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CAPUTO FRACTIONAL INTEGRAL INEQUALITIES VIA m-CONVEX FUNCTIONS

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ABSTRACT. In this paper, we prove the Hadamard type inequalities for m-convex functions via Caputo fractional derivatives and related inequalities. These results have some relationships with the Hadamard inequalities for convex functions and related inequalities.

1. INTRODUCTION

The concept of m-convexity was introduced by Toader in [19] and since then many properties, especially inequalities, have been obtained for this class of functions since they are defined (see, [9, 7, 13]).

Definition 1.1. A function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

If we take $m = 1$, then we recapture the concept of convex functions defined on $[0, b]$ and if we take $m = 0$, then we get the concept of starshaped functions on $[0, b]$. We recall that $f : [0, b] \to \mathbb{R}$ is called *starshaped* if

$$f(tx) \leq tf(x) \text{ for all } t \in [0, b] \text{ and } x \in [0, b].$$

Denote by $K_m(b)$ the set of the m-convex functions on $[0, b]$ for which $f(0) < 0$, then one has

$$K_1(b) \subset K_m(b) \subset K_0(b),$$

whenever $m \in (0, 1)$. Note that in the class $K_1(b)$ are only convex functions $f : [0, b] \to \mathbb{R}$ for which $f(0) \leq 0$ (see [7]).For more results and inequalities related to m-convex function one can consult for example [5, 10, 16] along with references.

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The Hadamard's inequality states that, if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

The Hadamard's inequality got attention of many mathematicians and many generalizations, extensions and refinements for different types of functions related to convex functions, have been proved so far for detail see, [2, 5, 7, 8, 10, 11, 16, 17].

Fractional calculus deals with the study of fractional order integrals and derivatives and their various applications. Different from classical or integer-order derivative, there are several kinds of definitions for fractional derivatives. These definitions are generally not equivalent with each other. Riemann-Liouville and Caputo are kinds of fractional derivatives.
In the realm of the fractional differential equations, Caputo derivative and Riemann–Liouville ones are mostly used. They generalize the ordinary integral and differential operators. However, the fractional derivatives have fewer properties than the corresponding classical ones. On the other hand, besides the smooth requirement, Caputo derivative does not coincide with the classical derivative [15].
Caputo-fractional derivatives are introduced by Italian mathematician Caputo in 1967.In the following we define Caputo fractional derivatives [14].

Definition 1.2. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, ...\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$${}^C D^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, x > a$$

and

$${}^C D^{\alpha}_{b-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)}\int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, x < b.$$

If $\alpha = n \in \{1, 2, 3, ...\}$ and usual derivative of f of order n exists, then Caputo fractional derivative $({}^C D^{\alpha}_{a+} f)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$({}^C D^{0}_{a+} f)(x) = ({}^C D^{0}_{b-} f)(x) = f(x)$$

where $n = 1$ and $\alpha = 0$.

For more results connected with Caputo-fractional derivatives one can consult [3, 1, 4].

In Section 2 we give the Hadamard inequality and some related inequalities for m-convex functions via Caputo-fractional derivatives. In Section 3 we give the Hadamard type inequalities for m-convex functions via Caputo-fractional derivatives

In the whole paper $C^n[a,b]$ denotes the space of n-times differentiable functions such that $f^{(n)}$ are continuous on $[a,b]$.

2. Hadamard type inequalities for m-convex functions via Caputo fractional derivatives

In this section we give Hadamard type inequalities for m-convex functions via caputo fractional derivatives and related fractional inequalities.

Theorem 2.1. *Let $f:[0,\infty)\to\mathbb{R}$ be a function such that $f\in C^n[a,b]$. Also let $f^{(n)}$ be positive and m-convex function on $[0,\infty)$, then for $0\le a\le mb$ the following inequalities hold*

$$\begin{aligned}
&f^{(n)}\left(\frac{a+mb}{2}\right)\\
&\le \frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left[{}^CD^{\alpha}_{a+}f(mb)+(-1)^n m^{n-\alpha+1}\,{}^CD^{\alpha}_{b-}f\left(\frac{a}{m}\right)\right]\\
&\le \frac{n-\alpha}{2(n-\alpha+1)}\left[f^{(n)}(a)-m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right]. \qquad (2)
\end{aligned}$$

