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OSTROWSKI TYPE INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some Ostrowski type inequalities for the k-g-fractional integrals of absolutely continuous functions. Some examples for a general exponential fractional integrals are also given.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\[2ex] 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

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If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1} g'(t)\, f(t)\, dt \\ &=: I_{a+,g}^{\alpha}f(x),\ \ a<x\le b\end{aligned} \tag{1.3}$$

and

$$\begin{aligned}S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1} g'(t)\, f(t)\, dt \\ &=: I_{b-,g}^{\alpha}f(x),\ \ a\le x<b,\end{aligned} \tag{1.4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100]

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\, f(t)\, dt, \tag{1.9}$$

for $a<x\le b$ and

$$E_{b-,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)\, f(t)\, dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)\, f(t)\, dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)\, f(t)\, dt,\ \ x\in[a,b). \tag{1.12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k+\lambda)} x^k, \ \ |x| < R, \ \text{with } R > 0 \tag{1.13}$$

for ρ, $\lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(x-t)^{\rho}) f(t)\,dt, \ \ x > a \tag{1.14}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(t-x)^{\rho}) f(t)\,dt, \ \ x < b \tag{1.15}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\,dt, \ \ x > a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\,dt, \ \ x < b \tag{1.17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\,dt, \ \ x > a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\,dt, \ \ x < b \tag{1.19}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\,dt, \tag{1.20}$$

for $0 < a < x \le b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\,dt, \tag{1.21}$$

for $0 < a \leq x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) \, dt, \ 0 < a < x \leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) \, dt, \ 0 < a \leq x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}^{\alpha}_{a+} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) \, dt, \ 0 < a < x \leq b, \tag{1.24}$$

$$\mathcal{L}^{\alpha}_{b-} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) \, dt, \ 0 < a \leq x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) \, dt, \ 0 < a < x \leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) \, dt, \ 0 < a \leq x < b. \tag{1.27}$$

For k and g as above, we consider the mixed operator

$$\begin{aligned} & S_{k,g,a+,b-} f(x) \\ & := \frac{1}{2} \left[S_{k,g,a+} f(x) + S_{k,g,b-} f(x) \right] \\ & = \frac{1}{2} \left[\int_a^x k(g(x) - g(t)) g'(t) f(t) \, dt + \int_x^b k(g(t) - g(x)) g'(t) f(t) \, dt \right] \end{aligned} \tag{1.28}$$

for the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$ and $x \in (a,b)$.

We use the *Lebesgue p-norms* defined as

$$\|h\|_{[c,d],\infty} := \operatorname*{essup}_{t \in [c,d]} |h(t)| < \infty \text{ provided } h \in L_\infty[c,d]$$

and

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(t)|^p \, dt \right)^{1/p} < \infty \text{ provided } h \in L_p[c,d], \ p \geq 1.$$

In the recent paper [20], we established the following trapezoid type inequalities for absolutely continuous functions:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f : [a,b] \to \mathbb{C}$ be* *an absolutely continuous on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x\in(a,b)$ we have*

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(x)-g(a))f(a)+\left[K(g(b)-g(x))\right]f(b)\right]\right| \tag{1.29}$$
$$\leq\frac{1}{2}\left[\int_a^x\left|K(g(x)-g(t))\right|\left|f'(t)\right|dt+\int_x^b\left|K(g(t)-g(x))\right|\left|f'(t)\right|dt\right]$$
$$\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,x],\infty}\left\|K(g(x)-g)\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],\infty}\left\|K(g-g(x))\right\|_{[x,b],1}\\ \quad \text{if } f'\in L_\infty[a,b]\ ;\\ \\ \left\|f'\right\|_{[a,x],p}\left\|K(g(x)-g)\right\|_{[a,x],q}+\left\|f'\right\|_{[x,b],p}\left\|K(g-g(x))\right\|_{[x,b],q}\\ \quad \text{if } f'\in L_p[a,b],\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,x],1}\left\|K(g(x)-g)\right\|_{[a,x],\infty}+\left\|f'\right\|_{[x,b],1}\left\|K(g-g(x))\right\|_{[x,b],\infty}\\ \quad \text{if } f'\in L_1[a,b];\end{cases}$$
$$\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,b],\infty}\left(\left\|K(g(x)-g)\right\|_{[a,x],1}+\left\|K(g-g(x))\right\|_{[x,b],1}\right)\\ \quad \text{if } f'\in L_\infty[a,b];\\ \\ \left\|f'\right\|_{[a,b],p}\left(\left\|K(g(x)-g)\right\|_{[a,x],q}^q+\left\|K(g-g(x))\right\|_{[x,b],q}^q\right)^{1/q}\\ \quad \text{if } f'\in L_p[a,b],\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,b],1}\max\left\{\left\|K(g(x)-g)\right\|_{[a,x],\infty},\left\|K(g-g(x))\right\|_{[x,b],\infty}\right\}\\ \quad \text{if } f'\in L_1[a,b].\end{cases}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [21]-[34] and the references therein.

Motivated by the above results, we establish in this paper some Ostrowski type inequalities for *k-g-fractional integrals* in the case of functions $f:[a,b]\to\mathbb{C}$ that are absolutely continuous on $[a,b]$ and g a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). In other words, we establish upper bounds for the quantity

$$\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|,\ x\in(a,b)$$

when the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval, $f:[a,b]\to\mathbb{C}$ is an absolutely continuous on $[a,b]$ and g is a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Some examples for a general exponential fractional integral are also given.

