**333**	**387**	**301**	213	13	235	43	261	2744
135	287	311	53	149	95	209	185	239	271	365	83	21	341	2744
161	373	349	227	137	35	269	103	169	27	211	285	319	79	2744
277	347	15	321	283	215	231	29	157	369	77	121	107	195	2744
353	257	245	383	111	65	39	127	199	281	163	315	181	25	2744
175	247	55	289	217	17	99	255	131	81	313	377	337	151	2744
2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744	2744

Example 16

In this case the magic sum is $\mathbf{S_{14\times 14} := 2744 = 14^3}$, and sum of numbers is $\mathbf{F_{196} := 38416 = 196^2 = 14^4}$. The middle block is a **pan diagonal magic square of order 4** with magic sum $\mathbf{S_{4\times 4} := 1384}$.

14. Magic Square of Order 15

According to (5) and (8), one can have a perfect square sum magic square of order 15 in two ways. One, using odd order numbers, and second with sequential values.

14.1 First Approach – Odd Numbers

Take $k = 15$ in (5), we get

$$1+3+5+...+(2\times 15^2-1)=15^4,$$

$$\Rightarrow \quad \mathbf{1+3+5+7+...+239+241 = 225^2 = 50625.} \tag{27}$$

According to values given in (27), the magic square of order 15 is given by

															3375
3	171	189	357	375	65	109	247	299	437	31	143	221	325	403	3375
339	387	15	153	201	277	449	77	95	259	311	415	43	121	233	3375
165	183	351	369	27	107	245	289	427	89	133	211	323	401	55	3375
381	9	177	195	333	439	67	119	257	275	413	41	145	223	301	3375
207	345	363	21	159	269	287	425	79	97	235	313	391	53	131	3375
61	113	251	295	433	33	141	219	327	405	5	169	187	359	377	3375
281	445	73	91	263	309	417	45	123	231	337	389	17	155	199	3375
103	241	293	431	85	135	213	321	399	57	167	185	349	367	29	3375
443	71	115	253	271	411	39	147	225	303	379	7	179	197	335	3375
265	283	421	83	101	237	315	393	51	129	209	347	365	19	157	3375
35	139	217	329	407	1	173	191	355	373	63	111	249	297	435	3375
307	419	47	125	229	341	385	13	151	203	279	447	75	93	261	3375
137	215	319	397	59	163	181	353	371	25	105	243	291	429	87	3375
409	37	149	227	305	383	11	175	193	331	441	69	117	255	273	3375
239	317	395	49	127	205	343	361	23	161	267	285	423	81	99	3375
3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

Example 17

In this case the magic sum is $\mathbf{S_{15\times 15} := 3375 = 15^3}$, and sum of numbers is $\mathbf{F_{225} := 50625 = 225^2 = 15^4}$. In this case each block of order 5 is a **pan diagonal magic square** forming a magic square of order 3:

			3375
1095	1157	1123	3375
1153	1125	1097	3375
1127	1093	1155	3375
3375	3375	3375	3375

Example 18

14.2 Second Approach – Sequential

According to formula (8), we have

$$K := T(n) - T(n-225), \quad n > 225,$$

i.e.,

$$K := \frac{n(n+1)}{2} - \frac{(n-225)(n-224)}{2}$$
$$:= 225n - 25200 = 225(n-112).$$

Take $n = 337$, we get a perfect square, i.e.,

$$K := T(337) - T(112) = 50625 = 225^2 = 15^4.$$

Simplifying, one get

$$\mathbf{113 + 114 + ... + 336 + 337 = 15^4 = 225^2 = 50625}. \qquad (28)$$

Thus, we have a perfect square sum for the 225 numbers starting from 113. According to values given in (28), the magic square of order 15 is given by

															3375
114	198	207	291	300	145	167	236	262	331	128	184	223	275	314	3375
282	306	120	189	213	251	337	151	160	242	268	320	134	173	229	3375
195	204	288	297	126	166	235	257	326	157	179	218	274	313	140	3375
303	117	201	210	279	332	146	172	241	250	319	133	185	224	263	3375
216	285	294	123	192	247	256	325	152	161	230	269	308	139	178	3375
143	169	238	260	329	129	183	222	276	315	115	197	206	292	301	3375
253	335	149	158	244	267	321	135	174	228	281	307	121	190	212	3375
164	233	259	328	155	180	219	273	312	141	196	205	287	296	127	3375
334	148	170	239	248	318	132	186	225	264	302	116	202	211	280	3375
245	254	323	154	163	231	270	309	138	177	217	286	295	122	191	3375
130	182	221	277	316	113	199	208	290	299	144	168	237	261	330	3375
266	322	136	175	227	283	305	119	188	214	252	336	150	159	243	3375
181	220	272	311	142	194	203	289	298	125	165	234	258	327	156	3375
317	131	187	226	265	304	118	200	209	278	333	147	171	240	249	3375
232	271	310	137	176	215	284	293	124	193	246	255	324	153	162	3375
3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375	3375

Example 19

In this case also the magic sum is $\mathbf{S_{15\times15} := 3375 = 15^3}$, and sum of number is $\mathbf{F_{225} := 50625 = 225^2 = 15^4}$. In this case each block of order 5 is a **pan diagonal magic square** forming a magic square of order 3:

			3375
1110	1141	1124	3375
1139	1125	1111	3375
1126	1109	1140	3375
3375	3375	3375	3375

Example 20

Still, there are more possibilities of getting sequential numbers for the magic square of order 15, where numbers sum is a perfect square. See below examples,

$$\mathbf{T(368) - T(143) = 144 + 145 + ... + 368 = 57600 = 240^2};$$
$$\mathbf{T(401) - T(176) = 177 + 178 + ... + 401 = 65025 = 255^2};$$
$$\mathbf{T(436) - T(211) = 212 + 213 + ... + 436 = 72900 = 270^2}.$$

15. Magic Square of Order 16

According to (8) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 16. Take $n = 256$ in (5) or $k = 16$ in (6), we get

$$1 + 3 + 5 + ... + (2\times16^2 - 1) = 16^4$$

$$\Rightarrow \quad \mathbf{1 + 3 + 5 + ... + 509 + 511 = 16^4 = 65536}. \tag{29}$$