Proof. Since $f^{(n)}$ is m-convex function, we have for $x,y\in[a,mb]$

$$f^{(n)}\left(\frac{x+my}{2}\right)\le\frac{f^{(n)}(x)+mf^{(n)}(y)}{2}. \qquad (3)$$

Using $x=ta+m(1-t)b$, $y=tb+\frac{1}{m}(1-t)a$, $t\in[0,1]$, then integrating over $[0,1]$ after multiplication with $t^{n-\alpha-1}$, we get

$$\frac{2}{n-\alpha}f^{(n)}\left(\frac{a+mb}{2}\right) \leq \int_0^1 t^{n-\alpha-1}f^{(n)}\left(ta+m(1-t)b\right)dt$$
$$+m\int_0^1 t^{n-\alpha-1}f^{(n)}\left(tb+\frac{1}{m}(1-t)a\right)dt$$
$$=\int_{mb}^{a}\left(\frac{mb-u}{mb-a}\right)^{n-\alpha-1} f^{(n)}(u)\frac{du}{a-mb}$$
$$+m^2\int_{\frac{a}{m}}^{b}\left(\frac{v-\frac{a}{m}}{b-\frac{a}{m}}\right)^{n-\alpha-1} f^{(n)}(v)\frac{dv}{mb-a}$$
$$=\Gamma(n-\alpha)\frac{1}{(mb-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^+}f(mb)+(-1)^n m^{n-\alpha-1}\,{}^{C}D^{\alpha}_{b-}f\left(\frac{a}{m}\right)\right].$$

From which we get first inequality in (2). For the prove of second inequality in (2) note that if $f^{(n)}$ is m-convex, then for $t\in[0,1]$, we get

$$f^{(n)}(ta+m(1-t)b)\leq tf^{(n)}(a)+m(1-t)f^{(n)}(b) \tag{4}$$

and

$$mf^{(n)}\left(tb+\frac{(1-t)}{m}a\right)\leq mtf^{(n)}(b)+m^2(1-t)f^{(n)}\left(\frac{a}{m^2}\right). \tag{5}$$

By adding the above two inequalities, then integrating the resulting inequality over $[0,1]$ after multiplication with $\frac{n-\alpha}{2}t^{n-\alpha-1}$, we get

$$\frac{n-\alpha}{2}\int_0^1 t^{n-\alpha-1}f^{(n)}\left(ta+m(1-t)b\right)dt$$
$$+\frac{n-\alpha}{2}m\int_0^1 t^{n-\alpha-1}f^{(n)}\left(tb+\frac{(1-t)}{m}a\right)dt$$
$$\leq\frac{n-\alpha}{2}\left[f^{(n)}(a)-m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\alpha}dt$$
$$+\frac{m(n-\alpha)}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\alpha-1}dt$$
$$=\frac{n-\alpha}{2(n-\alpha+1)}\left[f^{(n)}(a)-m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right].$$

From which we get second inequality in (2) □

Corollary 2.2. *If in Theorem 2.1, we take $m=1$, then inequality (2) becomes*

$$
\begin{aligned}
&f^{(n)}\left(\frac{a+b}{2}\right)\\
&\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a+}f(b)+{}^{C}D^{\alpha}_{b-}f(a)\right]\\
&\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned}
$$

In [11], above result is proved

Remark 2.3. If we take $\alpha = 0$, $n = 1$ along with $m = 1$ in Theorem 2.1, then we get inequality (1).

For the next result we need the following lemma.

Lemma 2.4. *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable function on (a, mb) such that $f \in C^n[a, mb]$ with $a < mb$. Also let $f^{(n+1)}$ be positive and m-convex function on $[a, mb]$. Then the following equality for Caputo fractional derivatives holds*

(6)

$$
\begin{aligned}
&\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{a+}f(mb)+(-1)^{n}{}^{C}D^{\alpha}_{mb-}f(a)]\\
&=\frac{mb-a}{2}\int_0^1[(1-t)^{n-\alpha}-t^{n-\alpha}]f^{(n+1)}(ta+m(1-t)b)dt.
\end{aligned}
$$