2. Some Identities for the Dual Operator $\breve{S}_{k,g,a+,b-}$

Observe that, using the definitions (1.1) and (1.2) we have

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt,\ x\in[a,b) \tag{2.1}$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt,\ x\in(a,b]. \tag{2.2}$$

Define also the mixed dual operator

$$\begin{aligned}
&\breve{S}_{k,g,a+,b-}f(x) \\
&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right] \\
&=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt\right]
\end{aligned} \tag{2.3}$$

for any $x\in(a,b)$.

Lemma 1. *With the above assumptions for k, g and if $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then we have for $x\in(a,b)$ that*

$$\begin{aligned}
\breve{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\
&\quad+\frac{1}{2}\gamma\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}\lambda\int_a^x K(g(t)-g(a))\,dt \\
&\quad+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[\lambda-f'(t)\right]dt
\end{aligned} \tag{2.4}$$

for any λ, $\gamma\in\mathbb{C}$.

In particular, we have the simple identity

$$\begin{aligned}
\breve{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\
&\quad+\frac{1}{2}\int_x^b K(g(b)-g(t))\,f'(t)\,dt-\frac{1}{2}\int_a^x K(g(t)-g(a))\,f'(t)\,dt
\end{aligned} \tag{2.5}$$

for $x\in(a,b)$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$\left[K(g(b)-g(t))\right]'=K'(g(b)-g(t))\,(g(b)-g(t))'=-k(g(b)-g(t))\,g'(t)$$

for $t\in(x,b)$ and

$$\left[K(g(t)-g(a))\right]'=K'(g(t)-g(a))\,(g(t)-g(a))'=k(g(t)-g(a))\,g'(t)$$

for $t\in(a,x)$.

Using the integration by parts formula, we have

$$\begin{aligned}
&\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt \\
&=-\int_x^b \left[K(g(b)-g(t))\right]'f(t)\,dt \\
&=-\left[K(g(b)-g(t))\,f(t)\Big|_x^b-\int_x^b K(g(b)-g(t))\,f'(t)\,dt\right] \\
&=K(g(b)-g(x))\,f(x)+\int_x^b K(g(b)-g(t))\,f'(t)\,dt
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned} &\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\ &=\int_a^x \left[K\left(g\left(t\right)-g\left(a\right)\right)\right]'f\left(t\right)dt \\ &= K\left(g\left(t\right)-g\left(a\right)\right)f\left(t\right)\Big|_a^x-\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt \\ &= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right)-\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt \end{aligned} \tag{2.7}$$

for any $x\in(a,b)$.

From (2.6) and (2.7) we have

$$\begin{aligned} &\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\ &= K\left(g\left(b\right)-g\left(x\right)\right)f\left(x\right)+\gamma\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)dt \\ &+\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)\left[f'\left(t\right)-\gamma\right]dt \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\ &= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right)-\lambda\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)dt \\ &-\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)\left[f'\left(t\right)-\lambda\right]dt \end{aligned} \tag{2.9}$$

for any $x\in(a,b)$.

If we add the equalities (2.8) and (2.9) and divide by 2 then we get the desired result (2.4). □

The above lemma provides several identities of interest, out of which we can mention the following:

Corollary 1. *With the assumption of Lemma 1 we have*

$$\begin{aligned} \breve{S}_{k,g,a+,b-}f\left(x\right) &= \frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)+K\left(g\left(x\right)-g\left(a\right)\right)\right]f\left(x\right) \\ &+\frac{1}{2}\left(\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)dt-\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)dt\right)f'\left(x\right) \\ &+\frac{1}{2}\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)\left[f'\left(t\right)-f'\left(x\right)\right]dt \\ &+\frac{1}{2}\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)\left[f'\left(x\right)-f'\left(t\right)\right]dt \end{aligned} \tag{2.10}$$

and

$$\breve{S}_{k,g,a+,b-}f(x)=\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)$$
$$+\frac{1}{2}f'(b)\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}f'(a)\int_a^x K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-f'(b)\right]dt$$
$$+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[f'(a)-f'(t)\right]dt \tag{2.11}$$

for $x\in(a,b)$.

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\,b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f&:=\breve{S}_{k,g,a+,b-}f(M_g(a,b))\\&=\frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt\\&\quad+\frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,f(t)\,dt.\end{aligned} \tag{2.12}$$

Using (2.4) and (2.5) we have the representations

$$\breve{P}_{k,g,a+,b-}f:=K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))$$
$$+\frac{1}{2}\gamma\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt-\frac{1}{2}\lambda\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_{M_g(a,b)}^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt$$
$$+\frac{1}{2}\int_a^{M_g(a,b)} K(g(t)-g(a))\left[\lambda-f'(t)\right]dt \tag{2.13}$$

for any λ, $\gamma \in \mathbb{C}$ and

$$\begin{aligned} \breve{P}_{k,g,a+,b-}f &:= K\left(\frac{g(b)-g(a)}{2}\right) f\left(M_g(a,b)\right) \\ &\quad + \frac{1}{2}\int_{M_g(a,b)}^{b} K\left(g(b)-g(t)\right) f'(t)\,dt \\ &\quad - \frac{1}{2}\int_{a}^{M_g(a,b)} K\left(g(t)-g(a)\right) f'(t)\,dt. \end{aligned} \tag{2.14}$$

