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# TRAPEZOID TYPE INEQUALITIES FOR THE GENERALIZED $k$-$g$-FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *$k$-$g$-right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some trapezoid type inequalities for the $k$-$g$-fractional integrals of absolutely continuous functions. Some examples for general exponential fractional integrals are also given.

## 1. Introduction

Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function $k$ is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then $k$ is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of $f$ by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *$k$-$g$-right-sided fractional integral* of $f$ by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

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If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where $\Gamma$ is the *Gamma function*, then

$$S_{k,g,a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)f(t)\,dt \\ =: I^{\alpha}_{a+,g}f(x),\ \ a<x\le b \tag{1.3}$$

$$S_{k,g,b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)f(t)\,dt \\ =: I^{\alpha}_{b-,g}f(x),\ \ a\le x<b, \tag{1.4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function $f$ with respect to another function $g$ on $[a,b]$ as defined in [22, p. 100]

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [22, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\le a<x\le b \tag{1.5}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ 0\le a<x\le b \tag{1.7}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"$\beta$-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)f(t)\,dt, \tag{1.9}$$

for $a<x\le b$ and

$$E^{\alpha}_{b-,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)f(t)\,dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *$k$-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)f(t)\,dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ \ x\in[a,b). \tag{1.12}$$

In [25], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k, \ \ |x| < R, \ R > 0 \tag{1.13}$$

for $\rho$, $\lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(x-t)^{\rho}\right) f(t)\,dt, \ \ x > a \tag{1.14}$$

where $\rho$, $\lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}\left(w(t-x)^{\rho}\right) f(t)\,dt, \ \ x < b \tag{1.15}$$

where $\rho$, $\lambda > 0$, $w \in \mathbb{R}$ and $f$ is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [23], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\,dt, \ \ x > a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\,dt, \ \ x < b \tag{1.17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\,dt, \ \ x > a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\,dt, \ \ x < b \tag{1.19}$$

where $\alpha \in (0,1)$.

Let $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\,dt, \tag{1.20}$$

for $0 < a < x \leq b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\,dt, \tag{1.21}$$

for $0 < a \leq x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) dt, \ 0 < a < x \leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) dt, \ 0 < a \leq x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) dt, \ 0 < a < x \leq b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) dt, \ 0 < a \leq x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) dt, \ 0 < a < x \leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) dt, \ 0 < a \leq x < b. \tag{1.27}$$

We also define the function $\mathbf{K} : [0, \infty) \to [0, \infty)$ by

$$\mathbf{K}(t) := \begin{cases} \int_0^t |k(s)| ds \text{ if } 0 < t, \\ \\ 0 \text{ if } t = 0. \end{cases}$$

We observe that if $k$ takes nonnegative values, as it does in some of the examples in Introduction, then $\mathbf{K}(t) = K(t)$ for $t \in [0, \infty)$.

In the recent paper [19] we obtained amongst other the following Ostrowski and trapezoid type inequalities for functions of bounded variation:

**Theorem 1.** *Assume that the kernel $k$ is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$ and $g$ be a strictly increasing function on $(a, b)$, having a continuous derivative $g'$ on $(a, b)$. Then we have the Ostrowski type* *inequality*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \tag{1.28}$$
$$\leq\frac{1}{2}\left[\int_x^b|k(g(t)-g(x))|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x|k(g(x)-g(t))|\bigvee_t^x(f)\,g'(t)\,dt\right]$$
$$\leq\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\ +\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\leq\frac{1}{2}\begin{cases}\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f);\\[2ex] \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

*and the trapezoid type inequality*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \tag{1.29}$$
$$\leq\frac{1}{2}\left[\int_a^x|k(g(x)-g(t))|\bigvee_a^t(f)\,g'(t)\,dt+\int_x^b|k(g(t)-g(x))|\bigvee_t^b(f)\,g'(t)\,dt\right]$$
$$\leq\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\leq\frac{1}{2}\begin{cases}\max\left\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f);\\[2ex] \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

*for any* $x\in(a,b)$.

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [20]-[33] and the references therein.

Motivated by the above results, we establish in this paper some trapezoid type inequalities for *k-g-fractional integrals* in the case of functions $f:[a,b]\to\mathbb{C}$ that are absolutely continuous on $[a,b]$ and $g$ a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Some examples for a general exponential fractional integral are also given.

