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NEW HERMITE-HADAMARD TYPE INEQUALITIES USING THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRAL

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Abstract. Several new inequalities are presented in this papers, as a continuation of the results given in previous papers, concerning the Hermite-Hadamard type inequalities for fractional integrals and for fractional integral operators. Thse results are established using four integral identities for n-time differentiable functions.

1. **Introduction**

The classical Hermite-Hadamard's inequality has been considered very useful in mathematical analysis being very intensely studied and generalized by many authors, like [28, 10, 9, 13, 1, 17, 21, 29, 15] and the references therein.

Many papers study the Riemann-Liouville fractionals integrals and give interesting generalizations of Hermite-Hadamard type inequalities using these kind of integrals, see for instance [12, 11, 13, 14, 15, 22, 19, 21, 17, 28, 29, 30, 31, 32, 24, 34, 3].

We will begin now by recalling the classical definition for the convex functions and then the definitions for other kind of convexities.

Definition 1. *A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on an interval I if the inequality*

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) \tag{1}$$

holds for all $x, y \in I$ and $t \in [0,1]$. The function f is said to be concave on I if the inequality (1) takes place in reversed direction.

It is necessary to recall below also the definition of fractionals integrals, see [12, 14, 13, 22, 23, 30] and then the definition of fractional integral operators. For other type of convexity see also the papers [25, 20]. The definition of s-convex function in the second sense was given in Breckner's paper [4] and the definition of s-convex functionin the first sense was introduced by Orlicz in [27].

Definition 2. *A function $f : [a,b] \to \mathbb{R}$ is said to be quasi-convex onl $[a,b]$ if*

$$f(tx + (1-t)y) \le \sup\{f(x), f(y)\}$$

holds for all $x, y \in [a,b]$ and $t \in [0,1]$.

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Definition 3. *A function $f : I \to \mathbb{R}$ is said to be P-convex on $[a, b]$ if it is nonnegative and for all $x, y \in I$ and $\lambda \in [9, 1]$*

$$f(tx + (1-t)y) \leq f(x) + f(y).$$

Definition 4. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-convex in the first sense on an interval I if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t^s)f(y)$$

holds for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 5. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-convex in the second sense on an interval I if the inequality*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$, $t \in [0, 1]$ and for some fixed $s \in (0, 1]$.

Definition 6. *A function $f : I \subset \mathbb{R}_+ \to \mathbb{R}_+$ is said to be s-Godunova-Levin functions of second kind on an interval I if the inequality*

$$f(tx + (1-t)y) \leq \frac{1}{t^s} f(x) + \frac{1}{(1-t)^s} f(y)$$

holds for all $x, y \in I$, $t \in (0, 1)$ and for some fixed $s \in [0, 1]$.

It is easy to see that for $s = 0$ s-Godunova-Levin functions of second kind are functions P-convex.

The classical Hermite-Hadamard's inequality for convex functions is

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Moreover, if the function f is concave then the inequality (2) hold in reversed direction.

Definition 7. *Let $f \in L[a, b]$. The Riemann-Liouville integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $\alpha \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$.

It is well-known that the beta function is defined when $a, b > 0$ by

$$R(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1}(1-t)^{b-1dt}.$$

The following class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \qquad (\rho,\ \lambda > 0;\ |x| < \mathbf{R}),$$

where the coefficients $\sigma(k)$, $(k \in \mathbb{N} = \mathbb{N} \cup \{0\})$ is a bounded sequence of positive real numbers and $\mathbf{R}$ is the set of real numbers, as in [24], was introduced in [33] and was used for giving in [3] the following left-sided and right-sided fractional integral operators from below:

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,a^+;w}\varphi)(x) = \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(x-t)^{\rho}]\varphi(t)dt, \quad (x > a > 0),$$

and

$$(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}\varphi)(x) = \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}[w(t-x)^{\rho}]\varphi(t)dt, \quad (0 < x < b),$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and $\varphi(t)$ is such that the integral on the right side exists. There are new integral inequalities for this operator, seet [24, 3, 34] and references therein.

It is important to mention that for example the classical Riemann-Liouville fractional integrals J^{α}_{a+} and J^{α}_{b-} of order α were obtained by setting $\lambda = \alpha$, $\sigma(0) = 1$ and $w = 0$ in previous integrals.