We can also consider

$$\begin{aligned} &\breve{M}_{k,g,a+,b-}f \\ &:= \frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g(b)-g(t)\right) g'(t) f(t)\,dt \\ &\quad + \frac{1}{2}\int_{a}^{\frac{a+b}{2}} k\left(g(t)-g(a)\right) g'(t) f(t)\,dt. \end{aligned} \tag{2.15}$$

Using (2.4) and (2.5) we have the representations

$$\begin{aligned} &\breve{M}_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right] f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{1}{2}\gamma\int_{\frac{a+b}{2}}^{b} K\left(g(b)-g(t)\right)dt - \frac{1}{2}\lambda\int_{a}^{\frac{a+b}{2}} K\left(g(t)-g(a)\right)dt \\ &\quad + \frac{1}{2}\int_{\frac{a+b}{2}}^{b} K\left(g(b)-g(t)\right)\left[f'(t)-\gamma\right]dt \\ &\quad + \frac{1}{2}\int_{a}^{\frac{a+b}{2}} K\left(g(t)-g(a)\right)\left[\lambda-f'(t)\right]dt \end{aligned} \tag{2.16}$$

for any λ, $\gamma \in \mathbb{C}$ and

$$\begin{aligned} &\breve{M}_{k,g,a+,b-}f \\ &= \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right] f\left(\frac{a+b}{2}\right) \\ &\quad + \frac{1}{2}\int_{\frac{a+b}{2}}^{b} K\left(g(b)-g(t)\right) f'(t)\,dt - \frac{1}{2}\int_{a}^{\frac{a+b}{2}} K\left(g(t)-g(a)\right) f'(t)\,dt. \end{aligned} \tag{2.17}$$

3. Inequalities in Terms of p-Norms of the Derivative

We have:

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then for any $x\in(a,b)$ we have the*

Ostrowski type inequality

$$
\begin{aligned}
(3.1)\quad & \left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\leq \frac{1}{2}\left[\int_x^b |K(g(b)-g(t))f'(t)|\,dt+\int_a^x |K(g(t)-g(a))f'(t)|\,dt\right] \\
&\leq \frac{1}{2}\begin{cases}
\|f'\|_{[a,x],\infty}\|K(g-g(a))\|_{[a,x],1}+\|f'\|_{[x,b],\infty}\|K(g(b)-g)\|_{[x,b],1} \\
\quad \textit{if } f'\in L_\infty[a,b], \\
\\
\|f'\|_{[a,x],p}\|K(g-g(a))\|_{[a,x],q}+\|f'\|_{[x,b],p}\|K(g(b)-g)\|_{[x,b],q}, \\
\textit{if } f'\in L_p[a,b],\ p,\ q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\\
\|f'\|_{[a,x],1}\|K(g-g(a))\|_{[a,x],\infty}+\|f'\|_{[x,b],1}\|K(g(b)-g)\|_{[x,b],\infty} \\
\textit{if } f'\in L_1[a,b],
\end{cases} \\
&\leq \frac{1}{2}\begin{cases}
\|f'\|_{[a,b],\infty}\left(\|K(g-g(a))\|_{[a,x],1}+\|K(g(b)-g)\|_{[x,b],1}\right) \\
\quad \textit{if } f'\in L_\infty[a,b]; \\
\\
\|f'\|_{[a,b],p}\left(\|K(g-g(a))\|_{[a,x],q}^q+\|K(g(b)-g)\|_{[x,b],q}^q\right)^{1/q} \\
\quad \textit{if } f'\in L_p[a,b],\ \textit{and } p,q>1 \textit{ with } \frac{1}{p}+\frac{1}{q}=1; \\
\\
\|f'\|_{[a,b],1}\max\left\{\|K(g-g(a))\|_{[a,x],\infty},\|K(g(b)-g)\|_{[x,b],\infty}\right\} \\
\quad \textit{if } f'\in L_1[a,b].
\end{cases}
\end{aligned}
$$

Proof. Using the identity (2.5) we have

$$
\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\quad\leq \frac{1}{2}\left|\int_x^b K(g(b)-g(t))f'(t)\,dt\right|+\frac{1}{2}\left|\int_a^x K(g(t)-g(a))f'(t)\,dt\right| \\
&\qquad\leq \frac{1}{2}\left[\int_x^b |K(g(b)-g(t))f'(t)|\,dt+\int_a^x |K(g(t)-g(a))f'(t)|\,dt\right],
\end{aligned}
$$

which proves the first inequality in (3.1).