According to values given in (29), the magic square of order 16 is given by

																4096
1	307	477	239	45	287	497	195	87	357	395	185	123	329	423	149	4096
463	253	19	289	483	209	63	269	409	171	69	375	437	135	105	347	4096
243	449	303	29	223	493	259	49	165	407	377	75	137	443	341	103	4096
317	15	225	467	273	35	205	511	363	89	183	389	327	117	155	425	4096
91	361	391	181	119	325	427	153	13	319	465	227	33	275	509	207	4096
405	167	73	379	441	139	101	343	451	241	31	301	495	221	51	257	4096
169	411	373	71	133	439	345	107	255	461	291	17	211	481	271	61	4096
359	85	187	393	331	121	151	421	305	3	237	479	285	47	193	499	4096
109	351	433	131	65	371	413	175	59	265	487	213	23	293	459	249	4096
419	145	127	333	399	189	83	353	501	199	41	283	473	235	5	311	4096
159	429	323	113	179	385	367	93	201	507	277	39	229	471	313	11	4096
337	99	141	447	381	79	161	403	263	53	219	489	299	25	247	453	4096
55	261	491	217	27	297	455	245	97	339	445	143	77	383	401	163	4096
505	203	37	279	469	231	9	315	431	157	115	321	387	177	95	365	4096
197	503	281	43	233	475	309	7	147	417	335	125	191	397	355	81	4096
267	57	215	485	295	21	251	457	349	111	129	435	369	67	173	415	4096
4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096	4096

Example 21

In this case the magic sum is $\mathbf{S_{16\times16} := 4096 = 16^3}$, and sum of numbers is $\mathbf{F_{256} := 65536 = 256^2 = 16^4}$. Each block of order 4 is also a **pan diagonal magic square** with magic sum $\mathbf{S_{4\times4} := 1024}$. Moreover, the above magic square is also a **bimagic** with bimagic sum $\mathbf{Sb_{16\times16} := 1398096}$.

16. Magic Square of Order 18

According to (7) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 18. Take $k = 18$ in (5), we get

$$1+3+5+...+(2\times18^2-1)=18^4$$

$$\Rightarrow \qquad \mathbf{1+3+5+...+645+647 = 18^4 = 104976}. \qquad (30)$$

According to values given in (30), below are two different ways of writing the magic square of order 18:

• First way:

																		5832
217	261	271	283	249	231	577	621	631	643	609	591	73	117	127	139	105	87	5832
273	229	285	243	257	225	633	589	645	603	617	585	129	85	141	99	113	81	5832
239	227	241	269	277	259	599	587	601	629	637	619	95	83	97	125	133	115	5832
279	247	223	263	235	265	639	607	583	623	595	625	135	103	79	119	91	121	5832
253	281	237	221	275	245	613	641	597	581	635	605	109	137	93	77	131	101	5832
251	267	255	233	219	287	611	627	615	593	579	647	107	123	111	89	75	143	5832
145	189	199	211	177	159	289	333	343	355	321	303	433	477	487	499	465	447	5832
201	157	213	171	185	153	345	301	357	315	329	297	489	445	501	459	473	441	5832
167	155	169	197	205	187	311	299	313	341	349	331	455	443	457	485	493	475	5832
207	175	151	191	163	193	351	319	295	335	307	337	495	463	439	479	451	481	5832
181	209	165	149	203	173	325	353	309	293	347	317	469	497	453	437	491	461	5832
179	195	183	161	147	215	323	339	327	305	291	359	467	483	471	449	435	503	5832
505	549	559	571	537	519	1	45	55	67	33	15	361	405	415	427	393	375	5832
561	517	573	531	545	513	57	13	69	27	41	9	417	373	429	387	401	369	5832
527	515	529	557	565	547	23	11	25	53	61	43	383	371	385	413	421	403	5832
567	535	511	551	523	553	63	31	7	47	19	49	423	391	367	407	379	409	5832
541	569	525	509	563	533	37	65	21	5	59	29	397	425	381	365	419	389	5832
539	555	543	521	507	575	35	51	39	17	3	71	395	411	399	377	363	431	5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832

Example 22

In this case the magic sum is $\mathbf{S_{18\times18} := 5832 = 18^3}$, and sum of all 324 numbers is $\mathbf{F_{324} := 104976 = 324^2 = 18^4}$. Each **block of order 6 is a magic square** constructed according to section 6 with different sums:

			5832					27
1512	3672	648	5832	$= 6^3 \times$	7	17	3	27
1080	1944	2808	5832		5	9	13	27
3240	216	2376	5832		15	1	11	27
5832	5832	5832	5832		27	27	27	27

Example 23

Interestingly, the same happens with the **sum of all members of each block of order 6** resulting again in a magic square of order 3:

			34992					27
9072	22032	3888	34992	$= 6^4 \times$	7	17	3	27
6480	11664	16848	34992		5	9	13	27
9072	22032	3888	34992		15	1	11	27
34992	34992	34992	34992		27	27	27	27

Example 24

• Second way:

Redistributing the total numbers 32992 in little different way, we get an alternative way of writing a magic square of order 18 with the same values given in (30). See below

																		5832
7	403	493	601	295	133	17	413	503	611	305	143	3	399	489	597	291	129	5832
511	115	619	241	367	79	521	125	629	251	377	89	507	111	615	237	363	75	5832
205	97	223	475	547	385	215	107	233	485	557	395	201	93	219	471	543	381	5832
565	277	61	421	169	439	575	287	71	431	179	449	561	273	57	417	165	435	5832
331	583	187	43	529	259	341	593	197	53	539	269	327	579	183	39	525	255	5832
313	457	349	151	25	637	323	467	359	161	35	647	309	453	345	147	21	633	5832
5	401	491	599	293	131	9	405	495	603	297	135	13	409	499	607	301	139	5832
509	113	617	239	365	77	513	117	621	243	369	81	517	121	625	247	373	85	5832
203	95	221	473	545	383	207	99	225	477	549	387	211	103	229	481	553	391	5832
563	275	59	419	167	437	567	279	63	423	171	441	571	283	67	427	175	445	5832
329	581	185	41	527	257	333	585	189	45	531	261	337	589	193	49	535	265	5832
311	455	347	149	23	635	315	459	351	153	27	639	319	463	355	157	31	643	5832
15	411	501	609	303	141	1	397	487	595	289	127	11	407	497	605	299	137	5832
519	123	627	249	375	87	505	109	613	235	361	73	515	119	623	245	371	83	5832
213	105	231	483	555	393	199	91	217	469	541	379	209	101	227	479	551	389	5832
573	285	69	429	177	447	559	271	55	415	163	433	569	281	65	425	173	443	5832
339	591	195	51	537	267	325	577	181	37	523	253	335	587	191	47	533	263	5832
321	465	357	159	33	645	307	451	343	145	19	631	317	461	353	155	29	641	5832
5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832	5832

Example 25

In this case also the magic sum is $\mathbf{S_{18\times18} := 5832 = 18^3}$, and sum of all 324 numbers is $\mathbf{F_{324} := 104976 = 324^2 = 18^4}$. Here also each **block of order 6 is a magic square** with different magic sums:

			5832					486
1932	1992	1908	5832	$=12\times$	161	166	159	486
1920	1944	1968	5832		160	162	164	486
1980	1896	1956	5832		165	158	163	486
5832	5832	5832	5832		486	486	486	486

Example 26

Similar to first way, here also, the **sum of all members of each block of order 6** results in a magic square of order 3:

			34992					486
11592	11952	11448	34992	$=72\times$	161	166	159	486
11520	11664	11808	34992		160	162	164	486
11880	11376	11736	34992		165	158	163	486
34992	34992	34992	34992		486	486	486	486

Example 27

17. Magic Square of Order 20

According to (7) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 18. Take $n=400$ in (5) or $k=20$ in (6), we get

$$1+3+5+...+(2\times 20^2-1)=20^4$$

$$\Rightarrow \quad \mathbf{1+3+5+...+397+399 = 20^4 = 160000}. \qquad (31)$$

According to values given in (31), the magic square of order 20 for odd numbers is given by

		8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000
	43	269	373	595	699	81	231	415	553	737	7	305	329	639	663	125	187	451	517	781	8000
8000	573	715	59	243	389	535	753	97	201	431	609	679	23	287	345	491	797	141	165	467	8000
8000	259	363	589	693	75	217	401	551	735	113	303	327	625	649	39	181	445	507	771	157	8000
8000	709	53	275	379	563	751	95	233	417	521	665	9	319	343	607	787	131	197	461	485	8000
8000	395	579	683	69	253	433	537	721	111	215	359	623	647	25	289	477	501	765	147	171	8000
8000	5	307	331	637	661	127	185	449	519	783	41	271	375	593	697	83	229	413	555	739	8000
8000	611	677	21	285	347	489	799	143	167	465	575	713	57	241	391	533	755	99	203	429	8000
8000	301	325	627	651	37	183	447	505	769	159	257	361	591	695	73	219	403	549	733	115	8000
8000	667	11	317	341	605	785	129	199	463	487	711	55	273	377	561	749	93	235	419	523	8000
8000	357	621	645	27	291	479	503	767	145	169	393	577	681	71	255	435	539	723	109	213	8000
8000	121	191	455	513	777	3	309	333	635	659	85	227	411	557	741	47	265	369	599	703	8000
8000	495	793	137	161	471	613	675	19	283	349	531	757	101	205	427	569	719	63	247	385	8000
8000	177	441	511	775	153	299	323	629	653	35	221	405	547	731	117	263	367	585	689	79	8000
8000	791	135	193	457	481	669	13	315	339	603	747	91	237	421	525	705	49	279	383	567	8000
8000	473	497	761	151	175	355	619	643	29	293	437	541	725	107	211	399	583	687	65	249	8000
8000	87	225	409	559	743	45	267	371	597	701	123	189	453	515	779	1	311	335	633	657	8000
8000	529	759	103	207	425	571	717	61	245	387	493	795	139	163	469	615	673	17	281	351	8000
8000	223	407	545	729	119	261	365	587	691	77	179	443	509	773	155	297	321	631	655	33	8000
8000	745	89	239	423	527	707	51	277	381	565	789	133	195	459	483	671	15	313	337	601	8000
8000	439	543	727	105	209	397	581	685	67	251	475	499	763	149	173	353	617	641	31	295	8000
	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000	8000

Example 28

In this case the magic square is pan diagonal and magic sum is $\mathbf{S_{20\times 20} := 8000 = 20^3}$. Also, the sum of all 400 numbers is $\mathbf{F_{400} := 160000 = 400^2 = 20^4}$. Moreover each block of order 5 a **pan diagonal magic square** with different magic sums:

		8000	8000	8000	8000
	1979	**2017**	1943	2061	8000
8000	1941	2063	1977	2019	8000
8000	2057	1939	2021	1983	8000
8000	2023	1981	2059	1937	8000
8000	8000	8000	8000	8000	8000

Example 29

The second block from up is a pan diagonal magic square of order with **magic sum 2017,** i.e., the year we are.

18. Magic Square of Order 21

According to (5) and (8), one can have a perfect square sum magic square of order 21 in two ways. One using odd order numbers and secondly having sequential values. Below are examples of both type.

18.1 First Approach – Odd Numbers

Take $k = 21$ in (5), we get

$$1+3+5+\ldots+(2\times 21^2-1)=21^4$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+\ldots+879+881 = 441^2 = 194481}. \tag{32}$$

According to values given in (32), we shall contruct in two different ways of magic square of order 21:

●First way:

																					9261
295	311	327	343	359	375	391	785	801	817	833	849	865	881	99	115	131	147	163	179	195	9261
373	389	307	309	325	341	357	863	879	797	799	815	831	847	177	193	111	113	129	145	161	9261
339	355	371	387	305	321	323	829	845	861	877	795	811	813	143	159	175	191	109	125	127	9261
319	335	337	353	369	385	303	809	825	827	843	859	875	793	123	139	141	157	173	189	107	9261
383	301	317	333	349	351	367	873	791	807	823	839	841	857	187	105	121	137	153	155	171	9261
363	365	381	299	315	331	347	853	855	871	789	805	821	837	167	169	185	103	119	135	151	9261
329	345	361	377	379	297	313	819	835	851	867	869	787	803	133	149	165	181	183	101	117	9261
197	213	229	245	261	277	293	393	409	425	441	457	473	489	589	605	621	637	653	669	685	9261
275	291	209	211	227	243	259	471	487	405	407	423	439	455	667	683	601	603	619	635	651	9261
241	257	273	289	207	223	225	437	453	469	485	403	419	421	633	649	665	681	599	615	617	9261
221	237	239	255	271	287	205	417	433	435	451	467	483	401	613	629	631	647	663	679	597	9261
285	203	219	235	251	253	269	481	399	415	431	447	449	465	677	595	611	627	643	645	661	9261
265	267	283	201	217	233	249	461	463	479	397	413	429	445	657	659	675	593	609	625	641	9261
231	247	263	279	281	199	215	427	443	459	475	477	395	411	623	639	655	671	673	591	607	9261
687	703	719	735	751	767	783	1	17	33	49	65	81	97	491	507	523	539	555	571	587	9261
765	781	699	701	717	733	749	79	95	13	15	31	47	63	569	585	503	505	521	537	553	9261
731	747	763	779	697	713	715	45	61	77	93	11	27	29	535	551	567	583	501	517	519	9261
711	727	729	745	761	777	695	25	41	43	59	75	91	9	515	531	533	549	565	581	499	9261
775	693	709	725	741	743	759	89	7	23	39	55	57	73	579	497	513	529	545	547	563	9261
755	757	773	691	707	723	739	69	71	87	5	21	37	53	559	561	577	495	511	527	543	9261
721	737	753	769	771	689	705	35	51	67	83	85	3	19	525	541	557	573	575	493	509	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