Proof. Since

$$
\begin{aligned}
&\frac{mb-a}{2}\int_0^1[(1-t)^{n-\alpha}-t^{n-\alpha}]f^{(n+1)}(ta+m(1-t)b)dt\\
&=\frac{mb-a}{2}\int_0^1(1-t)^{n-\alpha}f^{(n+1)}(ta+m(1-t)b)dt\\
&-\frac{mb-a}{2}\int_0^1 t^{n-\alpha}f^{(n+1)}(ta+m(1-t)b)dt.
\end{aligned} \tag{7}
$$

First term of right hand side is calculated as

$$
\begin{aligned}
&\frac{mb-a}{2}\int_0^1(1-t)^{n-\alpha}f^{(n+1)}(ta+m(1-t)b)dt\\
&=\frac{f^{(n)}(mb)}{2}+\frac{n-\alpha}{2}\int_{mb}^{a}\left(\frac{a-x}{a-mb}\right)^{n-\alpha+1}\frac{f^{(n)}(x)}{a-mb}dx\\
&=\frac{f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha-1)}{2(mb-a)^{n-\alpha}}(-1)^{n}{}^{C}D^{\alpha}_{mb-}f(a),
\end{aligned}
$$

while second term of right side is calculated as

$$\begin{aligned}
&-\frac{mb-a}{2}\int_0^1 t^{n-\alpha} f^{(n+1)}(ta+m(1-t)b)dt \\
&= \frac{f^{(n)}(a)}{2} - \frac{\alpha}{2}\int_{mb}^{a}\left(\frac{mb-x}{mb-a}\right)^{n-\alpha+1}\frac{f^{(n)}(x)}{a-mb}dx \\
&= \frac{f^{(n)}(a)}{2} - \frac{\Gamma(n-\alpha-1)}{2(mb-a)^{n-\alpha}}{}^{C}D^{\alpha}_{a^+}f(mb).
\end{aligned}$$

Now using these values of integrals in (7), we get the required result. □

Remark 2.5. If we take $m = 1$ in Lemma 2.4, then equality (6) becomes

$$\begin{aligned}
&\frac{f^{(n)}(a)+f^{(n)}(b)}{2} - \frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}[{}^{C}D^{\alpha}_{a^+}f(b) + (-1)^{n}{}^{C}D^{\alpha}_{b-}f(a)] \\
&\qquad = \frac{b-a}{2}\int_0^1[(1-t)^{n-\alpha} - t^{n-\alpha}]f^{(n+1)}(ta+(1-t)b)dt,
\end{aligned}$$

which is proved in [11]. If we take $\alpha = 0$,$n = 1$ along with $m = 1$ in Theorem 2.4, then equality (6) gives an equality (Lemma 2.1 [8]).

Using the above lemma we give following Hadamard-type inequality for m-convex function.

Theorem 2.6. *Let $f : [a, mb] \to \mathbb{R}$ be a differentiable function on (a, mb) such that $f \in C^n[a, mb]$ with $0 \le a < mb$. If $\left|f^{(n+1)}\right|$ is m-convex on $[a, mb]$, then the following inequality for Caputo fractional derivatives holds*

(8)
$$\begin{aligned}
&\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2} - \frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^+}f(mb) + (-1)^{n}{}^{C}D^{\alpha}_{mb-}f(a)\right]\right| \\
&\qquad \le \frac{mb-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)[f^{(n+1)}(a) + mf^{(n+1)}(b)].
\end{aligned}$$

Proof. Using Lemma 2.4 and m-convexity of $\left|f^{(n+1)}\right|$, we find

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}\left[{}^{C}D^{\alpha}_{a^+}f(mb)+{}^{C}D^{\alpha}_{mb-}f(a)\right]\right|$$
$$\leq \frac{mb-a}{2}\int_0^1 \left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left|f^{(n+1)}(ta+m(1-t)b)\right|dt$$
$$\leq \frac{mb-a}{2}\int_0^1 \left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt$$
$$= \frac{mb-a}{2}\left(\int_0^{\frac{1}{2}} \left|(1-t)^{n-\alpha}-t^{n-\alpha}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt\right.$$
(9)
$$\left.+\int_{\frac{1}{2}}^1 \left|t^{n-\alpha}-(1-t)^{n-\alpha}\right|\left[t\left|f^{(n+1)}(a)\right|+m(1-t)\left|f^{(n+1)}(b)\right|\right]dt\right).$$

