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OSTROWSKI AND TRAPEZOID TYPE INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some Ostrowski and trapezoid type inequalities for the k-g-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

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and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b). \tag{1.2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I^{\alpha}_{a+,g}f(x),\ \ a<x\le b\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}S_{k,g,b-}f(x)&=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I^{\alpha}_{b-,g}f(x),\ \ a\le x<b,\end{aligned} \tag{1.4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [21, p. 100]

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [21, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.5}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.7}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.9}$$

for $a<x\le b$ and

$$E^{\alpha}_{b-,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)\,f(t)\,dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ x\in[a,b). \tag{1.12}$$

In [24], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x):=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ |x|<R,\ \text{with } R>0 \tag{1.13}$$

for ρ, $\lambda>0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f(x):=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right)f(t)\,dt,\ x>a \tag{1.14}$$

where ρ, $\lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f(x):=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right)f(t)\,dt,\ x<b \tag{1.15}$$

where ρ, $\lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t)=t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [22], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.17}$$

where $\alpha\in(0,1)$.

We observe that for $k(t)=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t\in\mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b \tag{1.19}$$

where $\alpha\in(0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha>0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+}f(x):=\int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt, \tag{1.20}$$

for $0<a<x\leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha}f(x):=\int_{x}^{b}(g(t)-g(x))^{\alpha-1}\ln(g(t)-g(x))\,g'(t)\,f(t)\,dt, \tag{1.21}$$

for $0<a\leq x<b$, where $\alpha>0$. These are obtained from (1.11) and (1.12) for the kernel $k(t)=t^{\alpha-1}\ln t$, $t>0$.

For $\alpha=1$ we get

$$\mathcal{L}_{g,a+}f(x):=\int_{a}^{x}\ln(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ 0<a<x\leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-}f(x):=\int_{x}^{b}\ln(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ 0<a\leq x<b. \tag{1.23}$$

For $g(t)=t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha}f(x):=\int_{a}^{x}(x-t)^{\alpha-1}\ln(x-t)\,f(t)\,dt,\ 0<a<x\leq b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha}f(x):=\int_{x}^{b}(t-x)^{\alpha-1}\ln(t-x)\,f(t)\,dt,\ 0<a\leq x<b, \tag{1.25}$$

$$\mathcal{L}_{a+}f(x):=\int_{a}^{x}\ln(x-t)\,f(t)\,dt,\ 0<a<x\leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-}f(x):=\int_{x}^{b}\ln(t-x)\,f(t)\,dt,\ 0<a\leq x<b. \tag{1.27}$$

In the recent paper [18] we obtained the following Ostrowski and trapezoid type inequalities for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$.

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Also let g be a strictly increasing function on (a,b), having a continuous derivative g'*

on (a,b). *Then we have*

$$\begin{aligned}
&\left|I_{x-,g}^{\alpha}f(a)+I_{x+,g}^{\alpha}f(b)\right. \\
&\qquad\left.-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right]f(x)\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x(g(t)-g(a))^{\alpha-1}g'(t)\bigvee_t^x(f)\,dt+\int_x^b(g(b)-g(t))^{\alpha-1}g'(t)\bigvee_x^t(f)\,dt\right] \\
&\qquad\leq\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_a^x(f)+(g(b)-g(x))^{\alpha}\bigvee_x^b(f)\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\end{cases}
\end{aligned}\tag{1.28}$$

and

$$\begin{aligned}
&\left|I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right. \\
&\qquad\left.-\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]\right| \\
&\leq\frac{1}{\Gamma(\alpha)}\left[\int_a^x(g(x)-g(t))^{\alpha-1}g'(t)\bigvee_a^t(f)\,dt+\int_x^b(g(t)-g(x))^{\alpha-1}g'(t)\bigvee_t^b(f)\,dt\right] \\
&\qquad\leq\frac{1}{\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_a^x(f)+(g(b)-g(x))^{\alpha}\bigvee_x^b(f)\right] \\
&\leq\frac{1}{\Gamma(\alpha+1)}\begin{cases}\left[\frac{1}{2}(g(b)-g(a))+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right]^{\alpha}\bigvee_a^b(f); \\ \\ \left((g(x)-g(a))^{\alpha p}+(g(b)-g(x))^{\alpha p}\right)^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\left((g(x)-g(a))^{\alpha}+(g(b)-g(x))^{\alpha}\right)\end{cases}
\end{aligned}\tag{1.29}$$

for any $x\in(a,b)$.

For applications to the classical Riemann-Liouville fractional integrals, Hadamard fractional integrals and Harmonic fractional integrals see [18].

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [19]-[32] and the references therein.

Motivated by the above results, in this paper we establish some Ostrowski and trapezoid type inequalities for the k-g-fractional integrals of functions of bounded variation. Applications for mid-point and trapezoid inequalities are provided as well. Some examples for a general exponential fractional integral are also given.