## 2. Some Identities for the Operator $S_{k,g,a+,b-}$

For $k$ and $g$ as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned}(2.1)\quad & S_{k,g,a+,b-}f(x) \\ & := \frac{1}{2}\left[S_{k,g,a+}f(x) + S_{k,g,b-}f(x)\right] \\ & = \frac{1}{2}\left[\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt + \int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt\right]\end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

We have:

**Lemma 1.** *With the above assumptions for $k$, $g$ and if $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then we have for $x\in(a,b)$ that*

$$\begin{aligned}(2.2)\quad S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(x)-g(a))\,f(a) + \left[K(g(b)-g(x))\right]f(b)\right] \\ &+ \frac{1}{2}\lambda\int_a^x K(g(x)-g(t))\,dt - \frac{1}{2}\gamma\int_x^b K(g(t)-g(x))\,dt \\ &+ \frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-\lambda\right]dt + \frac{1}{2}\int_x^b K(g(t)-g(x))\left[\gamma-f'(t)\right]dt\end{aligned}$$

*for any $\lambda$, $\gamma\in\mathbb{C}$.*

*In particular, we have the simple identity*

$$\begin{aligned}(2.3)\quad S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(x)-g(a))\,f(a) + \left[K(g(b)-g(x))\right]f(b)\right] \\ &+ \frac{1}{2}\int_a^x K(g(x)-g(t))\,f'(t)\,dt - \frac{1}{2}\int_x^b K(g(t)-g(x))\,f'(t)\,dt\end{aligned}$$

*for $x\in(a,b)$.*

*Proof.* We have, by taking the derivative over $t$ and using the chain rule, that

$$\left[K(g(x)-g(t))\right]' = K'(g(x)-g(t))\,(g(x)-g(t))' = -k(g(x)-g(t))\,g'(t)$$

for $t\in(a,x)$ and

$$\left[K(g(t)-g(x))\right]' = K'(g(t)-g(x))\,(g(t)-g(x))' = k(g(t)-g(x))\,g'(t)$$

for $t\in(x,b)$.

Using the integration by parts formula, we have

$$\begin{aligned}(2.4)\quad & \int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt \\ &= -\int_a^x \left[K(g(x)-g(t))\right]' f(t)\,dt \\ &= -\left[K(g(x)-g(t))\,f(t)\Big|_a^x - \int_a^x K(g(x)-g(t))\,f'(t)\,dt\right] \\ &= K(g(x)-g(a))\,f(a) + \int_a^x K(g(x)-g(t))\,f'(t)\,dt\end{aligned}$$

and

$$\begin{aligned}
(2.5)\qquad & \int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= \int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right]' f\left(t\right)dt \\
&= \left[K\left(g\left(t\right)-g\left(x\right)\right)\right] f\left(t\right)\Big|_x^b - \int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right] f'\left(t\right)dt \\
&= \left[K\left(g\left(b\right)-g\left(x\right)\right)\right] f\left(b\right) - \int_x^b \left[K\left(g\left(t\right)-g\left(x\right)\right)\right] f'\left(t\right)dt
\end{aligned}$$

for any $x \in (a,b)$.

From (2.4) and (2.5) we get

$$\begin{aligned}
(2.6)\qquad & \int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\lambda\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)dt \\
&+ \int_a^x K\left(g\left(x\right)-g\left(t\right)\right)\left[f'\left(t\right)-\lambda\right]dt
\end{aligned}$$

and

$$\begin{aligned}
(2.7)\qquad & \int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= \left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)-\gamma\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)dt \\
&- \int_x^b K\left(g\left(t\right)-g\left(x\right)\right)\left[f'\left(t\right)-\gamma\right]dt
\end{aligned}$$

for any $x \in (a,b)$.

If we add the equalities (2.6) and (2.7) and divide by 2 then we get the desired result (2.2). □

The above lemma provides several identities of interest, out of which we can mention the following:

**Corollary 1.** *With the assumption of Lemma 1 we have*

$$\begin{aligned}
(2.8)\qquad S_{k,g,a+,b-}f\left(x\right) &= \frac{1}{2}\left[K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)\right] \\
&+ \frac{1}{2}\left(\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)dt-\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)dt\right)f'\left(x\right) \\
&+ \frac{1}{2}\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)\left[f'\left(t\right)-f'\left(x\right)\right]dt \\
&+ \frac{1}{2}\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)\left[f'\left(x\right)-f'\left(t\right)\right]dt
\end{aligned}$$

*and*

$$
\begin{aligned}
(2.9)\quad S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)\right] \\
&+\frac{1}{2}f'\left(a\right)\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)dt-\frac{1}{2}f'\left(b\right)\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)dt \\
&+\frac{1}{2}\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)\left[f'\left(t\right)-f'\left(a\right)\right]dt \\
&+\frac{1}{2}\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)\left[f'\left(b\right)-f'\left(t\right)\right]dt
\end{aligned}
$$

*for* $x\in(a,b)$.