By Hölder's integral inequality

$$
\left|\int_c^d u(t)v(t)\,dt\right|\leq\left(\int_c^d |u(t)|^p\,dt\right)^{1/p}\left(\int_c^d |v(t)|^q\,dt\right)^{1/q}
$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, and the sup-norm inequality we also have

$$\begin{aligned}&\int_a^x \left|K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)\right|dt\\&\leq\begin{cases}\left\|f'\right\|_{[a,x],\infty}\left\|K\left(g-g\left(a\right)\right)\right\|_{[a,x],1} & \text{if } f'\in L_\infty\left[a,b\right],\\ \\ \left\|f'\right\|_{[a,x],p}\left\|K\left(g-g\left(a\right)\right)\right\|_{[a,x],q}, & \text{if } f'\in L_p\left[a,b\right],\\ \text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; & \\ \\ \left\|f'\right\|_{[a,x],1}\left\|K\left(g-g\left(a\right)\right)\right\|_{[a,x],\infty} & \text{if } f'\in L_1\left[a,b\right],\end{cases}\end{aligned}$$

and

$$\begin{aligned}&\int_x^b \left|K\left(g\left(b\right)-g\left(t\right)\right)f'\left(t\right)\right|dt\\&\leq\begin{cases}\left\|f'\right\|_{[x,b],\infty}\left\|K\left(g\left(b\right)-g\right)\right\|_{[x,b],1} & \text{if } f'\in L_\infty\left[a,b\right],\\ \\ \left\|f'\right\|_{[x,b],p}\left\|K\left(g\left(b\right)-g\right)\right\|_{[x,b],q}, & \text{if } f'\in L_p\left[a,b\right],\\ \text{if } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1; & \\ \\ \left\|f'\right\|_{[x,b],1}\left\|K\left(g\left(b\right)-g\right)\right\|_{[x,b],\infty} & \text{if } f'\in L_1\left[a,b\right],\end{cases}\end{aligned}$$

which proves the second part of (3.1).

The last part follows by making use of the elementary Hölder type inequalities for positive real numbers c, d, u, $v\geq 0$

$$uc+vd\leq\begin{cases}\max\left\{u,v\right\}\left(c+d\right);\\ \\ \left(u^m+v^m\right)^{1/m}\left(c^n+d^n\right)^{1/n} & \text{with } m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1.\end{cases}\tag{3.2}$$

□

Remark 1. *Since*

$$\left|K\left(t\right)\right|=\left|\int_0^t k\left(s\right)ds\right|\leq\int_0^t\left|k\left(s\right)\right|ds=\mathbf{K}\left(t\right)\ \textit{for } t\in\left[0,\infty\right),$$

then

$$\begin{aligned}&\frac{1}{2}\left[\int_x^b\left|K\left(g\left(b\right)-g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt+\int_a^x\left|K\left(g\left(t\right)-g\left(a\right)\right)\right|\left|f'\left(t\right)\right|dt\right]\\&\leq\frac{1}{2}\left[\int_x^b\left|\mathbf{K}\left(g\left(b\right)-g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt+\int_a^x\left|\mathbf{K}\left(g\left(t\right)-g\left(a\right)\right)\right|\left|f'\left(t\right)\right|dt\right]\end{aligned}$$

and by using a similar argument to the one in the proof of Theorem 2 we get the chain of inequalities

$$
\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \quad (3.3)\\
&\leq\frac{1}{2}\left[\int_x^b\left|\mathbf{K}(g(b)-g(t))\right|\left|f'(t)\right|dt+\int_a^x\left|\mathbf{K}(g(t)-g(a))\right|\left|f'(t)\right|dt\right]\\
&\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,x],\infty}\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],\infty}\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],1}\\ \quad \text{if } f'\in L_\infty[a,b],\\ \\ \left\|f'\right\|_{[a,x],p}\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],q}+\left\|f'\right\|_{[x,b],p}\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],q},\\ \text{if } f'\in L_p[a,b],\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,x],1}\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],\infty}+\left\|f'\right\|_{[x,b],1}\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],\infty}\\ \text{if } f'\in L_1[a,b],\end{cases}\\
&\leq\frac{1}{2}\begin{cases}\left\|f'\right\|_{[a,b],\infty}\left(\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],1}+\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],1}\right)\\ \quad \text{if } f'\in L_\infty[a,b];\\ \\ \left\|f'\right\|_{[a,b],p}\left(\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],q}^q+\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],q}^q\right)^{1/q}\\ \quad \text{if } f'\in L_p[a,b],\ \text{and } p,q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left\|f'\right\|_{[a,b],1}\max\left\{\left\|\mathbf{K}(g-g(a))\right\|_{[a,x],\infty},\left\|\mathbf{K}(g(b)-g)\right\|_{[x,b],\infty}\right\}\\ \quad \text{if } f'\in L_1[a,b].\end{cases}
\end{aligned}
$$

We observe that, by Hölder's integral inequality we also have

$$
\begin{aligned}
\mathbf{K}(t)=\int_0^t|k(s)|\,ds&\leq\begin{cases}t\operatorname{essup}_{s\in[0,t]}|k(s)|\\ \\ t^{1/p}\left(\int_0^t|k(s)|^q\,ds\right)^{1/q},\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1\end{cases} \quad (3.4)\\
&\leq\begin{cases}t\left\|k\right\|_{[0,M],\infty}\ \text{ if } k\in L_\infty[0,M]\\ \\ t^{1/m}\left\|k\right\|_{[0,M],n},\ \text{ if } k\in L_n[0,M],\ m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}
\end{aligned}
$$

for $t\in[0,M]$, where $M>0$.