2. Some Identities for the Operator $S_{k,g,a+,b-}$

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} S_{k,g,a+,b-}f(x) &:= \frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right] \\ &= \frac{1}{2}\left[\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt+\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt\right] \end{aligned} \tag{2.1}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

The following two parameters representation for the operator $S_{k,g,a+,b-}$ holds:

Lemma 1. *With the above assumptions for k, g and f we have*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right] \\ &\quad+\frac{1}{2}\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &\quad+\frac{1}{2}\int_x^b k(g(t)-g(x))\,g'(t)\left[f(t)-\gamma\right]dt \end{aligned} \tag{2.2}$$

for any λ, $\gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$\left[K(g(x)-g(t))\right]'=K'(g(x)-g(t))\,(g(x)-g(t))'=-k(g(x)-g(t))\,g'(t)$$

for $t\in(a,x)$ and

$$\left[K(g(t)-g(x))\right]'=K'(g(t)-g(x))\,(g(t)-g(x))'=k(g(t)-g(x))\,g'(t)$$

for $t\in(x,b)$.

Therefore, for any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned} &\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &= \int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt-\lambda\int_a^x k(g(x)-g(t))\,g'(t)\,dt \\ &= S_{k,g,a+}f(x)+\lambda\int_a^x\left[K(g(x)-g(t))\right]'dt \\ &= S_{k,g,a+}f(x)+\lambda\left[K(g(x)-g(t))\right]\Big|_a^x=S_{k,g,a+}f(x)-\lambda K(g(x)-g(a)) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}(2.4)\qquad & \int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\gamma\right]dt \\ & =\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)dt \\ & =S_{k,g,b-}f\left(x\right)-\gamma\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]'dt \\ & =S_{k,g,b-}f\left(x\right)-\gamma\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]\Big|_x^b=S_{k,g,b-}f\left(x\right)-\gamma K\left(g\left(b\right)-g\left(x\right)\right)\end{aligned}$$

for $x\in(a,b)$.

If we add the equalities (2.3) and (2.4) and divide by 2 then we get the desired result (2.2). □

Corollary 1. *With the above assumptions for k, g and f we have the Ostrowski type identity*

$$\begin{aligned}(2.5)\qquad S_{k,g,a+,b-}f\left(x\right) & =\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)+K\left(g\left(x\right)-g\left(a\right)\right)\right]f\left(x\right) \\ & +\frac{1}{2}\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(x\right)\right]dt \\ & +\frac{1}{2}\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(x\right)\right]dt\end{aligned}$$

and the trapezoid type identity

$$\begin{aligned}(2.6)\qquad S_{k,g,a+,b-}f\left(x\right) & =\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)f\left(b\right)+K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)\right] \\ & +\frac{1}{2}\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt \\ & +\frac{1}{2}\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt\end{aligned}$$

for any $x\in(a,b)$.

For $x=\frac{a+b}{2}$ we can consider

$$\begin{aligned}(2.7)\qquad M_{k,g,a+,b-}f & :=S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \\ & =\frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\ & +\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g\left(t\right)-g\left(\frac{a+b}{2}\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

By (2.5) we have the representation

$$
\begin{aligned}
&M_{k,g,a+,b-}f \\
&= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right) \\
&+\frac{1}{2}\int_a^{\frac{a+b}{2}}k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt \\
&+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt
\end{aligned} \tag{2.8}
$$

and (2.6) we have

$$
\begin{aligned}
&M_{k,g,a+,b-}f \\
&= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)\right] \\
&+\frac{1}{2}\int_a^{\frac{a+b}{2}}k\left(g\left(\frac{a+b}{2}\right)-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt \\
&+\frac{1}{2}\int_{\frac{a+b}{2}}^b k\left(g(t)-g\left(\frac{a+b}{2}\right)\right)g'(t)\left[f(t)-f(b)\right]dt.
\end{aligned} \tag{2.9}
$$

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\, b\in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean.* If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean.* If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean.* If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p.* Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function.*

Using the g-mean of two numbers we can introduce

$$
\begin{aligned}
P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(M_g(a,b)\right) \\
&= \frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt \\
&+\frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt.
\end{aligned} \tag{2.10}
$$

Using (2.5) and (2.6) we have the representations

$$
\begin{aligned}
(2.11)\qquad & P_{k,g,a+,b-}f \\
&= K\left(\frac{g(b)-g(a)}{2}\right)f\left(M_g(a,b)\right) \\
&+\frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f\left(M_g(a,b)\right)\right]dt \\
&+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f\left(M_g(a,b)\right)\right]dt
\end{aligned}
$$

and

$$
\begin{aligned}
(2.12)\qquad & P_{k,g,a+,b-}f \\
&= K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2} \\
&+\frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)\left[f(t)-f(a)\right]dt \\
&+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)\left[f(t)-f(b)\right]dt.
\end{aligned}
$$