We recall below two identities given in a previous paper. In their demonstrations, the integration by parts and the induction will be used.

The following result is a generalization of Lemma 1 from [6] for fractional integral operators for functions n-time differentiable.

Lemma 1. *Let $f : [a,b] \to \mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0 < a < b$, $\lambda > n-1, x \in (a,b)$ and $t \in [0,1]$. If $f^{(n)} \in L[a,b]$ then the following equality for generalized fractional integrals holds:*

$$\int_0^1 t^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}t^{\rho}] f^{(n)}(tx + (1-t)a)dt +$$

$$+ \int_0^1 (1-t)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}(1-t)^{\rho}] f^{(n)}(tb + (1-t)x)dt =$$

$$= \sum_{k=1}^{n} \{ \frac{(-1)^{k-1}}{(x-a)^k} \mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^{\rho}] - \frac{1}{(b-x)^k} \mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^{\rho}] \} f^{(n-k)}(x) +$$

$$+ \frac{(-1)^n}{(x-a)^{\lambda+1}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w} f \right)(a) + \frac{1}{(b-x)^{\lambda+1}} \left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w} f \right)(b).$$

Next result is a generalization of Lemma 4 from [5] for fractional integral operators for functions n-time differentiable.

Lemma 2. *Let $f : [a,b] \to \mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0 < a < b$, $\lambda > n-1, x \in (a,b)$ and $t,\ r \in [0,1]$. If $f^{(n)} \in L[a,b]$ then the following equality for generalized fractional integrals holds:*

$$\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(1-r)^\rho (x-a)^\rho t^\rho] f^{(n)}(t(ra+(1-r)x)+(1-t)a)dt+$$

$$+\int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[wr^\rho (x-a)^\rho (1-t)^\rho] f^{(n)}(tx+(1-t)(ra+(1-r)x))dt+$$

$$+\int_0^1 t^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[w(1-r)^\rho (b-x)^\rho t^\rho] f^{(n)}(t(rx+(1-r)b)+(1-t)x)dt+$$

$$+\int_0^1 (1-t)^\lambda \mathcal{F}^\sigma_{\rho,\lambda+1}[wr^\rho (b-x)^\rho (1-t)^\rho] f^{(n)}(tb+(1-t)(rx+(1-r)b))dt=$$

$$=\sum_{k=1}^n \frac{(-1)^{k-1}}{(1-r)^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(1-r)^\rho (x-a)^\rho]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(1-r)^\rho (b-x)^\rho]\}-$$

$$-\sum_{k=1}^n \frac{1}{r^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^\sigma_{\rho,\lambda-k+2}[wr^\rho (x-a)^\rho]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^\sigma_{\rho,\lambda-k+2}[wr^\rho (b-x)^\rho]\}+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^\sigma_{\rho,\lambda-n+1,(ra+(1-r)x)^-;w}f)(a)+$$

$$+\frac{1}{r^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^\sigma_{\rho,\lambda-n+1,(ra+(1-r)x)^+;w}f)(x)+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^\sigma_{\rho,\lambda-n+1,(rx+(1-r)b)^-;w}f)(x)+$$

$$+\frac{1}{r^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^\sigma_{\rho,\lambda-n+1,(rx+(1-r)b)^+;w}f)(b).$$

In this paper the two new identities given above will be used below for several applications, like Hermite-Hadamard type inequalities for functions whose the n-time derivative iin absolute value of certain powers satisfies different type of convexities via Riemann-Liouville fractional integral operators and via fractional integrals.

2. **New Hermite-Hadamard type inequalities for fractional integral**

Next inequalities are satisfied for different type of convexities, but this time we used Lemma 1 or Lemma 2, see [7], modulus properties and the definition of convexities with the inequality from the proof of Theorem 1 from below.