If $g$ is a function which maps an interval $I$ of the real line to the real numbers, and is both continuous and injective then we can define the *$g$-mean of two numbers* $a$, $b\in I$ as

$$M_g\left(a,b\right):=g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* $p$. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g\left(a,b\right)=LME\left(a,b\right):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the $g$-mean of two numbers we can introduce

$$
\begin{aligned}
(2.10)\qquad P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right) \\
&= \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g\left(a\right)+g\left(b\right)}{2}-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&+\frac{1}{2}\int_{M_g(a,b)}^b k\left(g\left(t\right)-\frac{g\left(a\right)+g\left(b\right)}{2}\right)g'\left(t\right)f\left(t\right)dt.
\end{aligned}
$$

Using (2.2) and (2.3) we have the representations

$$
\begin{aligned}
P_{k,g,a+,b-}f &= K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2} \\
&\quad + \frac{1}{2}\lambda\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt \\
&\quad - \frac{1}{2}\gamma\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt \\
&\quad + \frac{1}{2}\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)[f'(t)-\lambda]\,dt \\
&\quad + \frac{1}{2}\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)[\gamma-f'(t)]\,dt
\end{aligned}
\tag{2.11}
$$

for any $\lambda$, $\gamma\in\mathbb{C}$ and

$$
\begin{aligned}
P_{k,g,a+,b-}f &= K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2} \\
&\quad + \frac{1}{2}\int_a^{M_g(a,b)} K\left(\frac{g(a)+g(b)}{2}-g(t)\right)f'(t)\,dt \\
&\quad - \frac{1}{2}\int_{M_g(a,b)}^b K\left(g(t)-\frac{g(a)+g(b)}{2}\right)f'(t)\,dt.
\end{aligned}
\tag{2.12}
$$

## 3. Inequalities in Terms of $p$-Norms of the Derivative

We use the *Lebesgue $p$-norms* defined as

$$\|h\|_{[c,d],\infty} := \operatorname*{essup}_{t\in[c,d]}|h(t)| < \infty \text{ provided } h\in L_\infty[c,d]$$

and

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(t)|^p\,dt\right)^{1/p} < \infty \text{ provided } h\in L_p[c,d],\ p\geq 1.$$

**Theorem 2.** *Assume that the kernel $k$ is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. Then for any $x\in(a,b)$ we have the* *trapezoid type inequality*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\right| \tag{3.1}$$
$$\leq \frac{1}{2}\left[\int_a^x |K(g(x)-g(t))|\,|f'(t)|\,dt+\int_x^b |K(g(t)-g(x))|\,|f'(t)|\,dt\right]$$
$$\leq \frac{1}{2}\begin{cases} \|f'\|_{[a,x],\infty}\|K(g(x)-g)\|_{[a,x],1}+\|f'\|_{[x,b],\infty}\|K(g-g(x))\|_{[x,b],1} \\ \quad \text{if } f'\in L_\infty[a,b]\ ; \\ \\ \|f'\|_{[a,x],p}\|K(g(x)-g)\|_{[a,x],q}+\|f'\|_{[x,b],p}\|K(g-g(x))\|_{[x,b],q} \\ \quad \text{if } f'\in L_p[a,b]\,,\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f'\|_{[a,x],1}\|K(g(x)-g)\|_{[a,x],\infty}+\|f'\|_{[x,b],1}\|K(g-g(x))\|_{[x,b],\infty} \\ \quad \text{if } f'\in L_1[a,b]\,; \end{cases}$$
$$\leq \frac{1}{2}\begin{cases} \|f'\|_{[a,b],\infty}\left(\|K(g(x)-g)\|_{[a,x],1}+\|K(g-g(x))\|_{[x,b],1}\right) \\ \quad \text{if } f'\in L_\infty[a,b]\,; \\ \\ \|f'\|_{[a,b],p}\left(\|K(g(x)-g)\|_{[a,x],q}^q+\|K(g-g(x))\|_{[x,b],q}^q\right)^{1/q} \\ \quad \text{if } f'\in L_p[a,b]\,,\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|f'\|_{[a,b],1}\max\left\{\|K(g(x)-g)\|_{[a,x],\infty}\,,\|K(g-g(x))\|_{[x,b],\infty}\right\} \\ \quad \text{if } f'\in L_1[a,b]\,. \end{cases}$$

*Proof.* Using the identity (2.3) we have

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\right|$$
$$\leq \frac{1}{2}\left|\int_a^x K(g(x)-g(t))f'(t)\,dt\right|+\frac{1}{2}\left|\int_x^b K(g(t)-g(x))f'(t)\,dt\right|$$
$$\leq \frac{1}{2}\left[\int_a^x |K(g(x)-g(t))f'(t)|\,dt+\int_x^b |K(g(t)-g(x))f'(t)|\,dt\right],$$

which proves the first inequality in (3.1).

By Hölder's integral inequality

$$\left|\int_c^d u(t)\,v(t)\,dt\right|\leq\left(\int_c^d |u(t)|^p\,dt\right)^{1/p}\left(\int_c^d |v(t)|^q\,dt\right)^{1/q}$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the sup-norm inequality we also have