Example 30

In this case the magic sum is $\mathbf{S_{21\times21} := 9261 = 21^3}$, and sum of numbers is $\mathbf{F_{441} := 194481 = 441^2 = 21^4}$. Each block of order 7 are **pan diagonal magic square** with different magic sums forming again a magic square of order 3:

			9261
2401	5831	1029	9261
1715	3087	4459	9261
5145	343	3773	9261
9261	9261	9261	9261

$= 7^3 \times$

			27
7	17	3	27
5	9	13	27
15	1	11	27
27	27	27	27

Example 31

Interestingly, the same happens with the **sum of all members of each block of order 7** resulting again in a magic square of order 3:

			64827
16807	40817	7203	64827
12005	21609	31213	64827
36015	2401	26411	64827
64827	64827	64827	64827

$= 7^4 \times$

			27
7	17	3	27
5	9	13	27
15	1	11	27
27	27	27	27

Example 32

●Second way:

Proceeding similar lines of order 18, we can also calculate second way of writing magic square of order 21:

																					9261
7	151	295	439	583	727	871	17	161	305	449	593	737	881	3	147	291	435	579	723	867	9261
709	853	115	133	277	421	565	719	863	125	143	287	431	575	705	849	111	129	273	417	561	9261
403	547	691	835	97	241	259	413	557	701	845	107	251	269	399	543	687	831	93	237	255	9261
223	367	385	529	673	817	79	233	377	395	539	683	827	89	219	363	381	525	669	813	75	9261
799	61	205	349	493	511	655	809	71	215	359	503	521	665	795	57	201	345	489	507	651	9261
619	637	781	43	187	331	475	629	647	791	53	197	341	485	615	633	777	39	183	327	471	9261
313	457	601	745	763	25	169	323	467	611	755	773	35	179	309	453	597	741	759	21	165	9261
5	149	293	437	581	725	869	9	153	297	441	585	729	873	13	157	301	445	589	733	877	9261
707	851	113	131	275	419	563	711	855	117	135	279	423	567	715	859	121	139	283	427	571	9261
401	545	689	833	95	239	257	405	549	693	837	99	243	261	409	553	697	841	103	247	265	9261
221	365	383	527	671	815	77	225	369	387	531	675	819	81	229	373	391	535	679	823	85	9261
797	59	203	347	491	509	653	801	63	207	351	495	513	657	805	67	211	355	499	517	661	9261
617	635	779	41	185	329	473	621	639	783	45	189	333	477	625	643	787	49	193	337	481	9261
311	455	599	743	761	23	167	315	459	603	747	765	27	171	319	463	607	751	769	31	175	9261
15	159	303	447	591	735	879	1	145	289	433	577	721	865	11	155	299	443	587	731	875	9261
717	861	123	141	285	429	573	703	847	109	127	271	415	559	713	857	119	137	281	425	569	9261
411	555	699	843	105	249	267	397	541	685	829	91	235	253	407	551	695	839	101	245	263	9261
231	375	393	537	681	825	87	217	361	379	523	667	811	73	227	371	389	533	677	821	83	9261
807	69	213	357	501	519	663	793	55	199	343	487	505	649	803	65	209	353	497	515	659	9261
627	645	789	51	195	339	483	613	631	775	37	181	325	469	623	641	785	47	191	335	479	9261
321	465	609	753	771	33	177	307	451	595	739	757	19	163	317	461	605	749	767	29	173	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

Example 33

In this case the magic sum is $\mathbf{S_{21\times21} := 9261 = 21^3}$, and sum of numbers is $\mathbf{F_{441} := 194481 = 441^2 = 21^4}$. Each block of order 7 is a **pan diagonal magic square** with different magic sums forming again a magic square of order 3:

			9261					1323
3073	3143	3045	9261	$=7\times$	439	449	435	1323
3059	3087	3115	9261		437	441	445	1323
3129	3031	3101	9261		447	433	443	1323
9261	9261	9261	9261		1323	1323	1323	1323

Example 34

Interestingly, the same happens with the **sum of all members of each block of order 7** resulting again in a magic square of order 3:

			64827					1323
21511	22001	21315	64827	$=7^2\times$	439	449	435	1323
21413	21609	21805	64827		437	441	445	1323
21903	21217	21707	64827		447	433	443	1323
64827	64827	64827	64827		1323	1323	1323	1323

Example 35

18.2 Second Approach – Sequential

According to formula (8), we have

$$K := T(n) - T(n-441), \qquad n > 441,$$

i.e.,

$$K := \frac{n(n+1)}{2} - \frac{(n-441)(n-440)}{2}$$
$$:= 441n - 97020 = 441(n-220).$$

Take $n = 661$, we get a perfect square, i.e.,

$$K := T(661) - T(220) = 194481 = 441^2 = 21^4.$$

Simplifying, one get

$$\mathbf{221 + 222 + ... + 660 + 661 = 21^4 = 441^2 = 194481}. \qquad (33)$$

Thus, we have a perfect square sum for the 441 numbers starting from 221 to 661. According to values given in (33), we can easily construct magic square of order 21 of sequential values giving sum of all numbers as prefect sum.

																					9261
368	376	384	392	400	408	416	613	621	629	637	645	653	661	270	278	286	294	302	310	318	9261
407	415	374	375	383	391	399	652	660	619	620	628	636	644	309	317	276	277	285	293	301	9261
390	398	406	414	373	381	382	635	643	651	659	618	626	627	292	300	308	316	275	283	284	9261
380	388	389	397	405	413	372	625	633	634	642	650	658	617	282	290	291	299	307	315	274	9261
412	371	379	387	395	396	404	657	616	624	632	640	641	649	314	273	281	289	297	298	306	9261
402	403	411	370	378	386	394	647	648	656	615	623	631	639	304	305	313	272	280	288	296	9261
385	393	401	409	410	369	377	630	638	646	654	655	614	622	287	295	303	311	312	271	279	9261
319	327	335	343	351	359	367	417	425	433	441	449	457	465	515	523	531	539	547	555	563	9261
358	366	325	326	334	342	350	456	464	423	424	432	440	448	554	562	521	522	530	538	546	9261
341	349	357	365	324	332	333	439	447	455	463	422	430	431	537	545	553	561	520	528	529	9261
331	339	340	348	356	364	323	429	437	438	446	454	462	421	527	535	536	544	552	560	519	9261
363	322	330	338	346	347	355	461	420	428	436	444	445	453	559	518	526	534	542	543	551	9261
353	354	362	321	329	337	345	451	452	460	419	427	435	443	549	550	558	517	525	533	541	9261
336	344	352	360	361	320	328	434	442	450	458	459	418	426	532	540	548	556	557	516	524	9261
564	572	580	588	596	604	612	221	229	237	245	253	261	269	466	474	482	490	498	506	514	9261
603	611	570	571	579	587	595	260	268	227	228	236	244	252	505	513	472	473	481	489	497	9261
586	594	602	610	569	577	578	243	251	259	267	226	234	235	488	496	504	512	471	479	480	9261
576	584	585	593	601	609	568	233	241	242	250	258	266	225	478	486	487	495	503	511	470	9261
608	567	575	583	591	592	600	265	224	232	240	248	249	257	510	469	477	485	493	494	502	9261
598	599	607	566	574	582	590	255	256	264	223	231	239	247	500	501	509	468	476	484	492	9261
581	589	597	605	606	565	573	238	246	254	262	263	222	230	483	491	499	507	508	467	475	9261
9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261	9261