We observe that

$$
\begin{aligned}
\int_a^x\left|\mathbf{K}(g(t)-g(a))\right|\left|f'(t)\right|dtt&\leq\begin{cases}\left\|k\right\|_{[0,g(x)-g(a)],\infty}\int_a^x(g(t)-g(a))\left|f'(t)\right|dt\\ \\ \left\|k\right\|_{[0,g(x)-g(a)],n}\int_a^x(g(t)-g(a))^{1/m}\left|f'(t)\right|dt,\\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}\\
&\leq\begin{cases}\left\|k\right\|_{[0,g(b)-g(a)],\infty}\int_a^x(g(t)-g(a))\left|f'(t)\right|dt\\ \\ \left\|k\right\|_{[0,g(b)-g(a)],n}\int_a^x(g(t)-g(a))^{1/m}\left|f'(t)\right|dt,\\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1\end{cases}
\end{aligned}
$$

and

$$\int_x^b \left|\mathbf{K}\left(g\left(b\right)-g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt \leq \begin{cases} \|k\|_{[0,g(b)-g(x)],\infty} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(x)],n} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$

$$\leq \begin{cases} \|k\|_{[0,g(b)-g(a)],\infty} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(a)],n} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1, \end{cases}$$

where $x\in(a,b)$.

Using the first bound in (3.3) we then get, for instance,

$$\left|\breve{S}_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)+K\left(g\left(x\right)-g\left(a\right)\right)\right]f\left(x\right)\right| \tag{3.5}$$
$$\leq \frac{1}{2}\left[\int_x^b \left|\mathbf{K}\left(g\left(b\right)-g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt+\int_a^x \left|\mathbf{K}\left(g\left(t\right)-g\left(a\right)\right)\right|\left|f'\left(t\right)\right|dt\right]$$
$$\leq \frac{1}{2}\begin{cases} \|k\|_{[0,g(x)-g(a)],\infty} \int_a^x \left(g\left(t\right)-g\left(a\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(x)-g(a)],n} \int_a^x \left(g\left(t\right)-g\left(a\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$
$$+\frac{1}{2}\begin{cases} \|k\|_{[0,g(b)-g(x)],\infty} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt \\ \\ \|k\|_{[0,g(b)-g(x)],n} \int_x^b \left(g\left(b\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt, \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$
$$\leq \frac{1}{2}\begin{cases} \|k\|_{[0,g(b)-g(a)],\infty} \\ \times\left[\int_a^x \left(g\left(t\right)-g\left(a\right)\right)\left|f'\left(t\right)\right|dt+\int_x^b \left(g\left(b\right)-g\left(t\right)\right)\left|f'\left(t\right)\right|dt\right] \\ \\ \|k\|_{[0,g(b)-g(a)],n} \\ \times\left[\int_a^x \left(g\left(t\right)-g\left(a\right)\right)^{1/m}\left|f'\left(t\right)\right|dt+\int_x^b \left(g\left(b\right)-g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt\right], \\ \text{if } m,n>1,\ \frac{1}{m}+\frac{1}{n}=1 \end{cases}$$

Observe that

$$
\begin{aligned}
&\int_a^x (g(t)-g(a))|f'(t)|\,dt+\int_x^b (g(b)-g(t))|f'(t)|\,dt \\
&\leq \begin{cases} (g(x)-g(a))\int_a^x |f'(t)|\,dt+(g(b)-g(x))\int_x^b |f'(t)|\,dt \\ \\ \sup_{t\in[a,x]}|f'(t)|\int_a^x (g(t)-g(a))\,dt+\sup_{t\in[a,x]}|f'(t)|\int_x^b (g(b)-g(t))\,dt \end{cases} \\
&\leq \begin{cases} (g(x)-g(a))\int_a^x |f'(t)|\,dt+(g(b)-g(x))\int_x^b |f'(t)|\,dt \\ \\ \sup_{t\in[a,x]}|f'(t)|\left(\int_a^x g(t)\,dt-g(a)(x-a)\right) \\ +\sup_{t\in[a,x]}|f'(t)|\left(g(b)(b-x)-\int_x^b g(t)\,dt\right) \end{cases} \\
&\leq \begin{cases} \max\{g(x)-g(a),g(b)-g(x)\}\int_a^b |f'(t)|\,dt \\ \\ \left(g(b)(b-x)-g(a)(x-a)+\int_a^x g(t)\,dt-\int_x^b g(t)\,dt\right)\sup_{t\in[a,b]}|f'(t)| \end{cases} \\
&= \begin{cases} \left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1} \\ \\ \left(g(b)(b-x)-g(a)(x-a)+\int_a^x g(t)\,dt-\int_x^b g(t)\,dt\right)\|f'\|_{[a,b],\infty}. \end{cases}
\end{aligned}
$$

We can state the following corollary that provides simple error bounds in terms of the functions involved:

Corollary 2. *With the assumptions of Theorem 2, we have*

$$
\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \\
&\qquad\leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],\infty} \\
&\quad\times\begin{cases} \left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1} \\ \\ \left(g(b)(b-x)-g(a)(x-a)+\int_a^x g(t)\,dt-\int_x^b g(t)\,dt\right)\|f'\|_{[a,b],\infty}. \end{cases}
\end{aligned} \tag{3.6}
$$

for $x\in(a,b)$.

