Received 14/02/17

INEQUALITIES FOR SYMMETRIZED OR ANTI-SYMMETRIZED INNER PRODUCTS OF COMPLEX-VALUED FUNCTIONS DEFINED ON AN INTERVAL

S. S. DRAGOMIR[1,2]

Abstract. For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}$, and defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]$$

and the *anti-symmetrical transform of* f on the interval $[a,b]$ denoted by $\tilde{f}$ and defined by

$$\tilde{f}:=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

We consider in this paper the inner products

$$\langle f,g\rangle_{\smile}:=\int_a^b\breve{f}(t)\,\overline{\breve{g}(t)}dt\text{ and }\langle f,g\rangle_{\sim}:=\int_a^b\tilde{f}(t)\,\overline{\tilde{g}(t)}dt,$$

the corresponding norms and establish their fundamental properties. Some Schwarz and Grüss' type inequalities are also provided.

1. Introduction

For a function $f:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform of* f on the interval $[a,b]$, denoted by $\breve{f}_{[a,b]}$ or simply $\breve{f}$, when the interval $[a,b]$ is implicit, as defined by

$$\breve{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]. \tag{1.1}$$

The *anti-symmetrical transform of* f on the interval $[a,b]$ is denoted by $\tilde{f}_{[a,b]}$, or simply $\tilde{f}$ and is defined by

$$\tilde{f}:=\frac{1}{2}\left[f(t)-f(a+b-t)\right],t\in[a,b].$$

It is obvious that for any function f we have $\breve{f}+\tilde{f}=f$. We observe that the symmetrical and anti-symmetrical transforms are *linear transforms*, namely

$$(\alpha f+\beta g)^{\smile}=\alpha\breve{f}+\beta\breve{g}$$

and

$$(\alpha f+\beta g)^{\sim}=\alpha\tilde{f}+\beta\tilde{g}$$

for any functions f, g and any scalars α, $\beta\in\mathbb{C}$.

1991 *Mathematics Subject Classification.* 26D15; 25D10.

Key words and phrases. Measurable functions, Integrable functions, Symmetrical and anti-symmetrical transforms, Inner products, Cauchy-Bunyakovsky-Schwarz inequality, Grüss' type inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 12, 17 pp.

We say that the function is *symmetrical a.e.* on the interval $[a,b]$ if

$$f(t)=f(a+b-t) \text{ for almost every } t\in[a,b]$$

and *anti-symmetrical a.e.* on the interval $[a,b]$ if

$$f(t)=-f(a+b-t) \text{ for almost every } t\in[a,b].$$

We observe that if the function is (Lebesgue) integrable on $[a,b]$, then by the change of variable $s=a+b-t$, $t\in[a,b]$ we have

$$\int_a^b \breve{f}(t)\,dt=\frac{1}{2}\left[\int_a^b f(t)\,dt+\int_a^b f(a+b-s)\,ds\right]=\int_a^b f(t)\,dt$$

and

$$\int_a^b \tilde{f}(t)\,dt=\frac{1}{2}\left[\int_a^b f(t)\,dt-\int_a^b f(a+b-s)\,ds\right]=0.$$

Assume that all functions below are measurable and the integrals involved are finite, then by considering the functionals

$$\langle f,g\rangle_{\smile}:=\int_a^b \breve{f}(t)\,\overline{\breve{g}(t)}dt \text{ and } \langle f,g\rangle_{\sim}:=\int_a^b \tilde{f}(t)\,\overline{\tilde{g}(t)}dt$$

we have

$$\langle \alpha f+\beta h,g\rangle_{\smile}=\alpha\langle f,g\rangle_{\smile}+\beta\langle h,g\rangle_{\smile},\ \ \langle g,f\rangle_{\smile}=\overline{\langle f,g\rangle_{\smile}}$$

for any scalars α, β and

$$\langle f,f\rangle_{\smile}\geq 0,$$

and the similar relations for the functional $\langle\cdot,\cdot\rangle_{\sim}$.

These show that the functionals $\langle\cdot,\cdot\rangle_{\smile}$ and $\langle\cdot,\cdot\rangle_{\sim}$ are nonnegative Hermitian forms. We also observe that if $\breve{f}\in L_2[a,b]$, the Hilbert space of Lebesgue square-integrable functions on $[a,b]$ and $\langle f,f\rangle_{\smile}=0$, then f must be *anti-symmetrical a.e.* on the interval $[a,b]$. Also, if $\tilde{f}\in L_2[a,b]$ and $\langle f,f\rangle_{\sim}=0$, then f must be *symmetrical a.e.* on the interval $[a,b]$.

We can define the equivalence relation " $\smile$ " by $f\smile g\Leftrightarrow f-g$ is *anti-symmetrical a.e.* on the interval $[a,b]$. Similarly, we have the equivalence relation " $\sim$ " by $f\sim g\Leftrightarrow f-g$ is *symmetrical a.e.* on the interval $[a,b]$.