$$\begin{aligned}
&\int_a^x \left|K\left(g\left(x\right)-g\left(t\right)\right)f'\left(t\right)\right|dt \\
&\leq \begin{cases}
\left\|f'\right\|_{[a,x],\infty}\left\|K\left(g\left(x\right)-g\right)\right\|_{[a,x],1} & \text{if } f'\in L_\infty\left[a,b\right], \\[2ex]
\left\|f'\right\|_{[a,x],p}\left\|K\left(g\left(x\right)-g\right)\right\|_{[a,x],q}, & \text{if } f'\in L_p\left[a,b\right], \\
\text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; & \\[2ex]
\left\|f'\right\|_{[a,x],1}\left\|K\left(g\left(x\right)-g\right)\right\|_{[a,x],\infty} & \text{if } f'\in L_1\left[a,b\right],
\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b \left|K\left(g\left(t\right)-g\left(x\right)\right)f'\left(t\right)\right|dt \\
&\leq \begin{cases}
\left\|f'\right\|_{[x,b],\infty}\left\|K\left(g-g\left(x\right)\right)\right\|_{[x,b],1} & \text{if } f'\in L_\infty\left[a,b\right], \\[2ex]
\left\|f'\right\|_{[x,b],p}\left\|K\left(g-g\left(x\right)\right)\right\|_{[x,b],q}, & \text{if } f'\in L_p\left[a,b\right], \\
\text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; & \\[2ex]
\left\|f'\right\|_{[x,b],1}\left\|K\left(g-g\left(x\right)\right)\right\|_{[x,b],\infty} & \text{if } f'\in L_1\left[a,b\right],
\end{cases}
\end{aligned}$$

which proves the second part of (3.1).

The last part follows by making use of the elementary Hölder type inequalities for positive real numbers $c,\, d,\, u,\, v\geq 0$

$$uc+vd\leq \begin{cases}
\max\left\{u,v\right\}\left(c+d\right); \\[2ex]
\left(u^m+v^m\right)^{1/m}\left(c^n+d^n\right)^{1/n} \text{ with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1.
\end{cases} \tag{3.2}$$

□

**Remark 1.** *Since*

$$\left|K\left(t\right)\right|=\left|\int_0^t k\left(s\right)ds\right|\leq\int_0^t\left|k\left(s\right)\right|ds=\mathbf{K}\left(t\right) \text{ for } t\in\left[0,\infty\right),$$

*then*

$$\begin{aligned}
&\frac{1}{2}\left[\int_a^x\left|K\left(g\left(x\right)-g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt+\int_x^b\left|K\left(g\left(t\right)-g\left(x\right)\right)\right|\left|f'\left(t\right)\right|dt\right] \\
&\leq\frac{1}{2}\left[\int_a^x\mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt+\int_x^b\mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt\right]
\end{aligned}$$

*and by using a similar argument to the one in the proof of Theorem 2 we get the chain of inequalities*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\right| \tag{3.3}$$
$$\leq\frac{1}{2}\left[\int_a^x \mathbf{K}(g(x)-g(t))\left|f'(t)\right|dt+\int_x^b \mathbf{K}(g(t)-g(x))\left|f'(t)\right|dt\right]$$
$$\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,x],\infty}\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],\infty}\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],1}\\ \text{if } f'\in L_\infty[a,b];\\ \\ \left\|f'\right\|_{[a,x],p}\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],q}+\left\|f'\right\|_{[x,b],p}\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],q}\\ \text{if } f'\in L_p[a,b],\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,x],1}\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],\infty}+\left\|f'\right\|_{[x,b],1}\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],\infty}\\ \text{if } f'\in L_1[a,b],\end{cases}$$
$$\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,b],\infty}\left[\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],1}+\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],1}\right]\\ \text{if } f'\in L_\infty[a,b],\\ \\ \left\|f'\right\|_{[a,b],p}\left[\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],q}^q+\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],q}^q\right]^{1/q}\\ \text{if } f'\in L_p[a,b],\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,b],1}\max\left\{\left\|\mathbf{K}(g(x)-g)\right\|_{[a,x],\infty},\left\|\mathbf{K}(g-g(x))\right\|_{[x,b],\infty}\right\}\\ \text{if } f'\in L_1[a,b],\ .\end{cases}$$

We observe that, by Hölder's integral inequality we also have

$$\mathbf{K}(t)=\int_0^t|k(s)|\,ds\leq\begin{cases}t\operatorname{essup}_{s\in[0,t]}|k(s)|\\ \\ t^{1/p}\left(\int_0^t|k(s)|^q\,ds\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases} \tag{3.4}$$
$$\leq\begin{cases}t\left\|k\right\|_{[0,M],\infty} \ \text{if } k\in L_\infty[0,M]\\ \\ t^{1/m}\left\|k\right\|_{[0,M],n},\ \text{if } k\in L_n[0,M],\ m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}$$

for $t\in[0,M]$, where $M>0$.