Remark 2. *If we take in the first branch of (3.6)* $x=M_g(a,b)$, *then we get*

$$
\begin{aligned}
&\left|\breve{P}_{k,g,a+,b-}f-K\left(\frac{g(b)-g(a)}{2}\right)f(M_g(a,b))\right| \\
&\qquad\leq \frac{1}{4}(g(b)-g(a))\|k\|_{[0,g(b)-g(a)],\infty}\|f'\|_{[a,b],1},
\end{aligned} \tag{3.7}
$$

where $\breve{P}_{k,g,a+,b-}f := \breve{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right),$ *while if we take* $x = \frac{a+b}{2}$ *in the second branch, then we get*

$$\left|\breve{M}_{k,g,a+,b-}f - \frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right) - g\left(a\right)\right) + K\left(g\left(b\right) - g\left(\frac{a+b}{2}\right)\right)\right] f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2}\left\|k\right\|_{[0,g(b)-g(a)],\infty} \times \left(\frac{g\left(b\right) - g\left(a\right)}{2}\left(b-a\right) + \int_a^{\frac{a+b}{2}} g\left(t\right)dt - \int_{\frac{a+b}{2}}^b g\left(t\right)dt\right)\left\|f'\right\|_{[a,b],\infty}. \tag{3.8}$$

Similarly, by using the second branch in (3.5), we have for $m,\, n > 1,\ \frac{1}{m} + \frac{1}{n} = 1$ that

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f\left(x\right) - \frac{1}{2}\left[K\left(g\left(b\right) - g\left(x\right)\right) + K\left(g\left(x\right) - g\left(a\right)\right)\right] f\left(x\right)\right| \\
&\leq \frac{1}{2}\left[\int_x^b \left|\mathbf{K}\left(g\left(b\right) - g\left(t\right)\right)\right|\left|f'\left(t\right)\right|dt + \int_a^x \left|\mathbf{K}\left(g\left(t\right) - g\left(a\right)\right)\right|\left|f'\left(t\right)\right|dt\right] \\
&\leq \frac{1}{2}\left\|k\right\|_{[0,g(x)-g(a)],n}\int_a^x \left(g\left(t\right) - g\left(a\right)\right)^{1/m}\left|f'\left(t\right)\right|dt \\
&\quad + \frac{1}{2}\left\|k\right\|_{[0,g(b)-g(x)],n}\int_x^b \left(g\left(b\right) - g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt \\
&\leq \frac{1}{2}\left\|k\right\|_{[0,g(b)-g(a)],n} \\
&\times \left[\int_a^x \left(g\left(t\right) - g\left(a\right)\right)^{1/m}\left|f'\left(t\right)\right|dt + \int_x^b \left(g\left(b\right) - g\left(t\right)\right)^{1/m}\left|f'\left(t\right)\right|dt\right]
\end{aligned} \tag{3.9}$$

for $x \in (a,b)$.

Using Hölder's integral inequality for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$ we have

$$
\begin{aligned}
&\int_a^x (g(x)-g(t))^{1/m}|f'(t)|\,dt+\int_x^b (g(b)-g(t))^{1/m}|f'(t)|\,dt\\
&\leq\left(\int_a^x (g(t)-g(a))^{p/m}\,dt\right)^{1/p}\left(\int_a^x |f'(t)|^q\,dt\right)^{1/q}\\
&+\left(\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\left(\int_x^b |f'(t)|^q\,dt\right)^{1/q}\\
&\leq\left[\left(\left(\int_a^x (g(t)-g(a))^{p/m}\,dt\right)^{1/p}\right)^p+\left(\left(\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\right)^p\right]^{1/p}\\
&\times\left[\left(\left(\int_a^x |f'(t)|^q\,dt\right)^{1/q}\right)^q+\left(\left(\int_x^b |f'(t)|^q\,dt\right)^{1/q}\right)^q\right]^{1/q}\\
&=\left(\int_a^x (g(t)-g(a))^{p/m}\,dt+\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\\
&\times\left(\int_a^x |f'(t)|^q\,dt+\int_x^b |f'(t)|^q\,dt\right)^{1/q}\\
&=\left(\int_a^x (g(t)-g(a))^{p/m}\,dt+\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\left(\int_a^b |f'(t)|^q\,dt\right)^{1/q}\\
&=\left(\int_a^x (g(t)-g(a))^{p/m}\,dt+\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\|f'\|_{[a,b],q}\,,
\end{aligned}
$$

where in the second inequality we used the Hölder's elementary inequality (3.2).

Therefore, we can state the following corollary that provided simple error bounds in terms of the functions involved.