Consider

$$\left|f^{(n+1)}(a)\right|\left[\int_0^{\frac{1}{2}} t(1-t)^{n-\alpha}dt-\int_0^{\frac{1}{2}} t^{n-\alpha+1}dt\right]$$
$$+m\left|f^{(n+1)}(b)\right|\left[\int_0^{\frac{1}{2}}(1-t)^{n-\alpha+1}dt-\int_0^{\frac{1}{2}}(1-t)t^{n-\alpha}dt\right]$$
$$=\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{(\frac{1}{2})^{n-\alpha+1}}{(n-\alpha+1)}\right]$$
$$+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\alpha+2)}-\frac{(\frac{1}{2})^{n-\alpha+1}}{(n-\alpha+1)}\right].$$

Similarly we have

$$\left|f^{(n+1)}(a)\right|\left[\int_{\frac{1}{2}}^1 t^{n-\alpha+1}dt-\int_{\frac{1}{2}}^1 t(1-t)^{n-\alpha}dt\right]$$
$$+m\left|f^{(n+1)}(b)\right|\left[\int_{\frac{1}{2}}^1(1-t)t^{n-\alpha}dt-\int_{\frac{1}{2}}^1(1-t)^{n-\alpha+1}dt\right]$$
$$=\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\alpha+2)}-\frac{(\frac{1}{2})^{n-\alpha+1}}{(n-\alpha+1)}\right]$$
$$+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{(\frac{1}{2})^{n-\alpha+1}}{(n-\alpha+1)}\right].$$

Therefore (9) becomes

$$\left|\frac{f^{(n)}(a)+f^{(n)}(mb)}{2}-\frac{\Gamma(n-\alpha+1)}{2(mb-a)^{n-\alpha}}[D^{\alpha}_{a^+}f(mb)+(-1)^n D^{\alpha}_{mb-}f(a)]\right|$$
$$\leq \frac{mb-a}{2}\left(\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}-\frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right.\right.$$
$$\left.+\frac{1}{n-\alpha+2}-\frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right]+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\alpha+1)(n-\alpha+2)}\right.$$
$$\left.\left.-\frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}+\frac{1}{n-\alpha+1}-\frac{(\frac{1}{2})^{n-\alpha+1}}{n-\alpha+1}\right]\right)$$
$$=\frac{mb-a}{2}\left(\left|f^{(n+1)}(a)\right|\left[\frac{1}{(n-\alpha+1)}-\frac{(\frac{1}{2})^{n-\alpha}}{(n-\alpha+1)}\right]+m\left|f^{(n+1)}(b)\right|\left[\frac{1}{(n-\alpha+1)}-\frac{(\frac{1}{2})^{n-\alpha}}{(n-\alpha+1}\right.\right.$$
$$=\frac{mb-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)\left[\left|f^{(n+1)}(a)\right|+m\left|f^{(n+1)}(b)\right|\right].$$

The proof is completed. □

Corollary 2.7. *If we take $m=1$ in Theorem 2.6, then inequality (8) becomes*

$$\left|\frac{f^{(n)}(a)+f^{(n)}(b)}{2}-\frac{\Gamma(n-\alpha+1)}{2(b-a)^{n-\alpha}}\left[{}^C D^{\alpha}_{a^+}f(b)+{}^C D^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq \frac{b-a}{2(n-\alpha+1)}\left(1-\frac{1}{2^{n-\alpha}}\right)[f^{(n+1)}(a)+f^{(n+1)}(b)],$$

which is proved in [11].

Remark 2.8. If we take $\alpha=0$, $n=1$ along with $m=1$ in Theorem 2.6, then inequality (8) gives an inequality, [8, Theorem 2.2]

Theorem 2.9. *Let $f:[a,b]\to\mathbb{R}$ be a positive function such that $f\in C^n[a,b]$, with $0\leq a<b$. Also let $f^{(n)}$ be m-convex function on $[a,b]$. Then the following inequalities for Caputo fractional derivatives hold*

(10)
$$f^{(n)}\left(\frac{a+mb}{2}\right)\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}$$
$$\left[({}^C D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\alpha+1}({}^C D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$\leq\frac{n-\alpha}{4(n-\alpha+1)}\left[f^{(n)}(a)-m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]+\frac{m}{2}\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right].$$