3. Some Identities for the Dual Operator $\breve{S}_{k,g,a+,b-}$

Observe that

$$
(3.1)\qquad S_{k,g,x+}f(b)=\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt,\ \ x\in[a,b)
$$

and

$$
(3.2)\qquad S_{k,g,x-}f(a)=\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt,\ \ x\in(a,b].
$$

Define also the mixed operator

$$
\begin{aligned}
(3.3)\qquad & \breve{S}_{k,g,a+,b-}f(x) \\
&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right] \\
&=\frac{1}{2}\left[\int_x^b k\left(g(b)-g(t)\right)g'(t)f(t)\,dt+\int_a^x k\left(g(t)-g(a)\right)g'(t)f(t)\,dt\right]
\end{aligned}
$$

for any $x\in(a,b)$.

Lemma 2. *With the above assumptions for k, g and f we have*

$$
\begin{aligned}
(3.4)\qquad \breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\lambda K\left(g(b)-g(x)\right)+\gamma K\left(g(x)-g(a)\right)\right] \\
&+\frac{1}{2}\int_a^x k\left(g(t)-g(a)\right)g'(t)\left[f(t)-\gamma\right]dt \\
&+\frac{1}{2}\int_x^b k\left(g(b)-g(t)\right)g'(t)\left[f(t)-\lambda\right]dt
\end{aligned}
$$

for any λ, $\gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$[K(g(b)-g(t))]' = K'(g(b)-g(t))(g(b)-g(t))' = -k(g(b)-g(t))g'(t)$$

for $t\in(x,b)$ and

$$[K(g(t)-g(a))]' = K'(g(t)-g(a))(g(t)-g(a))' = k(g(t)-g(a))g'(t)$$

for $t\in(a,x)$.

For any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned}\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]dt &\\ = \int_x^b k(g(b)-g(t))g'(t)f(t)dt - \lambda\int_x^b k(g(b)-g(t))g'(t)dt &\\ = S_{k,g,x+}f(b) + \lambda\int_x^b [K(g(b)-g(t))]' dt &\\ = S_{k,g,x+}f(b) - \lambda K(g(b)-g(x)) &\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]dt &\\ = \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma\int_a^x k(g(t)-g(a))g'(t)dt &\\ = \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma\int_a^x [K(g(t)-g(a))]' dt &\\ = \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma K(g(x)-g(a)) &\end{aligned} \tag{3.6}$$

for $x\in(a,b)$.

If we add the equalities (3.5) and (3.6) and divide by 2 then we get the desired result (3.4). □

Corollary 2. *With the assumptions of Lemma 2 we have the Ostrowski type identity*

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}[K(g(b)-g(x)) + K(g(x)-g(a))]f(x) \\ &+ \frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(x)]dt \\ &+ \frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(x)]dt\end{aligned} \tag{3.7}$$

and the trapezoid identity

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}[K(g(b)-g(x))f(b) + K(g(x)-g(a))f(a)] \\ &+ \frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-f(a)]dt \\ &+ \frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-f(b)]dt\end{aligned} \tag{3.8}$$

for $x\in(a,b)$.

For $x = \frac{a+b}{2}$ we can consider

$$\breve{M}_{k,g,a+,b-}f := \breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) = \frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt + \frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt. \tag{3.9}$$

Using the equalities (3.7) and (3.8), we have

$$\begin{aligned}\breve{M}_{k,g,a+,b-}f &= \frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(\frac{a+b}{2}\right)\right]dt\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\breve{M}_{k,g,a+,b-}f &= \frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)f\left(b\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)f\left(a\right)\right]\\ &+\frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(a\right)\right]dt\\ &+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(b\right)\right]dt.\end{aligned} \tag{3.11}$$

Using the g-mean of two numbers we can introduce

$$\breve{P}_{k,g,a+,b-}f := \breve{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right) = \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt + \frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt. \tag{3.12}$$

Using the equalities (3.7) and (3.8), we have

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f &= K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_g\left(a,b\right)\right)\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-f\left(M_g\left(a,b\right)\right)\right]dt\end{aligned} \tag{3.13}$$

and

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(b)+f(a)}{2} + \frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,[f(t)-f(a)]\,dt + \frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,[f(t)-f(b)]\,dt. \tag{3.14}$$

4. Trapezoid Functional $T_{k,g,a+,b-}$

We can also introduce the functional

$$T_{k,g,a+,b-}f := \frac{1}{2}\left[S_{k,g,a+}f(b) + S_{k,g,b-}f(a)\right] = \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,f(t)\,dt. \tag{4.1}$$

We have:

Lemma 3. *With the assumption of Lemma 1, we have*

$$T_{k,g,a+,b-}f = K(g(b)-g(a))\,\delta + \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,[f(t)-\delta]\,dt \tag{4.2}$$

for any $\delta \in \mathbb{C}$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,dt \\
&= \int_a^b k(g(b)-g(t))\,g'(t)\,dt + \int_a^b k(g(t)-g(a))\,g'(t)\,dt \\
&= -\int_a^b \left[K(g(b)-g(t))\right]' dt + \int_a^b \left[K(g(t)-g(a))\right]' dt \\
&= -\left.K(g(b)-g(t))\right|_a^b + \left.K(g(t)-g(a))\right|_a^b \\
&= K(g(b)-g(a)) + K(g(b)-g(a)) = 2K(g(b)-g(a)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,[f(t)-\delta]\,dt \\
&= \frac{1}{2}\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,f(t)\,dt \\
&- \frac{1}{2}\delta\int_a^b \left[k(g(b)-g(t)) + k(g(t)-g(a))\right] g'(t)\,dt \\
&= T_{k,g,a+,b-}f - \delta K(g(b)-g(a)),
\end{aligned}$$

which proves the desired equality (4.2). □

Corollary 3. *With the assumptions of Lemma 3 we have the Ostrowski type identity*

$$\begin{aligned} &T_{k,g,a+,b-}f \\ &= K(g(b)-g(a))\,f(x) \\ &+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\,[f(t)-f(x)]\,dt \end{aligned} \tag{4.3}$$

for any $x\in[a,b]$ and the trapezoid identity

$$\begin{aligned} &T_{k,g,a+,b-}f \\ &= K(g(b)-g(a))\,\frac{f(a)+f(b)}{2} \\ &+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\left[f(t)-\frac{f(a)+f(b)}{2}\right]dt. \end{aligned} \tag{4.4}$$

We observe that for $x=\frac{a+b}{2}$ we obtain from (4.3) that

$$\begin{aligned} &T_{k,g,a+,b-}f \\ &= K(g(b)-g(a))\,f\left(\frac{a+b}{2}\right) \\ &+\frac{1}{2}\int_a^b [k(g(b)-g(t))+k(g(t)-g(a))]\,g'(t)\left[f(t)-f\left(\frac{a+b}{2}\right)\right]dt. \end{aligned} \tag{4.5}$$

5. Inequalities for Functions of Bounded Variation

We considered the cumulative function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases} \int_0^t k(s)\,ds \text{ if } 0<t, \\ \\ 0 \text{ if } t=0. \end{cases}$$

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases} \int_0^t |k(s)|\,ds \text{ if } 0<t, \\ \\ 0 \text{ if } t=0. \end{cases}$$

We observe that if k takes nonnegative values on $(0,\infty)$, as it does in some of the examples in Introduction, then $\mathbf{K}(t)=K(t)$ for $t\in[0,\infty)$.

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the Ostrowski type* *inequality*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \tag{5.1}$$
$$\le\frac{1}{2}\left[\int_x^b\left|k(g(t)-g(x))\right|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x\left|k(g(x)-g(t))\right|\bigvee_t^x(f)\,g'(t)\,dt\right]$$
$$\le\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\quad+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\le\frac{1}{2}\begin{cases}\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f);\\ \\ \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

and the trapezoid type inequality

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \tag{5.2}$$
$$\le\frac{1}{2}\left[\int_a^x\left|k(g(x)-g(t))\right|\bigvee_a^t(f)\,g'(t)\,dt+\int_x^b\left|k(g(t)-g(x))\right|\bigvee_t^b(f)\,g'(t)\,dt\right]$$
$$\le\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\le\frac{1}{2}\begin{cases}\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f);\\ \\ \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\\ \times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\\ \times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

for any $x\in(a,b)$.

Proof. Using the equality (2.5) we have

$$\begin{aligned}
(5.3)\qquad & \left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\left|\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-f(x)\right]dt\right| \\
&+\frac{1}{2}\left|\int_x^b k(g(t)-g(x))\,g'(t)\left[f(t)-f(x)\right]dt\right| \\
&\leq \frac{1}{2}\int_a^x \left|k(g(x)-g(t))\,g'(t)\left[f(t)-f(x)\right]\right|dt \\
&+\frac{1}{2}\int_x^b \left|k(g(t)-g(x))\,g'(t)\left[f(t)-f(x)\right]\right|dt \\
&=\frac{1}{2}\int_a^x \left|k(g(x)-g(t))\right|\left|f(x)-f(t)\right|g'(t)\,dt \\
&+\frac{1}{2}\int_x^b \left|k(g(t)-g(x))\right|\left|f(t)-f(x)\right|g'(t)\,dt \\
&=:B(x)
\end{aligned}$$

for $x\in(a,b)$.