Corollary 3. *With the assumptions of Theorem 2, we have*

$$
\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|\\
&\leq\frac{1}{2}\|k\|_{[0,g(b)-g(a)],n}\\
&\times\left(\int_a^x (g(t)-g(a))^{p/m}\,dt+\int_x^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p}\|f'\|_{[a,b],q}
\end{aligned}
\tag{3.10}
$$

for $x\in(a,b)$, where $m,\ n>1,\ \frac{1}{m}+\frac{1}{n}=1$ and $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

If we take in (3.10) $x = M_g(a,b)$, then we get the simple inequality

$$\left|\breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right) f(M_g(a,b))\right| \le \frac{1}{2}\|k\|_{[0,g(b)-g(a)],n} \times \left(\int_a^{M_g(a,b)} (g(t)-g(a))^{p/m}\,dt + \int_{M_g(a,b)}^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p} \|f'\|_{[a,b],q}. \tag{3.11}$$

Also, if we take $m = p$ and $n = q$ in (3.10), then we get

$$\left|\breve{S}_{k,g,a+,b-}f(x) - \frac{1}{2}\left[K(g(b)-g(x)) + K(g(x)-g(a))\right] f(x)\right| \le \frac{1}{2}\|k\|_{[0,g(b)-g(a)],q} \times \left(g(b)(b-x) - g(a)(x-a) + \int_a^x g(t)\,dt - \int_x^b g(t)\,dt\right)^{1/p} \|f'\|_{[a,b],q} \tag{3.12}$$

for $x \in (a,b)$, while from (3.11) we get

$$\left|\breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right) f(M_g(a,b))\right| \le \frac{1}{2}\|k\|_{[0,g(b)-g(a)],q}\|f'\|_{[a,b],q} \times \left(g(b)(b-M_g(a,b)) - g(a)(M_g(a,b)-a) + \int_a^{M_g(a,b)} g(t)\,dt - \int_{M_g(a,b)}^b g(t)\,dt\right)^{1/p}. \tag{3.13}$$

Moreover, if we take in (3.10) $x = \frac{a+b}{2}$, then we get

$$\Bigg|\breve{M}_{k,g,a+,b-}f - \frac{1}{2}\left[K\left(g(b) - g\left(\frac{a+b}{2}\right)\right) + K\left(g\left(\frac{a+b}{2}\right) - g(a)\right)\right] f\left(\frac{a+b}{2}\right)\Bigg| \le \frac{1}{2}\|k\|_{[0,g(b)-g(a)],n} \times \left(\int_a^{\frac{a+b}{2}} (g(t)-g(a))^{p/m}\,dt + \int_{\frac{a+b}{2}}^b (g(b)-g(t))^{p/m}\,dt\right)^{1/p} \|f'\|_{[a,b],q} \tag{3.14}$$

where $m,\ n > 1$, $\frac{1}{m} + \frac{1}{n} = 1$ and $p,\ q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

If we take in (3.14) $m=p$ and $n=q$, then we get

$$\begin{aligned}(3.15)\quad & \Big| \breve{M}_{k,g,a+,b-}f \\ & -\frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right)\Big| \\ & \qquad\leq \frac{1}{2}\|k\|_{[0,g(b)-g(a)],q} \\ & \times\left(\frac{g(b)-g(a)}{2}(b-a)+\int_a^{\frac{a+b}{2}}g(t)\,dt-\int_{\frac{a+b}{2}}^b g(t)\,dt\right)^{1/p}\|f'\|_{[a,b],q}.\end{aligned}$$

4. Example for an Exponential Kernel

The above inequalities may be written for all the particular fractional integrals introduced in the introduction. We consider here only an example for a general exponential kernel that generalizes the transforms (1.16) and (1.17).

For $\alpha,\ \beta\in\mathbb{R}$ we consider the kernel $k(t):=\exp[(\alpha+\beta i)t]$, $t\in\mathbb{R}$. We have

$$K(t)=\frac{\exp[(\alpha+\beta i)t]-1}{(\alpha+\beta i)},\ \text{if } t\in\mathbb{R}$$

for $\alpha,\ \beta\neq 0$.

Also, we have

$$|k(s)|:=|\exp[(\alpha+\beta i)s]|=\exp(\alpha s)\ \text{ for } s\in\mathbb{R}$$

and

$$\mathbf{K}(t)=\int_0^t\exp(\alpha s)\,ds=\frac{\exp(\alpha t)-1}{\alpha}\ \text{if } 0<t,$$

for $\alpha\neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We have

$$\begin{aligned}(4.1)\quad \mathcal{G}^{\alpha+\beta i}_{g,a+,b-}f(x) &= \frac{1}{2}\int_x^b\exp[(\alpha+\beta i)(g(b)-g(t))]g'(t)f(t)\,dt \\ &+\frac{1}{2}\int_a^x\exp[(\alpha+\beta i)(g(t)-g(a))]g'(t)f(t)\,dt\end{aligned}$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}(4.2)\quad & \mathcal{H}^{\alpha+\beta i}_{h,a+,b-}f(x) \\ & :=\mathcal{G}^{\alpha+\beta i}_{\ln h,a+,b-}f(x) \\ & =\frac{1}{2}\left[\int_a^x\left(\frac{h(t)}{h(a)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt+\int_x^b\left(\frac{h(b)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned}$$

for $x\in(a,b)$.

Using the first part of (3.3), we have

$$\begin{aligned}(4.3)\quad & \Big|\mathcal{G}_{g,a+,b-}^{\alpha+\beta i}f(x) \\ & -\frac{1}{2}\left[\frac{\exp\left[(\alpha+\beta i)(g(b)-g(x))\right]+\exp\left[(\alpha+\beta i)(g(x)-g(a))\right]-2}{(\alpha+\beta i)}\right]f(x)\Big| \\ & \leq \frac{1}{2}\int_x^b\left[\frac{\exp(\alpha(g(b)-g(t)))-1}{\alpha}\right]|f'(t)|\,dt \\ & \quad+\frac{1}{2}\int_a^x\left[\frac{\exp(\alpha(g(t)-g(a)))-1}{\alpha}\right]|f'(t)|\,dt\end{aligned}$$

for $x\in(a,b)$.