We observe that

$$\int_a^x \mathbf{K}(g(x)-g(t))\left|f'(t)\right|dt\leq\begin{cases}\left\|k\right\|_{[0,g(x)-g(a)],\infty}\int_a^x(g(x)-g(t))\left|f'(t)\right|dt\\ \\ \left\|k\right\|_{[0,g(x)-g(a)],n}\int_a^x(g(x)-g(t))^{1/m}\left|f'(t)\right|dt,\\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}$$

$$\leq\begin{cases}\left\|k\right\|_{[0,g(b)-g(a)],\infty}\int_a^x(g(x)-g(t))\left|f'(t)\right|dt\\ \\ \left\|k\right\|_{[0,g(b)-g(a)],n}\int_a^x(g(x)-g(t))^{1/m}\left|f'(t)\right|dt,\\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}$$

and

$$\int_x^b \mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt \leq \begin{cases} \|k\|_{[0,g(b)-g(x)],\infty}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(x)],n}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$

$$\leq \begin{cases} \|k\|_{[0,g(b)-g(a)],\infty}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(a)],n}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1, \end{cases}$$

where $x\in(a,b)$.

Using the first bound in (3.3) we then get, for instance,

$$\left|S_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)\right]\right| \tag{3.5}$$
$$\leq \frac{1}{2}\left[\int_a^x \mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt+\int_x^b \mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt\right]$$
$$\leq \frac{1}{2}\begin{cases} \|k\|_{[0,g(x)-g(a)],\infty}\int_a^x \left(g\left(x\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(x)-g(a)],n}\int_a^x \left(g\left(x\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$
$$+\frac{1}{2}\begin{cases} \|k\|_{[0,g(b)-g(x)],\infty}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(x)],n}\int_x^b \left(g\left(t\right)-g\left(x\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$
$$\leq \frac{1}{2}\begin{cases} \|k\|_{[0,g(b)-g(a)],\infty} \\ \times\left(\int_a^x \left(g\left(x\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt+\int_x^b \left(g\left(t\right)-g\left(x\right)\right)\left|f'\left(t\right)\right|dt\right) \\ \\ \|k\|_{[0,g(b)-g(a)],n} \\ \times\left(\int_a^x \left(g\left(x\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt+\int_x^b \left(g\left(t\right)-g\left(x\right)\right)^{1/m}\left|f'\left(t\right)\right|dt\right) \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1. \end{cases}$$

Observe that

$$\begin{aligned}
&\int_a^x (g(x)-g(t))|f'(t)|\,dt+\int_x^b (g(t)-g(x))|f'(t)|\,dt\\
&\leq \begin{cases} (g(x)-g(a))\int_a^x |f'(t)|\,dt+(g(b)-g(x))\int_x^b |f'(t)|\,dt \\ \\ \sup_{t\in[a,x]}|f'(t)|\int_a^x (g(x)-g(t))\,dt+\sup_{t\in[a,x]}|f'(t)|\int_x^b (g(t)-g(x))\,dt \end{cases}\\
&= \begin{cases} (g(x)-g(a))\int_a^x |f'(t)|\,dt+(g(b)-g(x))\int_x^b |f'(t)|\,dt \\ \\ \sup_{t\in[a,x]}|f'(t)|\left(g(x)(x-a)-\int_a^x g(t)\,dt\right)dt \\ +\sup_{t\in[a,x]}|f'(t)|\left(\int_x^b g(t)\,dt-g(x)(b-x)\right)dt \end{cases}\\
&\leq \begin{cases} \max\{g(x)-g(a),g(b)-g(x)\}\int_a^b |f'(t)|\,dt \\ \\ \left(g(x)(2x-a-b)+\int_x^b g(t)\,dt-\int_a^x g(t)\,dt\right)\sup_{t\in[a,b]}|f'(t)| \end{cases}\\
&= \begin{cases} \left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1} \\ \\ \left(g(x)(2x-a-b)+\int_x^b g(t)\,dt-\int_a^x g(t)\,dt\right)\|f'\|_{[a,b],\infty}. \end{cases}
\end{aligned}$$

We can state the following corollary that provides simple error bounds in terms of the functions involved:

**Corollary 2.** *With the assumptions of Theorem 2, we have*

$$\begin{aligned}
(3.6)\quad &\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}[K(g(x)-g(a))f(a)+[K(g(b)-g(x))]f(b)]\right|\\
&\leq \|k\|_{[0,g(b)-g(a)],\infty}\begin{cases} \frac{1}{2}\left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1} \\ \\ \left(g(x)\left(x-\frac{a+b}{2}\right)+\frac{1}{2}\left(\int_x^b g(t)\,dt-\int_a^x g(t)\,dt\right)\right)\|f'\|_{[a,b],\infty} \end{cases}
\end{aligned}$$

*for $x\in(a,b)$.*

**Remark 2.** *If we take in the first branch of (3.6) $x=M_g(a,b)$, then we get*

$$\begin{aligned}
(3.7)\quad &\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}\right|\\
&\qquad\leq\frac{1}{4}(g(b)-g(a))\|k\|_{[0,g(b)-g(a)],\infty}\|f'\|_{[a,b],1},
\end{aligned}$$