Since f is of bounded variation, then

$$\left|f(x)-f(t)\right|\leq\bigvee_t^x(f)\leq\bigvee_a^x(f)\ \text{ for } a<t\leq x\leq b$$

and

$$\left|f(t)-f(x)\right|\leq\bigvee_x^t(f)\leq\bigvee_x^b(f)\ \text{ for } a\leq x\leq t<b.$$

Therefore

$$\begin{aligned}
B(x) &\leq \frac{1}{2}\int_a^x \left|k(g(x)-g(t))\right|\bigvee_t^x(f)\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b \left|k(g(t)-g(x))\right|\bigvee_x^t(f)\,g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x \left|k(g(x)-g(t))\right|g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b(f)\quad\int_x^b \left|k(g(t)-g(x))\right|g'(t)\,dt \\
&=:C(x)
\end{aligned}$$

for $x\in(a,b)$.

We have, by taking the derivative over t and using the chain rule, that

$$\left[\mathbf{K}(g(x)-g(t))\right]'=\mathbf{K}'(g(x)-g(t))\,(g(x)-g(t))'=-\left|k(g(x)-g(t))\right|g'(t)$$

for $t\in(a,x)$ and

$$\left[\mathbf{K}(g(t)-g(x))\right]'=\mathbf{K}'(g(t)-g(x))\,(g(t)-g(x))'=\left|k(g(t)-g(x))\right|g'(t)$$

for $t\in(x,b)$.

Then

$$\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt=-\int_a^x [\mathbf{K}(g(x)-g(t))]'\,dt=\mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))|\,g'(t)\,dt=\int_x^b [\mathbf{K}(g(t)-g(x))]'\,dt=\mathbf{K}(g(b)-g(x))$$

giving that

$$C(x)=\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\;\;+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right],$$

for $x\in(a,b)$, which proves the first and the second inequality in (5.1).

The last part of (4.2 is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq 0$

$$mc+nd\leq\begin{cases}\max\{m,n\}(c+d);\\ \\ (m^p+n^p)^{1/p}(c^q+d^q)^{1/q}\;\text{ with } p,\;q>1,\;\frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

Further, by the identity (2.6) we have, as above,

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right|\\
&\leq\frac{1}{2}\int_a^x |k(g(x)-g(t))|\,|f(t)-f(a)|\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\,|f(t)-f(b)|\,g'(t)\,dt\\
&\leq\frac{1}{2}\int_a^x |k(g(x)-g(t))|\bigvee_a^t(f)\,g'(t)\,dt\\
&+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\bigvee_t^b(f)\,g'(t)\,dt\\
&\leq\frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt\\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(t)-g(x))|\,g'(t)\,dt\\
&=\frac{1}{2}\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\frac{1}{2}\mathbf{K}(g(b)-g(x))\bigvee_x^b(f),
\end{aligned}$$

which proves (5.2). □

The following particular case for the functional

$$\begin{aligned}P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right)\\&= \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g\left(b\right)+g\left(a\right)}{2}-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(t\right)-\frac{g\left(b\right)+g\left(a\right)}{2}\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

is of interest:

Corollary 4. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}&\left|P_{k,g,a+,b-}f-K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)f\left(M_g\left(a,b\right)\right)\right| \\&\le \frac{1}{2}\int_{M_g(a,b)}^{b}\left|k\left(g\left(t\right)-\frac{g\left(b\right)+g\left(a\right)}{2}\right)\right|\bigvee_{M_g(a,b)}^{t}\left(f\right)g'\left(t\right)dt\\&+\frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g\left(b\right)+g\left(a\right)}{2}-g\left(t\right)\right)\right|\bigvee_{t}^{M_g(a,b)}\left(f\right)g'\left(t\right)dt\\&\le\frac{1}{2}\mathbf{K}\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\bigvee_b^b\left(f\right)\end{aligned}\tag{5.4}$$

and

$$\begin{aligned}&\left|P_{k,g,a+,b-}f-K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{f\left(b\right)+f\left(a\right)}{2}\right| \\&\le \frac{1}{2}\int_a^{M_g(a,b)}\left|k\left(\frac{g\left(b\right)+g\left(a\right)}{2}-g\left(t\right)\right)\right|\bigvee_{a}^{t}\left(f\right)g'\left(t\right)dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}\left|k\left(g\left(t\right)-\frac{g\left(b\right)+g\left(a\right)}{2}\right)\right|\bigvee_{t}^{b}\left(f\right)g'\left(t\right)dt\\&\le\frac{1}{2}\mathbf{K}\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\bigvee_b^b\left(f\right).\end{aligned}\tag{5.5}$$

We have:

Theorem 3. *With the assumptions of Theorem 2 we have the Ostrowski type inequality*

$$
\begin{aligned}
(5.6)\quad & \left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\le \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_t^x(f)\,g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_x^t(f)\,g'(t)\,dt \\
&\le \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right] \\
&\le \frac{1}{2}\begin{cases}
\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f); \\
\\
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned}
$$

and the trapezoid inequality

$$
\begin{aligned}
(5.7)\quad & \left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\le \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_a^t(f)\,g'(t)\,dt+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_t^b(f)\,g'(t)\,dt \\
&\le \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right] \\
&\le \frac{1}{2}\begin{cases}
\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f); \\
\\
\left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned}
$$

for any $x\in(a,b)$.