Proof. Since $f^{(n)}$ is m-convex, so we have

$$f^{(n)}\left(\frac{x+my}{2}\right)\leq\frac{f^{(n)}(x)+mf^{(n)}(y)}{2}. \tag{11}$$

Substituting $x=\frac{t}{2}a+m\frac{(2-t)}{2}b$ and $y=\frac{(2-t)}{2m}a+\frac{t}{2}b$ where $t\in[0,1]$. Now for all $x,y\in[a,b]$ we have,

$$2f^{(n)}\left(\frac{a+mb}{2}\right)\leq f^{(n)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)+mf^{(n)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right). \tag{12}$$

Taking product on both sides of above inequality with $t^{n-\alpha-1}$ and then integrating over $[0,1]$, we get

$$\begin{aligned}
&\frac{2}{n-\alpha}f^{(n)}\left(\frac{a+mb}{2}\right)\\
&\leq\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt\\
&+m\int_0^1 t^{n-\alpha-1}f^{(n)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\\
&=\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-u)\right)^{n-\alpha-1}f^{(n)}(u)\frac{(-2)du}{mb-a}\\
&+m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{\alpha-1}f^{(n)}(v)\frac{2dv}{mb-a}\\
&=\frac{2^{n-\alpha}\Gamma(n-\alpha)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right],
\end{aligned}$$

which gives

$$\begin{aligned}
f^{(n)}\left(\frac{a+mb}{2}\right)&\leq\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\\
&\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)+(-1)^n m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right].
\end{aligned} \tag{13}$$

Also m-convexity of $f^{(n)}$ gives,

$$\begin{aligned}
&f^{(n)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)+mf^{(n)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\\
&\leq\frac{t}{2}\left[f^{(n)}(a)-m^2f^{(n)}\left(\frac{a}{m^2}\right)\right]+m\left[f^{(n)}(b)+mf^{(n)}\left(\frac{a}{m^2}\right)\right],
\end{aligned}$$

Taking product with $t^{n-\alpha-1}$ on both sides of above inequality and then integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{t}{2}a + m\frac{2-t}{2}b\right)dt \\
&+ m\int_0^1 t^{n-\alpha-1} f^{(n)}\left(\frac{2-t}{2m}a + \frac{t}{2}b\right)dt \\
&\leq \frac{1}{2}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\alpha}dt \\
&+ m\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right]\int_0^1 t^{n-\alpha-1}dt \\
&\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-u)\right)^{n-\alpha-1} f^{(n)}(u)\frac{2du}{a-mb} \\
&+ m^2\int_{\frac{a}{m}}^{\frac{a+mb}{2m}}\left(\frac{2}{b-\frac{a}{m}}(v-\frac{a}{m})\right)^{n-\alpha-1} f^{(n)}(v)\frac{2dv}{mb-a} \\
&\leq \frac{1}{2(n-\alpha+1)}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] \\
&+ \frac{m}{n-\alpha}\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right]
\end{aligned}$$

which implies that

(14)
$$\begin{aligned}
&\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}} \\
&\left[({}^C D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb) + (-1)^n m^{n-\alpha+1}({}^C D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right] \\
&\leq \frac{n-\alpha}{4(n-\alpha+1)}\left[f^{(n)}(a) - m^2 f^{(n)}\left(\frac{a}{m^2}\right)\right] + \frac{m}{2}\left[f^{(n)}(b) + mf^{(n)}\left(\frac{a}{m^2}\right)\right].
\end{aligned}$$

Combining inequalities (13) and (14) we get the required inequality (10). □

Remark 2.10. If we put $m = 1$ in Theorem 2.9, then we get inequality in [12, Theorem 2].

Following lemma will be used for our next results.