*where $P_{k,g,a+,b-}f:=S_{k,g,a+,b-}f(M_g(a,b))$, while if we take $x=\frac{a+b}{2}$ in the second branch, then we get*

$$\begin{aligned}
(3.8)\quad &\left|S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)\right.\\
&\left.-\frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)\right]\right|\\
&\qquad\leq\frac{1}{2}\|k\|_{[0,g(b)-g(a)],\infty}\left(\int_{\frac{a+b}{2}}^b g(t)\,dt-\int_a^{\frac{a+b}{2}} g(t)\,dt\right)\|f'\|_{[a,b],\infty}.
\end{aligned}$$

Similarly, by using the second branch in (3.5), we have for $m,\, n>1,\ \frac{1}{m}+\frac{1}{n}=1$ that

$$\begin{aligned}
(3.9)\quad & \left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\right| \\
&\leq \frac{1}{2}\left[\int_a^x \mathbf{K}(g(x)-g(t))\left|f'(t)\right|dt+\int_x^b \mathbf{K}(g(t)-g(x))\left|f'(t)\right|dt\right] \\
&\leq \frac{1}{2}\left\|k\right\|_{[0,g(x)-g(a)],n}\int_a^x (g(x)-g(t))^{1/m}\left|f'(t)\right|dt \\
&+\frac{1}{2}\left\|k\right\|_{[0,g(b)-g(x)],n}\int_x^b (g(t)-g(x))^{1/m}\left|f'(t)\right|dt \\
&\leq \frac{1}{2}\left\|k\right\|_{[0,g(b)-g(a)],n} \\
&\times\left(\int_a^x (g(x)-g(t))^{1/m}\left|f'(t)\right|dt+\int_x^b (g(t)-g(x))^{1/m}\left|f'(t)\right|dt\right)
\end{aligned}$$

for $x\in(a,b)$.

Using Hölder's integral inequality for $p,\, q>1,\ \frac{1}{p}+\frac{1}{q}=1$ we have

$$\begin{aligned}
&\int_a^x (g(x)-g(t))^{1/m}\left|f'(t)\right|dt+\int_x^b (g(t)-g(x))^{1/m}\left|f'(t)\right|dt \\
&\leq \left(\int_a^x (g(x)-g(t))^{p/m}dt\right)^{1/p}\left(\int_a^x \left|f'(t)\right|^q dt\right)^{1/q} \\
&+\left(\int_x^b (g(t)-g(x))^{p/m}dt\right)^{1/p}\left(\int_x^b \left|f'(t)\right|^q dt\right)^{1/q} \\
&\leq \left[\left(\left(\int_a^x (g(x)-g(t))^{p/m}dt\right)^{1/p}\right)^p+\left(\left(\int_x^b (g(t)-g(x))^{p/m}dt\right)^{1/p}\right)^p\right]^{1/p} \\
&\times\left[\left(\left(\int_a^x \left|f'(t)\right|^q dt\right)^{1/q}\right)^q+\left(\left(\int_x^b \left|f'(t)\right|^q dt\right)^{1/q}\right)^q\right]^{1/q} \\
&=\left(\int_a^x (g(x)-g(t))^{p/m}dt+\int_x^b (g(t)-g(x))^{p/m}dt\right)^{1/p} \\
&\times\left(\int_a^x \left|f'(t)\right|^q dt+\int_x^b \left|f'(t)\right|^q dt\right)^{1/q} \\
&=\left(\int_a^b \left|g(x)-g(t)\right|^{p/m}dt\right)^{1/p}\left(\int_a^b \left|f'(t)\right|^q dt\right)^{1/q} \\
&=\left(\int_a^b \left|g(x)-g(t)\right|^{p/m}dt\right)^{1/p}\left\|f'\right\|_{[a,b],q},
\end{aligned}$$

where in the second inequality we used the Holder's elementary inequality (3.2).

Therefore, we can state the following corollary that provided simple error bounds in terms of the functions involved.

**Corollary 3.** *With the assumptions of Theorem 2, we have*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+[K(g(b)-g(x))]f(b)\right]\right| \leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],n}\left(\int_a^b |g(x)-g(t)|^{p/m}dt\right)^{1/p}\|f'\|_{[a,b],q} \tag{3.10}$$

*for* $x\in(a,b)$, *where* $m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1$ *and* $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

If we take in (3.10) $x=M_g(a,b)$, then we get the simple inequality

$$\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],n}\left(\int_a^b\left|\frac{g(b)+g(a)}{2}-g(t)\right|^{p/m}dt\right)^{1/p}\|f'\|_{[a,b],q}. \tag{3.11}$$

Also, if we take $m=p$ and $n=q$ in (3.10), then we get

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+[K(g(b)-g(x))]f(b)\right]\right| \leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],q}\left(\int_a^b |g(x)-g(t)|\,dt\right)^{1/p}\|f'\|_{[a,b],q} \tag{3.12}$$

for $x\in(a,b)$, while from (3.11) we get

$$\left|P_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}\right| \leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],q}\left(\int_a^b\left|\frac{g(b)+g(a)}{2}-g(t)\right|dt\right)^{1/p}\|f'\|_{[a,b],q}. \tag{3.13}$$

## 4. Example for an Exponential Kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction. We consider here only an example for a general exponential kernel that generalizes the transforms (1.16) and (1.17).