Example 36

In this case, the magic sum is $\mathbf{S_{21\times21} := 9261 = 21^3}$, and sum of number is $\mathbf{F_{441} := 50625 = 441^2 = 21^4}$. $\mathbf{F_{441} := 194481 = 441^2 = 21^4}$. In this case each block of order 7 is a pan diagonal magic square with different magic sums forming again a magic square of order 3:

			9261					27
2744	4459	2058	9261	$= 7^3 \times$	8	13	6	27
2401	3087	3773	9261		7	9	11	27
4116	1715	3430	9261		12	5	10	27
9261	9261	9261	9261		27	27	27	27

Example 37

Interestingly, the same happens with the **sum of all members of each block of order 7** resulting again in a magic square of order 3:

			64827
19208	31213	14406	64827
16807	21609	26411	64827
28812	12005	24010	64827
64827	64827	64827	64827

$= 7^4 \times$

			27
8	13	6	27
7	9	11	27
12	5	10	27
27	27	27	27

Example 38

Proceeding similar lines of order 18, we can also calculate second way of writing magic square of order 21:

Still, there are more possibilities of getting sequential numbers for the magic square of order 21, where numbers sum is a perfect square. See below examples,

$$\mathbf{T(704) - T(263) = 264 + 265 + ... + 704 = 213444 = 462^2};$$
$$\mathbf{T(749) - T(308) = 309 + 310 + ... + 749 = 233289 = 483^2};$$
$$\mathbf{T(796) - T(355) = 356 + 357 + ... + 796 = 254016 = 504^2}.$$

19. Magic Square of Order 22

According to (8) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 22. Take $n = 484$ in (5) or $k = 22$ in (6), we get

$$1 + 3 + 5 + ... + (2 \times 22^2 - 1) = 22^4$$
$$\Rightarrow \quad \mathbf{1 + 3 + 5 + ... + 965 + 967 = 22^4 = 234256}. \qquad (34)$$

According to values given in (34), the magic square of order 22 for odd numbers is given by

1	187	151	399	227	773	953	913	863	823	49	97	475	523	395	345	305	549	719	683	629	589
229	47	11	195	401	869	819	779	949	909	95	133	351	301	481	519	391	639	585	545	725	675
403	221	93	55	19	955	905	865	825	775	141	179	525	387	347	307	477	721	681	631	595	541
63	405	223	139	99	821	781	951	911	861	177	5	303	483	521	393	343	587	551	717	677	637
143	107	397	225	185	907	867	817	777	957	3	51	389	349	299	479	527	673	633	593	543	727
563	743	703	653	613	231	499	459	419	365	325	285	749	929	889	839	799	25	205	165	115	75
659	609	569	739	699	331	277	237	505	451	411	371	845	795	755	925	885	121	71	31	201	161
745	695	655	615	565	417	363	323	283	243	497	457	931	881	841	801	751	207	157	117	77	27
611	571	741	701	651	503	463	409	369	329	275	235	797	757	927	887	837	73	33	203	163	113
697	657	607	567	747	281	241	495	455	415	375	321	883	843	793	753	933	159	119	69	29	209
91	137	183	9	45	367	327	287	233	501	461	407	643	689	735	561	597	873	919	965	791	827
181	7	53	89	135	453	413	373	319	279	239	507	733	559	605	641	687	963	789	835	871	917
769	939	903	849	809	35	215	175	125	85	601	649	553	715	679	423	251	441	489	361	311	271
859	805	765	945	895	131	81	41	211	171	647	685	253	599	539	723	425	317	267	447	485	357
941	901	851	815	761	217	167	127	87	37	693	731	427	245	645	583	547	491	353	313	273	443
807	771	937	897	857	83	43	213	173	123	729	557	591	429	247	691	627	269	449	487	359	309
893	853	813	763	947	169	129	79	39	219	555	603	671	635	421	249	737	355	315	265	445	493
465	513	385	335	295	529	709	669	619	579	831	879	21	191	155	101	61	783	935	899	433	261
341	291	471	509	381	625	575	535	705	665	877	915	111	57	17	197	147	263	829	759	943	435
515	377	337	297	467	711	661	621	581	531	923	961	193	153	103	67	13	437	255	875	803	767
293	473	511	383	333	577	537	707	667	617	959	787	59	23	189	149	109	811	439	257	921	847
379	339	289	469	517	663	623	573	533	713	785	833	145	105	65	15	199	891	855	431	259	967

Example 39

In this case the magic sum $\mathbf{S_{22\times22} := 10648 = 22^3}$. Also, the sum of all 484 numbers is $\mathbf{F_{484} := 234256 = 484^2 = 22^4}$. There are 12 block of order 4 and 1 of order 7 are **magic squares**. These are given by **bold face letters**. The approach applied to construct this magic square is based on the procedure of H. White [9]. This approach is by used of computer program.