Lemma 2.11. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a,b]$, with $a < b$. Then the following equality for Caputo fractional derivatives holds*

$$\begin{aligned}
(15)\qquad & \frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\\
&\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\\
&=\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt\right.\\
&\left.-\int_0^1 t^{\alpha}f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right].
\end{aligned}$$

Proof. Since

$$\begin{aligned}
&\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)dt\right]\\
&=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f^{(n)}\left(\frac{a+mb}{2}\right)\right.\\
&\left.+\frac{2(n-\alpha)}{a-mb}\int_{mb}^{\frac{a+mb}{2}}\left(\frac{2}{mb-a}(mb-x)\right)^{n-\alpha-1}\frac{2}{mb-a}f^{(n)}(x)dx\right]\\
(16)\quad &=\frac{mb-a}{4}\left[-\frac{2}{mb-a}f^{(n)}\left(\frac{a+mb}{2}\right)\right.\\
&\left.+\frac{2^{n-\alpha+1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha+1}}({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right].
\end{aligned}$$

In the same way,

$$\begin{aligned}
&-\frac{mb-a}{4}\left[\int_0^1 t^{\alpha}f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)dt\right]\\
(17)\qquad &=-\frac{mb-a}{4}\left[\frac{2m}{mb-a}f^{(n)}\left(\frac{a+mb}{2m}\right)\right.\\
&\left.-(-1)^{n}\frac{2^{n-\alpha+1}m^{n-\alpha+1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha+1}}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right].
\end{aligned}$$

Adding inequalities (16) and (17) we get the required inequality (15). □

Remark 2.12. If we put $m=1$ in above Lemma, then we get equality in [12, Lemma 1].

With the help of above lemma we prove the following Hadamard-type inequality.

Theorem 2.13. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in C^{n+1}[a,b]$, with $a < b$. If $|f(n+1)|^q$ is also m-convex on $[a,b]$ for $q \geq 1$, then the following inequality for Caputo fractional derivatives holds*

(18)

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\right.$$
$$\left.+(-1)^n m^{n-\alpha+1}({}^C D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq \frac{mb-a}{4(n-\alpha+1)}\left(\frac{1}{2(n-\alpha+2)}\right)^{\frac{1}{q}}$$
$$\left[\left((n-\alpha+1)\,|f^{(n+1)}(a)|^q+m\,(n-\alpha+3)\,|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\,(n-\alpha+3)\,|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+(n-\alpha+1)\,|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right].$$

Proof. By using Lemma 2.11, m-convexity of $|f^{(n+1)}|^q$ and taking $q = 1$, we have

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^C D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)+\right.\right.$$
$$\left.(-1)^n m^{n-\alpha+1}({}^C D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq \frac{mb-a}{4}\int_0^1 t^{n-\alpha}\left(\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt\right.$$
$$\left.+\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|\right)dt.$$
$$=\frac{mb-a}{4}\left(\frac{m}{n-\alpha+1}\left[|f^{(n+1)}(b)|+|f^{(n+1)}\left(\frac{a}{m^2}\right)|\right]\right.$$
$$\left.+\frac{\left[|f^{(n+1)}(a)|-m|f^{(n+1)}\left(\frac{a}{m^2}\right)|+|f^{(n+1)}(b)|-m|f^{(n+1)}(b)|\right]}{2(n-\alpha+2)}\right).$$

Using Lemma 2.11 and for $q > 1$, we have

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\\
&\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq\frac{mb-a}{4}\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt\\
&+\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt.
\end{aligned}$$

Applying power mean inequality we have

$$\begin{aligned}
&\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\\
&\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq\frac{mb-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}\left[\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right.\\
&\left.+\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right]
\end{aligned}$$

Also from m-convexity of $|f^{(n+1)}|^q$, we have

$$\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.$$
$$\left.+(-1)^n m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq \frac{mb-a}{4}\left(\frac{1}{n-\alpha+1}\right)^{\frac{1}{p}}$$
$$\left[\left[\int_0^1 t^{n-\alpha}\left(\frac{t}{2}|f^{(n+1)}(a)|^q+m\frac{2-t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{n-\alpha}\left(m\frac{2-t}{2}|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+\frac{t}{2}|f^{(n+1)}(b)|^q\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4(n-\alpha+1)}\left(\frac{1}{2(n-\alpha+2)}\right)^{\frac{1}{q}}$$
$$\left[\left((n-\alpha+1)\,|f^{(n+1)}(a)|^q+m\,(n-\alpha+3)\,|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\,(n-\alpha+3)\,|f^{(n+1)}\left(\frac{a}{m^2}\right)|^q+(n-\alpha+1)\,|f^{(n+1)}(b)|^q\right)^{\frac{1}{q}}\right],$$

which is required. □

Remark 2.14. If we put $m=1$ in Theorem 2.13, then we get inequality in [12, Theorem 5].