For $\alpha,\ \beta\in\mathbb{R}$ we consider the kernel $k(t):=\exp[(\alpha+\beta i)t]$, $t\in\mathbb{R}$. We have

$$K(t)=\frac{\exp[(\alpha+\beta i)t]-1}{(\alpha+\beta i)},\ \text{if } t\in\mathbb{R}$$

for $\alpha,\ \beta\neq 0$.

Also, we have

$$|k(s)|:=|\exp[(\alpha+\beta i)s]|=\exp(\alpha s)\ \text{ for } s\in\mathbb{R}$$

and

$$\mathbf{K}(t)=\int_0^t\exp(\alpha s)\,ds=\frac{\exp(\alpha t)-1}{\alpha}\ \text{if } 0<t,$$

for $\alpha\neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and $g$ be a strictly increasing function on $(a,b)$, having a continuous derivative $g'$ on $(a,b)$. We have

$$\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(x)=\frac{1}{2}\int_a^x\exp\left[(\alpha+\beta i)(g(x)-g(t))\right]g'(t)f(t)\,dt+\frac{1}{2}\int_x^b\exp\left[(\alpha+\beta i)(g(t)-g(x))\right]g'(t)f(t)\,dt \tag{4.1}$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then we can consider the following operator as well

$$\begin{aligned}&\kappa_{h,a+,b-}^{\alpha+\beta i}f(x)\\&:=\mathcal{E}_{\ln h,a+,b-}^{\alpha+\beta i}f(x)\\&=\frac{1}{2}\left[\int_a^x\left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt+\int_x^b\left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned} \tag{4.2}$$

for $x\in(a,b)$.

From the first part of (3.3) we have

$$\begin{aligned}&\Bigg|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(x)-\\&-\frac{1}{2}\left[\frac{\{\exp[(\alpha+\beta i)(g(b)-g(x))]-1\}f(a)+\{\exp[(\alpha+\beta i)(g(x)-g(a))]-1\}f(b)}{(\alpha+\beta i)}\right]\Bigg|\\&\le\frac{1}{2}\int_a^x\left[\frac{\exp(\alpha(g(x)-g(t)))-1}{\alpha}\right]|f'(t)|\,dt\\&\qquad+\frac{1}{2}\int_x^b\left[\frac{\exp(\alpha(g(t)-g(x)))-1}{\alpha}\right]|f'(t)|\,dt\end{aligned} \tag{4.3}$$

for $x\in(a,b)$.

If we denote

$$\begin{aligned}\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i}f&:=\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(M_g(a,b))\\&=\frac{1}{2}\int_a^{M_g(a,b)}\exp\left[(\alpha+\beta i)\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right]g'(t)f(t)\,dt\\&+\frac{1}{2}\int_{M_g(a,b)}^b\exp\left[(\alpha+\beta i)\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right]g'(t)f(t)\,dt,\end{aligned}$$

then by (4.3) we get

$$\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(a)+f(b)}{2}\right| \tag{4.4}$$
$$\leq\frac{1}{2}\int_a^{M_g(a,b)}\left[\frac{\exp\left(\alpha\left(\frac{g(b)+g(a)}{2}-g(t)\right)\right)-1}{\alpha}\right]|f'(t)|\,dt$$
$$+\frac{1}{2}\int_{M_g(a,b)}^b\left[\frac{\exp\left(\alpha\left(g(t)-\frac{g(b)+g(a)}{2}\right)\right)-1}{\alpha}\right]|f'(t)|\,dt.$$

Assume that $\alpha>0$, then

$$\|k\|_{[0,g(b)-g(a)],\infty}=\sup_{s\in[0,g(b)-g(a)]}\exp(\alpha s)=\exp(\alpha[g(b)-g(a)])$$

and by (3.6) we have

$$\left|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i}f(x)-\right. \tag{4.5}$$
$$\left.-\frac{1}{2}\left[\frac{\{\exp[(\alpha+\beta i)(g(b)-g(x))]-1\}f(a)+\{\exp[(\alpha+\beta i)(g(x)-g(a))]-1\}f(b)}{(\alpha+\beta i)}\right]\right|$$
$$\leq\exp(\alpha[g(b)-g(a)])\begin{cases}\frac{1}{2}\left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1}\\ \\ \left(g(x)\left(x-\frac{a+b}{2}\right)+\frac{1}{2}\left(\int_x^b g(t)\,dt-\int_a^x g(t)\,dt\right)\right)\|f'\|_{[a,b],\infty}\end{cases}$$

for $x\in(a,b)$.