20. Magic Square of Order 24

According to (7) we don't have sequential magic square with number sum a perfect square. We will obtain only odd numbers magic square of order 24. Take $k = 24$ in (5), we get

$$1+3+5+...+(2\times 24^2-1) = 24^4$$
$$\Rightarrow \quad \mathbf{1+3+5+...+1149+1151 = 24^4 = 331776}. \tag{35}$$

There are many ways of constructing magic square of order 24 with odd order numbers, for example, $8\times3, 3\times8, 4\times6$ or 6×4. Instead of using all the four ways, we shall use only two. In the case of 8×3, we can work with bimagic square of order 8 as given in Section 8. See below:

● First way 8×3:

277	727	637	79	475	1105	979	313	287	737	647	89	485	1115	989	323	273	723	633	75	471	1101	975	309
457	1123	961	331	223	781	583	133	467	1133	971	341	233	791	593	143	453	1119	957	327	219	777	579	129
7	709	799	205	385	907	1033	547	17	719	809	215	395	917	1043	557	3	705	795	201	381	903	1029	543
403	889	1051	529	61	655	853	151	413	899	1061	539	71	665	863	161	399	885	1047	525	57	651	849	147
673	43	169	835	871	421	511	1069	683	53	179	845	881	431	521	1079	669	39	165	831	867	417	507	1065
925	367	565	1015	691	25	187	817	935	377	575	1025	701	35	197	827	921	363	561	1011	687	21	183	813
763	241	115	601	1141	439	349	943	773	251	125	611	1151	449	359	953	759	237	111	597	1137	435	345	939
1087	493	295	997	745	259	97	619	1097	503	305	1007	755	269	107	629	1083	489	291	993	741	255	93	615
275	725	635	77	473	1103	977	311	279	729	639	81	477	1107	981	315	283	733	643	85	481	1111	985	319
455	1121	959	329	221	779	581	131	459	1125	963	333	225	783	585	135	463	1129	967	337	229	787	589	139
5	707	797	203	383	905	1031	545	9	711	801	207	387	909	1035	549	13	715	805	211	391	913	1039	553
401	887	1049	527	59	653	851	149	405	891	1053	531	63	657	855	153	409	895	1057	535	67	661	859	157
671	41	167	833	869	419	509	1067	675	45	171	837	873	423	513	1071	679	49	175	841	877	427	517	1075
923	365	563	1013	689	23	185	815	927	369	567	1017	693	27	189	819	931	373	571	1021	697	31	193	823
761	239	113	599	1139	437	347	941	765	243	117	603	1143	441	351	945	769	247	121	607	1147	445	355	949
1085	491	293	995	743	257	95	617	1089	495	297	999	747	261	99	621	1093	499	301	1003	751	265	103	625
285	735	645	87	483	1113	987	321	271	721	631	73	469	1099	973	307	281	731	641	83	479	1109	983	317
465	1131	969	339	231	789	591	141	451	1117	955	325	217	775	577	127	461	1127	965	335	227	785	587	137
15	717	807	213	393	915	1041	555	1	703	793	199	379	901	1027	541	11	713	803	209	389	911	1037	551
411	897	1059	537	69	663	861	159	397	883	1045	523	55	649	847	145	407	893	1055	533	65	659	857	155
681	51	177	843	879	429	519	1077	667	37	163	829	865	415	505	1063	677	47	173	839	875	425	515	1073
933	375	573	1023	699	33	195	825	919	361	559	1009	685	19	181	811	929	371	569	1019	695	29	191	821
771	249	123	609	1149	447	357	951	757	235	109	595	1135	433	343	937	767	245	119	605	1145	443	353	947
1095	501	303	1005	753	267	105	627	1081	487	289	991	739	253	91	613	1091	497	299	1001	749	263	101	623

Example 40

In this case the magic sum $\mathbf{S_{24\times24} := 13824 = 24^3}$. Also, the sum of all 576 numbers is $\mathbf{F_{576} := 331776 = 576^2 = 24^4}$.

The construction of above magic square is based on the procedure of making combinations of magic square of order 8 with order 3. The procedure used is the one given in section 8 of bimagic square of order 8 combining with order 3 given in section 3. Below are the details of each block of order 8 forming a magic square of order 3:

			13824					864
4592	4672	4560	13824	$=2^4\times$	287	292	285	864
4576	4608	4640	13824		286	288	290	864
4656	4544	4624	13824		291	284	289	864
13824	13824	13824	13824		864	864	864	864

Example 41

The magic squares formed by **sum of all members each block of order 8** forms again a magic square of order 3:

			110592					864
36736	37376	36480	110592	$=2^7\times$	287	292	285	864
36608	36864	37120	110592		286	288	290	864
37248	36352	36992	110592		291	284	289	864
110592	110592	110592	110592		864	864	864	864

Example 42

• Second way 6×4:

433	793	1	937	477	837	45	981	487	847	55	991	499	859	67	1003	465	825	33	969	447	807	15	951
73	865	505	721	117	909	549	765	127	919	559	775	139	931	571	787	105	897	537	753	87	879	519	735
1081	145	649	289	1125	189	693	333	1135	199	703	343	1147	211	715	355	1113	177	681	321	1095	159	663	303
577	361	1009	217	621	405	1053	261	631	415	1063	271	643	427	1075	283	609	393	1041	249	591	375	1023	231
489	849	57	993	445	805	13	949	501	861	69	1005	459	819	27	963	473	833	41	977	441	801	9	945
129	921	561	777	85	877	517	733	141	933	573	789	99	891	531	747	113	905	545	761	81	873	513	729
1137	201	705	345	1093	157	661	301	1149	213	717	357	1107	171	675	315	1121	185	689	329	1089	153	657	297
633	417	1065	273	589	373	1021	229	645	429	1077	285	603	387	1035	243	617	401	1049	257	585	369	1017	225
455	815	23	959	443	803	11	947	457	817	25	961	485	845	53	989	493	853	61	997	475	835	43	979
95	887	527	743	83	875	515	731	97	889	529	745	125	917	557	773	133	925	565	781	115	907	547	763
1103	167	671	311	1091	155	659	299	1105	169	673	313	1133	197	701	341	1141	205	709	349	1123	187	691	331
599	383	1031	239	587	371	1019	227	601	385	1033	241	629	413	1061	269	637	421	1069	277	619	403	1051	259
495	855	63	999	463	823	31	967	439	799	7	943	479	839	47	983	451	811	19	955	481	841	49	985
135	927	567	783	103	895	535	751	79	871	511	727	119	911	551	767	91	883	523	739	121	913	553	769
1143	207	711	351	1111	175	679	319	1087	151	655	295	1127	191	695	335	1099	163	667	307	1129	193	697	337
639	423	1071	279	607	391	1039	247	583	367	1015	223	623	407	1055	263	595	379	1027	235	625	409	1057	265
469	829	37	973	497	857	65	1001	453	813	21	957	437	797	5	941	491	851	59	995	461	821	29	965
109	901	541	757	137	929	569	785	93	885	525	741	77	869	509	725	131	923	563	779	101	893	533	749
1117	181	685	325	1145	209	713	353	1101	165	669	309	1085	149	653	293	1139	203	707	347	1109	173	677	317
613	397	1045	253	641	425	1073	281	597	381	1029	237	581	365	1013	221	635	419	1067	275	605	389	1037	245
467	827	35	971	483	843	51	987	471	831	39	975	449	809	17	953	435	795	3	939	503	863	71	1007
107	899	539	755	123	915	555	771	111	903	543	759	89	881	521	737	75	867	507	723	143	935	575	791
1115	179	683	323	1131	195	699	339	1119	183	687	327	1097	161	665	305	1083	147	651	291	1151	215	719	359
611	395	1043	251	627	411	1059	267	615	399	1047	255	593	377	1025	233	579	363	1011	219	647	431	1079	287

Example 43

In this case also the magic sum $\mathbf{S_{24\times24}:=13824=24^3}$. Also, the sum of all 576 numbers is $\mathbf{F_{576}:=331776=576^2=24^4}$.