Proposition 1. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and t $f^{(n)} \in L[a,b]$ with $a, b \in I^0,\ 0 < a < b$. If $|f^{(n)}|^q$ is convex on $[a,b]$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality takes place:*

$$|I(f,x,a,b,\alpha,n)| = |\sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2) f^{(n-k)}(x) \left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}} \right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)]| \leq$$

$$\leq \frac{1}{(\alpha+1)^{\frac{1}{p}} (\alpha+2)^{\frac{1}{q}}} \{(x-a) \left(|f^{(n)}(x)|^q + \frac{1}{\alpha+1} |f^{(n)}(a)|^q \right)^{\frac{1}{q}} + (b-x) \left(\frac{1}{\alpha+1} |f^{(n)}(b)|^q + |f^{(n)}(x)| \right)^{\frac{1}{q}} \}$$

Proposition 2. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0,\ 0 < a < b$. If $|f^{(n)}|^q$ is quasi-convex on $[a,b]$ for some fixed $q > 1$, where $\frac{1}{p} + \frac{1}{q} = 1$ then the following inequality holds:*

$$|I(f,x,a,b,\alpha,n)| = |\sum_{k=2}^{n} \alpha(\alpha-1)...(\alpha-k+2) f^{(n-k)}(x) \left(\frac{(-1)^{k-1}}{(x-a)^{k-1}} - \frac{1}{(b-x)^{k-1}} \right) +$$

$$+\Gamma(\alpha+1)[\frac{(-1)^n}{(x-a)^\alpha} J_{x^-}^{\alpha-n+1} f(a) + \frac{1}{(b-x)^\alpha} J_{x^+}^{\alpha-n+1} f(b)]| \leq$$

$$\leq \frac{1}{\alpha+1} \{(x-a) \sup\{|f^{(n)}(x)|, |f^{(n)}(a)|\} + (b-x) \sup\{|f^{(n)}(b)|, |f^{(n)}(x)|\}\}$$

Proposition 3. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0,\ 0 < a < b,\ \lambda \in (0,1),\ x \in [a,b]$. If $|f^{(n)}|^q$ is P-convex on $[a,b]$ then the following inequality holds:*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \leq \frac{1}{(\alpha+1)^{\frac{1}{p}}} \frac{1}{(\alpha+1)^{\frac{1}{q}}} \{(1-\lambda)(x-a)(|f^{(n)}(\lambda a + (1-\lambda)x)|^q + |f^{(n)}(a)|^q)^{\frac{1}{q}} +$$

$$+\lambda(x-a)(|f^{(n)}(x)|^q + |f^{(n)}(\lambda a + (1-\lambda)x)|^q)^{\frac{1}{q}} +$$

$$+(1-\lambda)(b-x)(|f^{(n)}(\lambda x + (1-\lambda)b)|^q + |f^{(n)}(x)|^q)^{\frac{1}{q}} +$$

$$+\lambda(b-x)(|f^{(n)}(b)|^q + |f^{(n)}(\lambda x + (1-\lambda)b)|^q)^{\frac{1}{q}}\},$$

where $\alpha > n-1$.

Proposition 4. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0,1)$, $x \in [a,b]$. If $|f^{(n)}|^q$ is quasi-convex on $[a,b]$ then the following inequality holds:*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \leq \frac{1}{\alpha+1}\{(1-\lambda)(x-a)\sup\{|f^{(n)}(\lambda a+(1-\lambda)x)|, |f^{(n)}(a)|\}+$$
$$+\lambda(x-a)\sup\{|f^{(n)}(x)|, |f^{(n)}(\lambda a+(1-\lambda)x)|\}+$$
$$+(1-\lambda)(b-x)\sup\{|f^{(n)}(\lambda x+(1-\lambda)b)|, |f^{(n)}(x)|\}+$$
$$+\lambda(b-x)\sup\{|f^{(n)}(b)|, |f^{(n)}(\lambda x+(1-\lambda)b)|\}\},$$

where $\alpha > n-1$.

Theorem 1. *Let $n \in \mathbb{N}^*$ and $f : I \subset \mathbb{R} \to \mathbb{R}$ be o function such that $f^{(n)}$ exists on the interior I^0 of an interval I and $f^{(n)} \in L[a,b]$ with $a, b \in I^0$, $0 < a < b$, $\lambda \in (0,1)$, $x \in [a,b]$. If $|f^{(n)}|^q$ is s-convex in the first sense on $[a,b]$ and $\alpha > n-1$. then the following inequality takes place:*