Theorem 2.15. *Let $f:[a,b]\to\mathbb{R}$ be a function such that $f\in C^{n+1}[a,b]$, with $a<b$. If $|f^{(n+1)}|^q$ is m-convex on $[a,b]$ for $q>1$,* *then the following inequality for Caputo fractional derivatives holds*

$$
\begin{aligned}
(19)\quad & \Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\\
&\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left(\frac{|f^{(n+1)}(a)|^{q}+3m|f^{(n+1)}(b)|^{q}}{4}\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{3m|f^{(n+1)}\left(\frac{a}{m^{2}}\right)|^{q}+|f^{(n+1)}(b)|^{q}}{4}\right)^{\frac{1}{q}}\right]\\
&\leq \frac{mb-a}{4}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right.\\
&\left.+3m\left(|f^{(n+1)}\left(\frac{a}{m^{2}}\right)|+|f^{(n+1)}(b)|\right)\right],
\end{aligned}
$$

with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.11 we have

$$
\begin{aligned}
& \Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\\
&\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]\\
&-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|\\
&\leq \frac{mb-a}{4}\left[\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|dt\right.\\
&\left.+\int_{0}^{1}t^{n-\alpha}\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|dt\right].
\end{aligned}
$$

Applying *Hölder's* inequality, we get

$$\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.$$
$$\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4}\left[\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{t}{2}a+m\frac{2-t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1 t^{np-\alpha p}dt\right]^{\frac{1}{p}}\left[\int_0^1\left|f^{(n+1)}\left(\frac{2-t}{2m}a+\frac{t}{2}b\right)\right|^{q}dt\right]^{\frac{1}{q}}\right].$$

Since $|f^{(n+1)}|^{q}$ is m-convex, so we have

$$\Bigg|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.$$
$$\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\Bigg|$$
$$\leq\frac{mb-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}$$
$$\left[\left[\int_0^1\left(\frac{t}{2}|f^{(n+1)}(a)|^{q}+m\frac{2-t}{2}|f^{(n+1)}(b)|^{q}\right)dt\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\int_0^1\left(m\frac{2-t}{2}|f^{(n+1)}(\frac{a}{m^2})|^{q}+\frac{t}{2}|f^{(n+1)}(b)|^{q}\right)dt\right]^{\frac{1}{q}}\right]$$
$$=\frac{mb-a}{4}\left(\frac{1}{np-\alpha p+1}\right)^{\frac{1}{p}}$$
$$\left[\left[\frac{|f^{(n+1)}(a)|^{q}+3m|f^{(n+1)}(b)|^{q}}{4}\right]^{\frac{1}{q}}+\left[\frac{3m|f^{(n+1)}\left(\frac{a}{m^2}\right)|^{q}+|f^{(n+1)}(b)|^{q}}{4}\right]^{\frac{1}{q}}\right].$$

For second inequality of (19), we apply Minkowski's inequality which gives

$$\left|\frac{2^{n-\alpha-1}\Gamma(n-\alpha+1)}{(mb-a)^{n-\alpha}}\left[({}^{C}D^{\alpha}_{(\frac{a+mb}{2})+}f)(mb)\right.\right.$$
$$\left.+(-1)^{n}m^{n-\alpha+1}({}^{C}D^{\alpha}_{(\frac{a+mb}{2m})-}f)\left(\frac{a}{m}\right)\right]$$
$$\left.-\frac{1}{2}\left[f^{(n)}\left(\frac{a+mb}{2}\right)+mf^{(n)}\left(\frac{a+mb}{2m}\right)\right]\right|$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[\left[|f^{(n+1)}(a)|^{q}+3m|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[3m|f^{(n+1)}\left(\frac{a}{m^{2}}\right)|^{q}+|f^{(n+1)}(b)|^{q}\right]^{\frac{1}{q}}\right]$$
$$\leq\frac{mb-a}{16}\left(\frac{4}{np-\alpha p+1}\right)^{\frac{1}{p}}\left[|f^{(n+1)}(a)|+|f^{(n+1)}(b)|\right.$$
$$\left.+3m\left(|f^{(n+1)}\left(\frac{a}{m^{2}}\right)|+|f^{(n+1)}(b)|\right)\right].$$

□

Remark 2.16. If we put $m = 1$ in above theorem, then we get inequality in [12, Theorem 6].

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