The construction of above magic square is based on the procedure of making combinations of magic square of order 4 with order 6. The procedure used is the one given in section 4. Each block of order 4 is perfect pan diagonal magic square of order 4 with different magic sums. These magic sums again make a magic square of order 6:

						13824
2164	2340	2380	2428	2292	2220	13824
2388	2212	2436	2268	2324	2196	13824
2252	2204	2260	2372	2404	2332	13824
2412	2284	2188	2348	2236	2356	13824
2308	2420	2244	2180	2396	2276	13824
2300	2364	2316	2228	2172	2444	13824
13824	13824	13824	13824	13824	13824	13824

$= 4 \times$

						3456
541	585	595	607	573	555	3456
597	553	609	567	581	549	3456
563	551	565	593	601	583	3456
603	571	547	587	559	589	3456
577	605	561	545	599	569	3456
575	591	579	557	543	611	3456
3456	3456	3456	3456	3456	3456	3456

Example 44

The magic squares formed by **sum of all members each block of order 4** forms again a magic square of order 6:

						55296
8656	9360	9520	9712	9168	8880	55296
9552	8848	9744	9072	9296	8784	55296
9008	8816	9040	9488	9616	9328	55296
9648	9136	8752	9392	8944	9424	55296
9232	9680	8976	8720	9584	9104	55296
9200	9456	9264	8912	8688	9776	55296
55296	55296	55296	55296	55296	55296	55296

$= 4^2 \times$

						3456
541	585	595	607	573	555	3456
597	553	609	567	581	549	3456
563	551	565	593	601	583	3456
603	571	547	587	559	589	3456
577	605	561	545	599	569	3456
575	591	579	557	543	611	3456
3456	3456	3456	3456	3456	3456	3456

Example 45

Interestingly, the magic sum of second magic square in both the cases is **sequential digits 3456**.

21. Magic Square of Order 25

According to (5) and (8), one can have a perfect square sum magic square of order 25 in two ways. One using odd order numbers and secondly having sequential values. Below are examples of both type.

21.1 First Approach – Odd Numbers

Take $n = 25^2 = 625$ in (5) or $k = 25$ in (6), we get

$$1+3+5+...+(2\times 25^2-1)=21^4$$

$$\Rightarrow \qquad \mathbf{1+3+5+7+...+1247+1249 = 625^2 = 390625}\,. \tag{36}$$

According to values given in (36), the magic square of order 25 is given by

1	337	613	949	1225	885	1161	247	273	559	469	545	821	1107	183	1093	129	405	731	767	677	953	1039	65	391
913	1249	25	301	637	297	573	859	1185	211	1121	157	483	519	845	705	781	1067	143	429	89	365	691	977	1003
325	601	937	1213	49	1159	235	261	597	873	533	819	1145	171	457	117	443	729	755	1081	991	1027	53	389	665
1237	13	349	625	901	561	897	1173	209	285	195	471	507	833	1119	779	1055	131	417	743	353	689	965	1041	77
649	925	1201	37	313	223	259	585	861	1197	807	1133	169	495	521	431	717	793	1079	105	1015	91	377	653	989
1069	145	421	707	783	693	979	1005	81	367	27	303	639	915	1241	851	1187	213	299	575	485	511	847	1123	159
721	757	1083	119	445	55	381	667	993	1029	939	1215	41	327	603	263	599	875	1151	237	1147	173	459	535	811
133	419	745	771	1057	967	1043	79	355	681	341	627	903	1239	15	1175	201	287	563	899	509	835	1111	197	473
795	1071	107	433	719	379	655	981	1017	93	1203	39	315	641	927	587	863	1199	225	251	161	497	523	809	1135
407	733	769	1095	121	1031	67	393	679	955	615	941	1227	3	339	249	275	551	887	1163	823	1109	185	461	547
877	1153	239	265	591	451	537	813	1149	175	1085	111	447	723	759	669	995	1021	57	383	43	329	605	931	1217
289	565	891	1177	203	1113	199	475	501	837	747	773	1059	135	411	71	357	683	969	1045	905	1231	17	343	629
1191	227	253	589	865	525	801	1137	163	499	109	435	711	797	1073	983	1019	95	371	657	317	643	929	1205	31
553	889	1165	241	277	187	463	549	825	1101	761	1097	123	409	735	395	671	957	1033	69	1229	5	331	617	943
215	291	577	853	1189	849	1125	151	487	513	423	709	785	1061	147	1007	83	369	695	971	631	917	1243	29	305
685	961	1047	73	359	19	345	621	907	1233	893	1179	205	281	567	477	503	839	1115	191	1051	137	413	749	775
97	373	659	985	1011	921	1207	33	319	645	255	581	867	1193	229	1139	165	491	527	803	713	799	1075	101	437
959	1035	61	397	673	333	619	945	1221	7	1167	243	279	555	881	541	827	1103	189	465	125	401	737	763	1099
361	697	973	1009	85	1245	21	307	633	919	579	855	1181	217	293	153	489	515	841	1127	787	1063	149	425	701
1023	59	385	661	997	607	933	1219	45	321	231	267	593	879	1155	815	1141	177	453	539	449	725	751	1087	113
493	529	805	1131	167	1077	103	439	715	791	651	987	1013	99	375	35	311	647	923	1209	869	1195	221	257	583
1105	181	467	543	829	739	765	1091	127	403	63	399	675	951	1037	947	1223	9	335	611	271	557	883	1169	245
517	843	1129	155	481	141	427	703	789	1065	975	1001	87	363	699	309	635	911	1247	23	1183	219	295	571	857
179	455	531	817	1143	753	1089	115	441	727	387	663	999	1025	51	1211	47	323	609	935	595	871	1157	233	269
831	1117	193	479	505	415	741	777	1053	139	1049	75	351	687	963	623	909	1235	11	347	207	283	569	895	1171

Example 43

In this case the magic sum is $\mathbf{S_{25\times25} := 15625 = 25^3}$, and sum of numbers is $\mathbf{F_{625} := 390625 = 625^2 = 25^4}$. In this the magic square is pan diagonal and bimagic with bimagic sum $\mathbf{Sb_{25\times25} := 13020825}$. Moreover each block of order 5 is a pan diagonal magic square constant magic sum $\mathbf{S_{5\times5} := 3125 = 5^5}$.