$$|\mathcal{I}(f,x,a,b,\lambda,\alpha,n)| \leq \frac{1}{(\alpha+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)[\frac{1}{\alpha+s+1}|f^{(n)}(\lambda a+(1-\lambda)x)|^q+$$
$$+\frac{s}{(\alpha+1)(\alpha+s+1)}|f^{(n)}(a)|^q]^{\frac{1}{q}}+\lambda(x-a)[B(\alpha+1,s+1)|f^{(n)}(x)|^q+$$
$$+(\frac{1}{\alpha+1}-B(\alpha+1,s+1))|f^{(n)}(\lambda a+(1-\lambda)x)|^q]^{\frac{1}{q}}+$$
$$+(1-\lambda)(b-x)[\frac{1}{\alpha+s+1}|f^{(n)}(\lambda x+(1-\lambda)b)|^q+\frac{s}{(\alpha+1)(\alpha+s+1)}|f^{(n)}(x)|^q]^{\frac{1}{q}}+$$
$$+\lambda(b-x)[B(\alpha+1,s+1)|f^{(n)}(b)|^q+(\frac{1}{\alpha+1}-B(\alpha+1,s+1))|f^{(n)}(\lambda x+(1-\lambda)b)|^q]^{\frac{1}{q}}\}.$$

Proof. From Lemma 2 see [7] and properties of modulus we have,

$$|I(f,x,a,b,\lambda,\alpha,n)| \leq (1-\lambda)(x-a)\int_0^1 t^\alpha|f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)|dt+$$
$$+\lambda(x-a)\int_0^1(1-t)^\alpha|f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))|dt+$$
$$+(1-\lambda)(b-x)\int_0^1 t^\alpha|f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)|dt+$$
$$+\lambda(b-x)\int_0^1(1-t)^\alpha|f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))|dt.$$

Then we obtain,

$$|I(f,x,a,b,\lambda,\alpha,n)| \leq$$
$$\leq (1-\lambda)(x-a)\left(\int_0^1 t^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 t^\alpha|f^{(n)}(t(\lambda a+(1-\lambda)x)+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$
$$+\lambda(x-a)\left(\int_0^1(1-t)^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^\alpha|f^{(n)}(tx+(1-t)(\lambda a+(1-\lambda)x))|dt\right)^{\frac{1}{q}}+$$
$$+(1-\lambda)(b-x)\left(\int_0^1 t^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1 t^\alpha|f^{(n)}(t(\lambda x+(1-\lambda)b)+(1-t)x)|dt\right)^{\frac{1}{q}}+$$

$$+\lambda(b-x)\left(\int_0^1(1-t)^\alpha dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^\alpha|f^{(n)}(tb+(1-t)(\lambda x+(1-\lambda)b))|dt\right)^{\frac{1}{q}}\leq$$

$$\leq\frac{1}{(\alpha+1)^{\frac{1}{p}}}\{(1-\lambda)(x-a)[\int_0^1 t^\alpha(t^s|f^{(n)}(\lambda a+(1-\lambda)x)|^q+(1-t^s)|f^{(n)}(a)|^q)dt]^{\frac{1}{q}}+$$

$$+\lambda(x-a)[\int_0^1(1-t)^\alpha(t^s|f^{(n)}(x)|^q+(1-t^s)|f^{(n)}(\lambda a+(1-\lambda)x)|^q)dt]^{\frac{1}{q}}+$$

$$+(1-\lambda)(b-x)[\int_0^1 t^\alpha(t^s|f^{(n)}(\lambda x+(1-\lambda)b)|^q+(1-t^s)|f^{(n)}(x)|^q)dt]^{\frac{1}{q}}+$$

$$+\lambda(b-x)[\int_0^1(1-t)^\alpha(t^s|f^{(n)}(b)|^q+(1-t^s)|f^{(n)}(\lambda x+(1-\lambda)b)|^q)dt]^{\frac{1}{q}}\}.$$

By calculus taking into account of the properties of function Euler beta we obtain the desired inequality.

∎

3. Hermite-Hadamard type inequalities for fractional integral operators

We also obtain using Lemma 1, the following results for n-time differentiable functions whose absolute value is convex via fractional integral operator. These new inequalities improve results from [8], Theorem 1, 2 and 3.

Theorem 2. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b,\ \lambda>n-1, x\in(a,b)$ and $t\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is convex on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^n(-1)^{k-1}f^{(n-k)}(x)\{\frac{\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(x-a)^\rho]}{(x-a)^k}-\frac{\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(b-x)^\rho]}{(b-x)^k}\}+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$

$$\leq\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[w(x-a)^\rho]|f^{(n)}(x)|+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[w(x-a)^\rho]|f^{(n)}(a)|+$$

$$+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[w(b-x)^\rho]|f^{(n)}(x)|+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[w(b-x)^\rho]|f^{(n)}(b)|,$$

where

$$\sigma_2(k)=\frac{\sigma(k)}{\lambda+\rho k+2},\quad\sigma_3(k)=\frac{\sigma(k)}{(\lambda+\rho k+1)(\lambda+\rho k+2)}$$

and $\rho,\ \lambda>0,\quad w\in\mathbb{R}$.