In particular,

$$\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(a)+f(b)}{2}\right| \tag{4.6}$$
$$\leq\frac{1}{4}\exp(\alpha[g(b)-g(a)])(g(b)-g(a))\|f'\|_{[a,b],1}.$$

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on $(a,b)$, having a continuous derivative $h'$ on $(a,b)$, then by (4.6) we get

$$\left|\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\frac{f(a)+f(b)}{2}\right| \tag{4.7}$$
$$\leq\frac{1}{4}\left(\frac{h(b)}{h(a)}\right)^{\alpha}\ln\left(\frac{h(b)}{h(a)}\right)\|f'\|_{[a,b],1},$$

where $\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f:=\overline{\mathcal{E}}_{\ln h,a+,b-}^{\alpha+\beta i}f$.

Furthermore, for $n>1$, a real number, we have

$$\|k\|_{[0,g(b)-g(a)],n}=\left(\int_0^{g(b)-g(a)}\exp(n\alpha s)\,ds\right)^{1/n}=\left(\frac{\exp(n\alpha(g(b)-g(a)))-1}{n\alpha}\right)^{1/n}.$$

Using the inequality (3.10) we have for $\alpha$, $\beta \neq 0$ that

$$\begin{aligned} &\Big|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) - \\ &-\frac{1}{2}\left[\frac{\{\exp[(\alpha+\beta i)(g(b)-g(x))]-1\} f(a) + \{\exp[(\alpha+\beta i)(g(x)-g(a))]-1\} f(b)}{(\alpha+\beta i)}\right]\Bigg| \\ &\leq \frac{1}{2}\left(\frac{\exp(n\alpha(g(b)-g(a)))-1}{n\alpha}\right)^{1/n}\left(\int_a^b |g(x)-g(t)|^{p/m} dt\right)^{1/p} \|f'\|_{[a,b],q} \end{aligned} \tag{4.8}$$

for $x \in (a,b)$, where $m$, $n > 1$, $\frac{1}{m}+\frac{1}{n}=1$ and $p$, $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$.

In particular,

$$\begin{aligned} &\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f - \frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(a)+f(b)}{2}\right| \\ &\leq \frac{1}{2}\left(\frac{\exp(n\alpha(g(b)-g(a)))-1}{n\alpha}\right)^{1/n} \\ &\times\left(\int_a^b \left|\frac{g(b)+g(a)}{2}-g(t)\right|^{p/m} dt\right)^{1/p} \|f'\|_{[a,b],q}. \end{aligned} \tag{4.9}$$

If we take $m=p$ and $n=q$ with $p$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, then by (4.8) we have

$$\begin{aligned} &\Big|\mathcal{E}_{g,a+,b-}^{\alpha+\beta i} f(x) - \\ &-\frac{1}{2}\left[\frac{\{\exp[(\alpha+\beta i)(g(b)-g(x))]-1\} f(a) + \{\exp[(\alpha+\beta i)(g(x)-g(a))]-1\} f(b)}{(\alpha+\beta i)}\right]\Bigg| \\ &\leq \frac{1}{2}\left(\frac{\exp(q\alpha(g(b)-g(a)))-1}{q\alpha}\right)^{1/q}\left(\int_a^b |g(x)-g(t)| dt\right)^{1/p} \|f'\|_{[a,b],q} \end{aligned} \tag{4.10}$$

and by (4.9) we get

$$\begin{aligned} &\left|\overline{\mathcal{E}}_{g,a+,b-}^{\alpha+\beta i} f - \frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}\frac{f(a)+f(b)}{2}\right| \\ &\leq \frac{1}{2}\left(\frac{\exp(q\alpha(g(b)-g(a)))-1}{q\alpha}\right)^{1/q} \\ &\times\left(\int_a^b \left|\frac{g(b)+g(a)}{2}-g(t)\right| dt\right)^{1/p} \|f'\|_{[a,b],q}. \end{aligned} \tag{4.11}$$

## References

[1] R. P. Agarwal, M.-J. Luo and R. K. Raina, On Ostrowski type inequalities, *Fasc. Math.* **56** (2016), 5-27.

[2] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[3] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[4] A. O. Akdemir, Inequalities of Ostrowski's type for $m$- and $(\alpha, m)$-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383
[5] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378
[6] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.
[7] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized $s$-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.
[8] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.
[9] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.
[10] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.
[11] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.
[12] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]
[13] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.
[14] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].
[15] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].
[16] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].
[17] S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].
[18] S. S. Dragomir, Further Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 84. [Online http://rgmia.org/papers/v20/v20a84.pdf].
[19] S. S. Dragomir, Ostrowski and trapezoid type inequalities for the generalized $k$-$g$-fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. .
[20] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106
[21] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized $(s, m, \varphi)$-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.
[22] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0.
[23] M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type Inequalities for convex functions via fractional integrals, Preprint arXiv:1701.00092.
[24] M. A. Noor, K. I. Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic $h$-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445
[25] R. K. Raina , On generalized Wright's hypergeometric functions and fractional calculus operators, *East Asian Math. J.*, **21**(2)(2005), 191-203.
[26] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[27] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.
[28] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.
[29] M. Tunç, On new inequalities for $h$-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.
[30] M. Tunç, Ostrowski type inequalities for $m$- and $(\alpha, m)$-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.
[31] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.
[32] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.
[33] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

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