21.2 Second Approach – Sequential

According to formula (7), we have

$$K := T(n) - T(n-625), \quad n > 625,$$

i.e.,

$$K := \frac{n(n+1)}{2} - \frac{(n-625)(n-624)}{2}$$
$$:= 625n - 195000 = 625(n-312).$$

Take $n = 937$, we get a perfect square, i.e.,

$$K(937) := T(937) - T(312) = 390625 = 625^2 = 25^4.$$

Simplifying, one get

$$\mathbf{313 + 314 + ... + 936 + 937 = 25^4 = 625^2 = 390625}. \tag{37}$$

Thus, we have a perfect square sum for the 625 numbers starting from 313 to 937. According to values given in (37), we can easily construct magic square of order 25 of sequential values giving sum of all numbers as prefect sum.

313	481	619	787	925	755	893	436	449	592	547	585	723	866	404	859	377	515	678	696	651	789	832	345	508
769	937	325	463	631	461	599	742	905	418	873	391	554	572	735	665	703	846	384	527	357	495	658	801	814
475	613	781	919	337	892	430	443	611	749	579	722	885	398	541	371	534	677	690	853	808	826	339	507	645
931	319	487	625	763	593	761	899	417	455	410	548	566	729	872	702	840	378	521	684	489	657	795	833	351
637	775	913	331	469	424	442	605	743	911	716	879	397	560	573	528	671	709	852	365	820	358	501	639	807
847	385	523	666	704	659	802	815	353	496	326	464	632	770	933	738	906	419	462	600	555	568	736	874	392
673	691	854	372	535	340	503	646	809	827	782	920	333	476	614	444	612	750	888	431	886	399	542	580	718
379	522	685	698	841	796	834	352	490	653	483	626	764	932	320	900	413	456	594	762	567	730	868	411	549
710	848	366	529	672	502	640	803	821	359	914	332	470	633	776	606	744	912	425	438	393	561	574	717	880
516	679	697	860	373	828	346	509	652	790	620	783	926	314	482	437	450	588	756	894	724	867	405	543	586
751	889	432	445	608	538	581	719	887	400	855	368	536	674	692	647	810	823	341	504	334	477	615	778	921
457	595	758	901	414	869	412	550	563	731	686	699	842	380	518	348	491	654	797	835	765	928	321	484	627
908	426	439	607	745	575	713	881	394	562	367	530	668	711	849	804	822	360	498	641	471	634	777	915	328
589	757	895	433	451	406	544	587	725	863	693	861	374	517	680	510	648	791	829	347	927	315	478	621	784
420	458	601	739	907	737	875	388	556	569	524	667	705	843	386	816	354	497	660	798	628	771	934	327	465
655	793	836	349	492	322	485	623	766	929	759	902	415	453	596	551	564	732	870	408	838	381	519	687	700
361	499	642	805	818	773	916	329	472	635	440	603	746	909	427	882	395	558	576	714	669	712	850	363	531
792	830	343	511	649	479	622	785	923	316	896	434	452	590	753	583	726	864	407	545	375	513	681	694	862
493	661	799	817	355	935	323	466	629	772	602	740	903	421	459	389	557	570	733	876	706	844	387	525	663
824	342	505	643	811	616	779	922	335	473	428	446	609	752	890	720	883	401	539	582	537	675	688	856	369
559	577	715	878	396	851	364	532	670	708	638	806	819	362	500	330	468	636	774	917	747	910	423	441	604
865	403	546	584	727	682	695	858	376	514	344	512	650	788	831	786	924	317	480	618	448	591	754	897	435
571	734	877	390	553	383	526	664	707	845	800	813	356	494	662	467	630	768	936	324	904	422	460	598	741
402	540	578	721	884	689	857	370	533	676	506	644	812	825	338	918	336	474	617	780	610	748	891	429	447
728	871	409	552	565	520	683	701	839	382	837	350	488	656	794	624	767	930	318	486	416	454	597	760	898

Example 44

In this case, also the magic sum is $\mathbf{S_{25\times25} := 15625 = 25^3}$, and sum of numbers is $\mathbf{F_{25^2} := 390625 = 625^2 = 25^4}$. The magic square is pan diagonal and bimagic with bimagic sum $\mathbf{Sb_{25\times25} := 10579425}$. Moreover each block of order 5 is a pan diagonal magic square constant magic sum $\mathbf{S_{5\times5} := 3125 = 5^5}$.

Still, there are more possibilities of getting sequential numbers for the magic square of order 25, where the numbers sum is a perfect square. See below examples,

$$\mathbf{T(988) - T(363) = 364 + 365 + ... + 988 = 422500 = 650^2};$$
$$\mathbf{T(1041) - T(416) = 417 + 418 + ... + 1041 = 455625 = 675^2};$$
$$\mathbf{T(1096) - T(471) = 472 + 473 + ... + 1096 = 490000 = 700^2}.$$

22. Summary

Summarizing the results obtained above, we lead to following interesting table:

Order of Magic Square	Number of Elements	Magic Sum, $S_{k\times k}$	Sum of all Numbers, F_{k^2}
3	3^2	3^3	3^4
4	4^2	4^3	4^4
5	5^2	5^3	5^4
6	6^2	6^3	6^4
7	7^2	7^3	7^4
8	8^2	8^3	8^4
9	9^2	9^3	9^4
10	10^2	10^3	10^4
11	11^2	11^3	11^4
12	12^2	12^3	12^4
13	13^2	13^3	13^4
14	14^2	14^3	14^4
15	15^2	15^3	15^4
16	16^2	16^3	16^4
17	17^2	17^3	17^4
18	18^2	18^3	18^4
19	19^2	19^3	19^4
20	20^2	20^3	20^4
21	21^2	21^3	21^4
22	22^2	22^3	22^4
23	23^2	23^3	23^4
24	24^2	24^3	24^4
25	25^2	25^3	25^4
...	...	...	...

Table 1

The above Table 1 is up to magic square of order 25. According to formulas given in Section 2, we can extend it for higher orders. From the above Table 1, we derive the following general values:

There exists magic squares following properties:	
Order of Magic Square:	k
Number of Elements:	k^2
Magic Sum:	k^3
Sum of all numbers:	k^4
This is true for all order magic squares, i.e., k = 3, 4, 5, ...	

Table 2

The examples of magic squares written above are the adaptations of author's work on ***intervally distributed magic squares*** studied in Taneja (2015a, 2015b, 2015c). For other kind of magic squares refer Taneja (2015d, 2015e).

The examples given for the magic square of orders 8, 9, 16 and 25 are ***bimagic squares***. The order 14 magic square is based on the approach due to Zhu [10], and the order 22 is based on the approach due to White [9].

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