Proof. Using the properties of modulus, Lemma 1 and that $|f^{(n)}|$ is convex function we get:

$$|\sum_{k=1}^n(-1)^{k-1}f^{(n-k)}(x)\{\frac{\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(x-a)^\rho]}{(x-a)^k}-\frac{\mathcal{F}^\sigma_{\rho,\lambda-k+2}[w(b-x)^\rho]}{(b-x)^k}\}+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^\sigma_{\rho,\lambda-n+1,x^+;w}f\right)(b)|=$$

$$=|I_1+I_2|\leq\int_0^1 t^\lambda|\mathcal{F}^\sigma_{\rho,\lambda+1}[w(x-a)^\rho t^\rho]f^{(n)}(tx+(1-t)a)|dt+$$

$$+\int_0^1 (1-t)^\lambda |\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^\rho(1-t)^\rho] f^{(n)}(tb+(1-t)x)|dt \leq$$

$$\leq \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(|f^{(n)}(x)| \int_0^1 t^{\lambda+\rho k+1} dt + |f^{(n)}(a)| \int_0^1 t^{\lambda+\rho k}(1-t)dt \right) +$$

$$+\sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k (b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)} \left(|f^{(n)}(b)| \int_0^1 (1-t)^{\lambda+\rho k} t dt + |f^{(n)}(x)| \int_0^1 (1-t)^{\lambda+\rho k+1} dt \right).$$

From here by easily calculus we get the desired inequality. We mention that for the integral, $\int_0^1 (1-t)^{\lambda+\rho k} t dt$ we changed the variable $1-t$ and denoted by u and then the compute the new integral obtained.
∎

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$, $s\in(0,1]$ and $t\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is s-convex in the second sense on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}(-1)^{k-1} f^{(n-k)}(x)\{\frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^\rho]}{(x-a)^k} - \frac{\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^\rho]}{(b-x)^k}\}+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a) + \frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)| \leq$$

$$\leq \mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+1}[w(x-a)^\rho]|f^{(n)}(x)| + \mathcal{F}^{\sigma_{5,s}}_{\rho,\lambda+1}[w(x-a)^\rho]|f^{(n)}(a)|+$$

$$+\mathcal{F}^{\sigma_{5,s}}_{\rho,\lambda+1}[w(b-x)^\rho|f^{(n)}(b)| + \mathcal{F}^{\sigma_{4,s}}_{\rho,\lambda+1}[w(b-x)^\rho|f^{(n)}(x)|,$$

where

$$\sigma_{4,s}(k) = \frac{\sigma(k)}{\lambda+\rho k+s+1}, \quad \sigma_{5,s}(k) = \sigma(k) B(s+1, \lambda+\rho k+1)$$

and $\rho,\ \lambda>0,\quad w\in\mathbb{R},\ s\in(0,1]$ and $B(.,.)$ is Euler beta function.

Proof. We use the same method as in Theorem 2, but this time we apply the definition of s-convex function in the second sense. ∎

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$ and $t,r\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|$ is convex on (a,b) then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}\frac{(-1)^{k-1}}{(1-r)^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(1-r)^\rho(x-a)^\rho]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(1-r)^\rho(b-x)^\rho]\}-$$

$$-\sum_{k=1}^{n}\frac{1}{r^k}\{\frac{f^{(n-k)}(ra+(1-r)x)}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[wr^\rho(x-a)^\rho]+$$

$$+\frac{f^{(n-k)}(rx+(1-r)b)}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[wr^\rho(b-x)^\rho]\}+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^-;w}f)(a)+$$

$$+\frac{1}{r^{\lambda+1}(x-a)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(ra+(1-r)x)^+;w}f)(x)+$$

$$+\frac{(-1)^n}{(1-r)^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^-;w}f)(x)+$$

$$+\frac{1}{r^{\lambda+1}(b-x)^{\lambda+1}}(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,(rx+(1-r)b)^+;w}f)(b)| \leq$$