If we denote

$$\begin{aligned}\bar{\mathcal{G}}_{g,a+,b-}^{\alpha+\beta i}f &:= \mathcal{G}_{g,a+,b-}^{\alpha+\beta i}f(M_g(a,b)) \\ &= \frac{1}{2}\int_{M_g(a,b)}^b\exp\left[(\alpha+\beta i)(g(b)-g(t))\right]g'(t)f(t)\,dt \\ &+\frac{1}{2}\int_a^{M_g(a,b)}\exp\left[(\alpha+\beta i)(g(t)-g(a))\right]g'(t)f(t)\,dt\end{aligned}$$

then by (4.3) we get

$$\begin{aligned}(4.4)\quad & \left|\bar{\mathcal{G}}_{g,a+,b-}^{\alpha+\beta i}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}f(M_g(a,b))\right| \\ & \leq \frac{1}{2}\int_{M_g(a,b)}^b\left[\frac{\exp(\alpha(g(b)-g(t)))-1}{\alpha}\right]|f'(t)|\,dt \\ & \quad+\frac{1}{2}\int_a^{M_g(a,b)}\left[\frac{\exp(\alpha(g(t)-g(a)))-1}{\alpha}\right]|f'(t)|\,dt.\end{aligned}$$

Assume that $\alpha>0$, then

$$\|k\|_{[0,g(b)-g(a)],\infty}=\sup_{s\in[0,g(b)-g(a)]}\exp(\alpha s)=\exp(\alpha[g(b)-g(a)])$$

and by (3.6) we have

$$\begin{aligned}(4.5)\quad & \Big|\mathcal{G}_{g,a+,b-}^{\alpha+\beta i}f(x) \\ & -\frac{1}{2}\left[\frac{\exp\left[(\alpha+\beta i)(g(b)-g(x))\right]+\exp\left[(\alpha+\beta i)(g(x)-g(a))\right]-2}{(\alpha+\beta i)}\right]f(x)\Big| \\ & \leq \frac{1}{2}\exp(\alpha[g(b)-g(a)]) \\ & \times\begin{cases}\left(\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(a)+g(b)}{2}\right|\right)\|f'\|_{[a,b],1} \\ \\ \left(g(b)(b-x)-g(a)(x-a)+\int_a^x g(t)\,dt-\int_x^b g(t)\,dt\right)\|f'\|_{[a,b],\infty}\end{cases}\end{aligned}$$

for $x\in(a,b)$.

In particular,

$$\left|\bar{\mathcal{G}}_{g,a+,b-}^{\alpha+\beta i}f-\frac{\exp\left[(\alpha+\beta i)\frac{g(b)-g(a)}{2}\right]-1}{(\alpha+\beta i)}f\left(M_{g}\left(a,b\right)\right)\right| \le\frac{1}{4}\exp\left(\alpha\left[g\left(b\right)-g\left(a\right)\right]\right)\left(g\left(b\right)-g\left(a\right)\right)\left\|f'\right\|_{[a,b],1}. \tag{4.6}$$

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then by (4.6) we get

$$\left|\bar{\mathcal{H}}_{h,a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}f\left(M_{g}\left(a,b\right)\right)\right| \le\frac{1}{4}\left(\frac{h\left(b\right)}{h\left(a\right)}\right)^{\alpha}\ln\left(\frac{h\left(b\right)}{h\left(a\right)}\right)\left\|f'\right\|_{[a,b],1}, \tag{4.7}$$

where $\bar{\mathcal{H}}_{h,a+,b-}^{\alpha+\beta i}f:=\bar{\mathcal{G}}_{\ln h,a+,b-}^{\alpha+\beta i}f$.

References

[1] R. P. Agarwal, M.-J. Luo and R. K. Raina, On Ostrowski type inequalities, *Fasc. Math.* **56** (2016), 5-27.

[2] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[3] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[4] A. O. Akdemir, Inequalities of Ostrowski's type for m- and (α,m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[5] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[6] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[7] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[8] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[9] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.

[10] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[11] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[12] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]

[13] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[14] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[15] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[16] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[17] S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[18] S. S. Dragomir, Further Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 84. [Online http://rgmia.org/papers/v20/v20a84.pdf].

[19] S. S. Dragomir, Ostrowski and trapezoid type inequalities for the generalized k-g-fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. .

[20] S. S. Dragomir, Trapezoid type inequalities for the generalized k-g-fractional integrals of absolutely continuous functions, *RGMIA Res. Rep. Coll.* **20** (2017), Art.

[21] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[22] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[23] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0.

[24] M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type Inequalities for convex functions via fractional integrals, Preprint arXiv:1701.00092.

[25] M. A. Noor, K. I. Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445

[26] R. K. Raina , On generalized Wright's hypergeometric functions and fractional calculus operators, *East Asian Math. J.*, **21**(2)(2005), 191-203.

[27] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[28] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[29] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[30] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[31] M. Tunç, Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[32] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[33] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[34] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

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