Proof. Using the identity (3.7) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(x)|\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(t)-f(x)|\,g'(t)\,dt \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_t^x(f)\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_x^t(f)\,g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt \\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right],
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.6).

By the identity (3.8) we have

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\,|f(t)-f(a)|\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\,|f(b)-f(t)|\,g'(t)\,dt \\
&\leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\bigvee_a^t(f)\,g'(t)\,dt \\
&+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\bigvee_t^b(f)\,g'(t)\,dt \\
&\leq \frac{1}{2}\bigvee_a^x(f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt \\
&+\frac{1}{2}\bigvee_x^b(f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt \\
&=\frac{1}{2}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right]
\end{aligned}$$

for any $x\in(a,b)$, which proves (5.7). □

Also, we have the particular inequalities for

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right)\\ &= \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt\\ &+\frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Corollary 5. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}(5.8)\quad &\left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{f\left(b\right)+f\left(a\right)}{2}\right|\\ &\leq\frac{1}{2}\int_{a}^{M_g(a,b)}\left|k\left(g\left(t\right)-g\left(a\right)\right)\right|\bigvee_{t}^{M_g(a,b)}\left(f\right)g'\left(t\right)dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b}\left|k\left(g\left(b\right)-g\left(t\right)\right)\right|\bigvee_{M_g(a,b)}^{t}\left(f\right)g'\left(t\right)dt\\ &\leq\frac{1}{2}\mathbf{K}\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\bigvee_{b}^{b}\left(f\right)\end{aligned}$$

and

$$\begin{aligned}(5.9)\quad &\left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{f\left(b\right)+f\left(a\right)}{2}\right|\\ &\leq\frac{1}{2}\int_{a}^{M_g(a,b)}\left|k\left(g\left(t\right)-g\left(a\right)\right)\right|\bigvee_{a}^{t}\left(f\right)g'\left(t\right)dt\\ &+\frac{1}{2}\int_{M_g(a,b)}^{b}\left|k\left(g\left(b\right)-g\left(t\right)\right)\right|\bigvee_{t}^{b}\left(f\right)g'\left(t\right)dt\\ &\leq\frac{1}{2}\mathbf{K}\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\bigvee_{b}^{b}\left(f\right).\end{aligned}$$

Finally, we have the following result for the trapezoid functional

$$\begin{aligned}T_{k,g,a+,b-}f &:= \frac{1}{2}\left[S_{k,g,a+}f\left(b\right)+S_{k,g,b-}f\left(a\right)\right]\\ &=\frac{1}{2}\int_{a}^{b}\left[k\left(g\left(b\right)-g\left(t\right)\right)+k\left(g\left(t\right)-g\left(a\right)\right)\right]g'\left(t\right)f\left(t\right)dt.\end{aligned}$$

Theorem 4. *With the assumptions of Theorem 2 we have the trapezoid type inequality*

$$(5.10)\quad\left|T_{k,g,a+,b-}f-K\left(g\left(b\right)-g\left(a\right)\right)\frac{f\left(a\right)+f\left(b\right)}{2}\right|\leq\frac{1}{2}\mathbf{K}\left(g\left(b\right)-g\left(a\right)\right)\bigvee_{a}^{b}\left(f\right).$$

Proof. From the identity (4.4) we have

$$
\begin{aligned}
&\left| T_{k,g,a+,b-}f - K\left(g\left(b\right) - g\left(a\right)\right)\frac{f\left(a\right) + f\left(b\right)}{2} \right| \qquad (5.11)\\
&\leq \frac{1}{2}\int_a^b \left| k\left(g\left(b\right) - g\left(t\right)\right) + k\left(g\left(t\right) - g\left(a\right)\right)\right| \left| f\left(t\right) - \frac{f\left(a\right) + f\left(b\right)}{2}\right| g'\left(t\right) dt \\
&\leq \frac{1}{2}\int_a^b \left[\left| k\left(g\left(b\right) - g\left(t\right)\right)\right| + \left| k\left(g\left(t\right) - g\left(a\right)\right)\right|\right] \left| f\left(t\right) - \frac{f\left(a\right) + f\left(b\right)}{2}\right| g'\left(t\right) dt \\
&=: D.
\end{aligned}
$$