$$\leq \mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[(1-r)^\rho(x-a)^\rho w]|f^{(n)}(ra+(1-r)x)|+$$

$$+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[(1-r)^\rho(x-a)^\rho w]|f^{(n)}(a)|+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[r^\rho(x-a)^\rho w]|f^{(n)}(x)|+$$

$$+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[r^\rho(x-a)^\rho w]|f^{(n)}(ra+(1-r)x)|+$$

$$+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[(1-r)^\rho(b-x)^\rho w]|f^{(n)}(rx+(1-r)b)|+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[(1-r)^\rho(b-x)^\rho w]|f^{(n)}(x)|+$$

$$+\mathcal{F}^{\sigma_3}_{\rho,\lambda+1}[r^\rho(b-x)^\rho w]|f^{(n)}(b)|+\mathcal{F}^{\sigma_2}_{\rho,\lambda+1}[r^\rho(b-x)^\rho w]|f^{(n)}(rx+(1-r)b)|,$$

where

$$\sigma_2(k)=\frac{\sigma(k)}{\lambda+\rho k+2},\quad \sigma_3(k)=\frac{\sigma(k)}{(\lambda+\rho k+1)(\lambda+\rho k+2)}$$

and $\rho,\ \lambda>0,\quad w\in\mathbb{R}$.

Proof. We use the same method as in Theorem 2, we shall apply Lemma 2 and the definition of the convex functions. ∎

We will present below two new Hermite-Hadamard type inequalities for functions whose n-order derivative absolute value are s-convex in the second sense, using the Riemann-Liouville fractional integral operators, which generalize inequalities from [6] and [5].

Theorem 5. *Let* $f:[a,b]\to\mathbb{R}$ *be an n-time differentiable mapping on* (a,b) *with* $0<a<b$, $\lambda>n-1, x\in(a,b)$, $s\in(0,1]$ *and* $t\in[0,1]$. *If* $f^{(n)}\in L[a,b]$ *and* $|f^{(n)}|^q$ *is s-convex in the second sense on* (a,b), $q>1$, *with* $\frac{1}{p}+\frac{1}{q}=1$, *then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^\rho]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^\rho]\}f^{(n-k)}(x)+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$

$$\leq\frac{1}{(s+1)^{\frac{1}{q}}}\{\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[w(x-a)^\rho][|f^{(n)}(x)|^q+|f^{(n)}(a)|^q]^{\frac{1}{q}}+\mathcal{F}^{\sigma_6}_{\rho,\lambda+1}[w(b-x)^\rho][|f^{(n)}(b)|^q+|f^{(n)}(x)|^q]^{\frac{1}{q}}\}$$

, where

$$\sigma_6(k)=\frac{\sigma(k)}{[(\lambda+\rho k)p+1]^{\frac{1}{p}}}$$

and $\rho,\ \lambda>0,\ w\in\mathbb{R}$.

Proof. By Lemma 1, we obtain:

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^{\rho}]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^{\rho}]\}f^{(n-k)}(x)+$$
$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$
$$\leq\int_0^1 t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(x-a)^{\rho}t^{\rho}]|f^{(n)}(tx+(1-t)a)|dt+$$
$$+\int_0^1(1-t)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}[w(b-x)^{\rho}(1-t)^{\rho}]|f^{(n)}(tb+(1-t)x)|dt$$
$$\leq\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1 t^{\rho k+\lambda}|f^{(n)}(tx+(1-t)a)|dt+$$
$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1(1-t)^{\rho k+\lambda}|f^{(n)}(tb+(1-t)x)|dt.$$

From Holder's inequality and then by s-convexity of $|f^{(n)}|^q$ we get:

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^{\rho}]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^{\rho}]\}f^{(n-k)}(x)+$$
$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$
$$\leq\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\int_0^1 t^{(\rho k+\lambda)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{(n)}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$
$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\int_0^1(1-t)^{(\rho k+\lambda)p}dt\right)^{\frac{1}{p}}\left(\int_0^1|f^{(n)}(tb+(1-t)x)|^q dt\right)^{\frac{1}{q}}\leq$$
$$\leq\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\frac{1}{[p(\lambda+\rho k)+1]^{\frac{1}{p}}}[\int_0^1(t^s|f^{(n)}(x)|^q+(1-t)^s|f^{(n)}(a)|^q)dt]^{\frac{1}{q}}+$$
$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\frac{1}{[p(\lambda+\rho k)+1]^{\frac{1}{p}}}[\int_0^1(t^s|f^{(n)}(b)|^q+(1-t)^s|f^{(n)}(x)|^q)dt]^{\frac{1}{q}}.$$