Since $f : [a,b] \to \mathbb{C}$ is of bounded variation, then for any $t \in [a,b]$ we have

$$
\begin{aligned}
\left| f\left(t\right) - \frac{f\left(a\right) + f\left(b\right)}{2}\right| &= \left| \frac{f\left(t\right) - f\left(a\right) + f\left(t\right) - f\left(b\right)}{2}\right| \\
&\leq \frac{1}{2}\left[\left| f\left(t\right) - f\left(a\right)\right| + \left| f\left(b\right) - f\left(t\right)\right|\right] \leq \frac{1}{2}\bigvee_a^b \left(f\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D &\leq \frac{1}{4}\bigvee_a^b \left(f\right)\int_a^b \left[\left| k\left(g\left(b\right) - g\left(t\right)\right)\right| + \left| k\left(g\left(t\right) - g\left(a\right)\right)\right|\right] g'\left(t\right) dt \\
&= \frac{1}{4}\bigvee_a^b \left(f\right)\left[\mathbf{K}\left(g\left(b\right) - g\left(a\right)\right) + \mathbf{K}\left(g\left(b\right) - g\left(a\right)\right)\right] = \frac{1}{2}\mathbf{K}\left(g\left(b\right) - g\left(a\right)\right)\bigvee_a^b \left(f\right),
\end{aligned}
$$

which proves the desired result (5.10). □

6. Example for an Exponential Kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction.

If we take, for instance $k(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then we recapture the results for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as outlined in [18].

For α, $\beta \in \mathbb{R}$ we consider the kernel $k(t) := \exp\left[(\alpha + \beta i)t\right]$, $t \in \mathbb{R}$. We have

$$
K\left(t\right) = \frac{\exp\left[\left(\alpha + \beta i\right)t\right] - 1}{\left(\alpha + \beta i\right)}, \text{ if } t \in \mathbb{R}
$$

for α, $\beta \neq 0$.

Also, we have

$$
\left| k\left(s\right)\right| := \left|\exp\left[\left(\alpha + \beta i\right)s\right]\right| = \exp\left(\alpha s\right) \text{ for } s \in \mathbb{R}
$$

and

$$
\mathbf{K}\left(t\right) = \int_0^t \exp\left(\alpha s\right) ds = \frac{\exp\left(\alpha t\right) - 1}{\alpha} \text{ if } 0 < t,
$$

for $\alpha \neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We have

$$\begin{aligned}\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(x)&=\frac{1}{2}\int_a^x\exp\left[(\alpha+\beta i)(g(x)-g(t))\right]g'(t)f(t)\,dt\\&+\frac{1}{2}\int_x^b\exp\left[(\alpha+\beta i)(g(t)-g(x))\right]g'(t)f(t)\,dt\end{aligned}\tag{6.1}$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}&\kappa_{h,a+,b-}^{\alpha+\beta i}f(x)\\&:=\mathcal{E}_{\ln h,a+,b-}^{\alpha+\beta i}f(x)\\&=\frac{1}{2}\left[\int_a^x\left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt+\int_x^b\left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned}\tag{6.2}$$

for $x\in(a,b)$.

By using the inequality (5.1) we have for $x\in(a,b)$ that

$$\begin{aligned}&\Bigg|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(x)\\&-\frac{1}{2}\left[\frac{\exp\left[(\alpha+\beta i)(g(b)-g(x))\right]-1+\exp\left[(\alpha+\beta i)(g(x)-g(a))\right]-1}{(\alpha+\beta i)}\right]f(x)\Bigg|\\&\leq\frac{1}{2}\left[\int_x^b\exp\left(\alpha(g(t)-g(x))\right)g'(t)\bigvee_x^t(f)\,dt+\int_a^x\exp\left(\alpha(g(x)-g(t))\right)g'(t)\bigvee_t^x(f)\,dt\right]\\&\leq\frac{1}{2}\left[\frac{\exp\left(\alpha(g(b)-g(x))\right)-1}{\alpha}\bigvee_x^b(f)+\frac{\exp\left(\alpha(g(x)-g(a))\right)-1}{\alpha}\bigvee_a^x(f)\right]\\&\leq\frac{1}{2}\begin{cases}\max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha},\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]\left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\\text{with }p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]\left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}\end{aligned}\tag{6.3}$$

for $\alpha,\ \beta\in\mathbb{R}$ with $\alpha\neq0$.