By calculus we find the desired inequality.
∎

Theorem 6. *Let $f:[a,b]\to\mathbb{R}$ be an n-time differentiable mapping on (a,b) with $0<a<b$, $\lambda>n-1, x\in(a,b)$, $s\in(0,1]$ and $t\in[0,1]$. If $f^{(n)}\in L[a,b]$ and $|f^{(n)}|^q$ is s-convex in the second sense on (a,b), $q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality for generalized fractional integral operators takes place:*

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^{\rho}]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^{\rho}]\}f^{(n-k)}(x)+$$
$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\leq$$
$$\leq\mathcal{F}^{\sigma_{7,s}}_{\rho,\lambda+1}[w(x-a)^{\rho}]+\mathcal{F}^{\sigma_{8,s}}_{\rho,\lambda+1}[w(b-x)^{\rho}]$$

, *where*

$$\sigma_{7,s}(k)=\frac{\sigma(k)}{(\lambda+\rho k+1)^{\frac{1}{p}}}[B(1,\lambda+\rho k+s+1)|f^{(n)}(x)|^q+B(\lambda+\rho k+1,s+1)|f^{(n)}(a)|^q]^{\frac{1}{q}}$$

$$\sigma_{8,s}(k)=\frac{\sigma(k)}{(\lambda+\rho k+1)^{\frac{1}{p}}}[B(\lambda+\rho k+1,s+1)|f^{(n)}(b)|^q+B(1,\lambda+\rho k+s+1)|f^{(n)}(x)|^q]^{\frac{1}{q}}$$

and $\rho,\ \lambda>0,\ \ w\in\mathbb{R}$.

Proof. From Lemma 1 and properties of modulus, like before we get

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^\rho]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^\rho]\}f^{(n-k)}(x)+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\le$$

$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1 t^{\rho k+\lambda}|f^{(n)}(tx+(1-t)a)|dt+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\int_0^1(1-t)^{\rho k+\lambda}|f^{(n)}(tb+(1-t)x)|dt.$$

Now, using power -mean inequality and s-convexity of $|f^{(n)}|^q$, we have,

$$|\sum_{k=1}^{n}\{\frac{(-1)^{k-1}}{(x-a)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(x-a)^\rho]-\frac{1}{(b-x)^k}\mathcal{F}^{\sigma}_{\rho,\lambda-k+2}[w(b-x)^\rho]\}f^{(n-k)}(x)+$$

$$+\frac{(-1)^n}{(x-a)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^-;w}f\right)(a)+\frac{1}{(b-x)^{\lambda+1}}\left(\mathcal{J}^{\sigma}_{\rho,\lambda-n+1,x^+;w}f\right)(b)|\le$$

$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\int_0^1 t^{\rho k+\lambda}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^{\rho k+\lambda}|f^{(n)}(tx+(1-t)a)|^q dt\right)^{\frac{1}{q}}+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\left(\int_0^1(1-t)^{\rho k+\lambda}dt\right)^{\frac{1}{p}}\left(\int_0^1(1-t)^{\rho k+\lambda}|f^{(n)}(tb+(1-t)x)|^q dt\right)^{\frac{1}{q}}\le$$

$$\le\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(x-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\frac{1}{(\lambda+\rho k+1)^{\frac{1}{p}}}[\int_0^1 t^{\lambda+\rho k}.(t^s|f^{(n)}(x)|^q+(1-t)^s|f^{(n)}(a)|^q)dt]^{\frac{1}{q}}+$$

$$+\sum_{k=0}^{\infty}\frac{\sigma(k)|w|^k(b-x)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\frac{1}{(\lambda+\rho k+1)^{\frac{1}{p}}}[\int_0^1(1-t)^{\lambda+\rho k}.(t^s|f^{(n)}(b)|^q+(1-t)^s|f^{(n)}(x)|^q)dt]^{\frac{1}{q}}.$$

By easy calculus we establish the desired inequality, using the properties of function Euler beta.

∎

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