By using the inequality (5.2) we also have for $x \in (a,b)$ that

$$\left|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2}\left[\frac{(\exp[(\alpha+\beta i)(g(b)-g(x))]-1) f(b) + (\exp[(\alpha+\beta i)(g(x)-g(a))]-1) f(a)}{(\alpha+\beta i)}\right]\right| \tag{6.4}$$
$$\leq \frac{1}{2}\left[\int_a^x \exp(\alpha(g(t)-g(x))) g'(t) \bigvee_a^t (f)\, dt + \int_x^b \exp(\alpha(g(x)-g(t))) g'(t) \bigvee_t^b (f)\, dt\right]$$
$$\leq \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \bigvee_x^b (f) + \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \bigvee_a^x (f)\right]$$
$$\leq \frac{1}{2}\begin{cases} \max\left\{\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}, \frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right\} \bigvee_a^b (f); \\ \\ \left[\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p + \left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p\right]^{1/p} \\ \times \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\frac{\exp(\alpha(g(b)-g(x)))-1+\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right] \\ \times \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases}$$

for α, $\beta \in \mathbb{R}$ with $\alpha \neq 0$.

If we denote

$$\begin{aligned} \overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f &:= \mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(M_g(a,b)) \\ &= \frac{1}{2}\int_a^x \exp\left[(\alpha+\beta i)\left(\frac{g(b)+g(a)}{2} - g(t)\right)\right] g'(t) f(t)\, dt \\ &+ \frac{1}{2}\int_x^b \exp\left[(\alpha+\beta i)\left(g(t) - \frac{g(b)+g(a)}{2}\right)\right] g'(t) f(t)\, dt \end{aligned}$$

then by (5.4) and (5.5) we have the simpler results

$$\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f - \frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)} f(M_g(a,b))\right| \tag{6.5}$$
$$\leq \frac{1}{2}\int_{M_g(a,b)}^b \exp\left(\alpha\left(g(t) - \frac{g(b)+g(a)}{2}\right)\right) g'(t) \bigvee_{M_g(a,b)}^t (f)\, dt$$
$$+ \frac{1}{2}\int_a^{M_g(a,b)} \exp\left(\alpha\left(\frac{g(b)+g(a)}{2} - g(t)\right)\right) g'(t) \bigvee_t^{M_g(a,b)} (f)\, dt$$
$$\leq \frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha} \bigvee_b^b (f)$$

and

$$\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i}f-\frac{\exp\left[(\alpha+\beta i)\,\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f\left(b\right)+f\left(a\right)}{2}\right| \tag{6.6}$$
$$\leq\frac{1}{2}\int_{a}^{M_{g}(a,b)}\exp\left(\alpha\left(g\left(t\right)-\frac{g\left(b\right)+g\left(a\right)}{2}\right)\right)g'\left(t\right)\bigvee_{a}^{t}\left(f\right)dt$$
$$+\frac{1}{2}\int_{M_{g}(a,b)}^{b}\exp\left(\alpha\left(\frac{g\left(b\right)+g\left(a\right)}{2}-g\left(t\right)\right)\right)g'\left(t\right)\bigvee_{t}^{b}\left(f\right)dt$$
$$\leq\frac{1}{2}\frac{\exp\left(\alpha\left(\frac{g(b)-g(a)}{2}\right)\right)-1}{\alpha}\bigvee_{b}^{b}\left(f\right).$$

In particular, if we take in (6.5) and (6.6) $g=\ln t$, $t\in[a,b]\subset(0,\infty)$, then by using the notation $G(\gamma,\delta):=\sqrt{\gamma\delta}$ for the *geometric mean* of the positive real numbers γ, $\delta>0$ we have

$$\left|\bar{\kappa}_{a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}f\left(G\left(a,b\right)\right)\right| \tag{6.7}$$
$$\leq\frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G\left(a,b\right)}\right)^{\alpha}\frac{1}{t}\bigvee_{G(a,b)}^{t}\left(f\right)dt$$
$$+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G\left(a,b\right)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{G(a,b)}\left(f\right)dt$$
$$\leq\frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}\left(f\right)$$

and

$$\left|\bar{\kappa}_{a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{b}{a}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\frac{f\left(b\right)+f\left(a\right)}{2}\right| \tag{6.8}$$
$$\leq\frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{G\left(a,b\right)}{t}\right)^{\alpha}\frac{1}{t}\bigvee_{t}^{b}\left(f\right)dt$$
$$+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{t}{G\left(a,b\right)}\right)^{\alpha}\frac{1}{t}\bigvee_{a}^{t}\left(f\right)dt$$
$$\leq\frac{1}{2}\frac{\left(\frac{b}{a}\right)^{\alpha}-1}{\alpha}\bigvee_{b}^{b}\left(f\right),$$

where

$$\bar{\kappa}_{a+,b-}^{\alpha+\beta i}f:=\frac{1}{2}\int_{G(a,b)}^{b}\left(\frac{t}{G\left(a,b\right)}\right)^{\alpha+\beta i}\frac{1}{t}f\left(t\right)dt+\frac{1}{2}\int_{a}^{G(a,b)}\left(\frac{G\left(a,b\right)}{t}\right)^{\alpha+\beta i}\frac{1}{t}f\left(t\right)dt.$$

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