We define the linear space of measurable functions $L_2^{\smile}[a,b]$ as the collections of all " $\smile$ "-classes of measurable functions for which $\int_a^b\left|\breve{f}(t)\right|^2 dt<\infty$, and in a similar way the space $L_2^{\sim}[a,b]$. In this situation $\langle\cdot,\cdot\rangle_{\smile}$ becomes a proper inner product on $L_2^{\smile}[a,b]$ and $\langle\cdot,\cdot\rangle_{\sim}$ a proper inner product on $L_2^{\sim}[a,b]$. Therefore $\|\cdot\|_{\smile}:=\langle\cdot,\cdot\rangle_{\smile}^{1/2}$ and $\|\cdot\|_{\sim}:=\langle\cdot,\cdot\rangle_{\sim}^{1/2}$ are norms on $L_2^{\smile}[a,b]$ and $L_2^{\sim}[a,b]$, respectively.

In what follows we establish some fundamental properties for these inner products. Some Schwarz and Grüss' type inequalities are also provided. For recent results in connection to Grüss' inequality, see [1]-[12], [14]-[18], [20]-[27] and the references therein.

2. Some Fundamental Properties

We have:

Theorem 1. *If $f,\ g\in L_2[a,b]$ then $f,\ g\in L_2^{\smile}[a,b]$, we have the representations*

$$\begin{aligned}\langle f,g\rangle_{\smile}&=\frac{1}{2}\left[\int_a^b f(t)\overline{g(t)}+\int_a^b f(a+b-t)\overline{g(t)}dt\right]\\&=\frac{1}{2}\left[\int_a^b f(t)\overline{g(t)}+\int_a^b f(t)\overline{g(a+b-t)}dt\right]\\&=\int_a^b f(t)\overline{\breve{g}(t)}dt=\int_a^b\breve{f}(t)\overline{g(t)}dt,\end{aligned}\tag{2.1}$$

$$\begin{aligned}\|f\|_{\smile}^2&=\frac{1}{2}\left[\int_a^b|f(t)|^2+\int_a^b f(t)\overline{f(a+b-t)}dt\right]\\&=\int_a^b f(t)\overline{\breve{f}(t)}dt=\int_a^b\breve{f}(t)\overline{f(t)}dt\end{aligned}\tag{2.2}$$

and the inequalities

$$\begin{aligned}&\left(\frac{1}{b-a}\int_a^b|f(t)|\,dt\right)^2\\&\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b|f(t)|^2+\frac{1}{b-a}\int_a^b f(t)\overline{f(a+b-t)}dt\right]\\&\le\frac{1}{b-a}\int_a^b|f(t)|^2,\end{aligned}\tag{2.3}$$

$$\begin{aligned}&\left|\int_a^b f(t)\overline{g(t)}+\int_a^b f(a+b-t)\overline{g(t)}dt\right|^2\\&\le\left[\int_a^b|f(t)|^2+\int_a^b f(t)\overline{f(a+b-t)}dt\right]\\&\times\left[\int_a^b|g(t)|^2+\int_a^b g(t)\overline{g(a+b-t)}dt\right].\end{aligned}\tag{2.4}$$

Proof. We have by the definition of $\langle\cdot,\cdot\rangle_{\smile}$ that

$$\begin{aligned}\langle f,g\rangle_{\smile}&=\frac{1}{4}\int_a^b[f(t)+f(a+b-t)]\overline{[g(t)+g(a+b-t)]}dt\\&=\frac{1}{4}\int_a^b\Big[f(t)\overline{g(t)}+f(a+b-t)\overline{g(t)}\\&+f(t)\overline{g(a+b-t)}+f(a+b-t)\overline{g(a+b-t)}\Big]dt\end{aligned}\tag{2.5}$$

$$= \frac{1}{4}\left[\int_a^b f(t)\overline{g(t)}dt + \int_a^b f(a+b-t)\overline{g(t)}dt \right.$$
$$\left. + \int_a^b f(t)\overline{g(a+b-t)}dt + \int_a^b f(a+b-t)\overline{g(a+b-t)}dt\right],$$

for any $f,\, g \in L_2[a,b]$.

Using the change of variable $s = a+b-t,\, t \in [a,b]$, we have

$$\int_a^b f(t)\overline{g(a+b-t)}dt = \int_a^b f(a+b-t)\overline{g(t)}dt$$

and

$$\int_a^b f(a+b-t)\overline{g(a+b-t)}dt = \int_a^b f(t)\overline{g(t)}dt$$

and by (2.5) we get the first equality in (2.1). The rest is obvious.

The equality (2.2) follows by (2.1) for $g = f$. Also, from (2.2) we observe that $\int_a^b f(t)\overline{f(a+b-t)}dt$ is a real number for any $f \in L_2[a,b]$.

If $f \in L_2[a,b]$, then by Cauchy-Bunyakovsky-Schwarz inequality we have

$$\begin{aligned}
\|f\|_{\smile}^2 &= \frac{1}{4}\int_a^b |f(t)+f(a+b-t)|^2\, dt \\
&\geq \frac{1}{4(b-a)}\left|\int_a^b [f(t)+f(a+b-t)]\, dt\right|^2 \\
&= \frac{1}{4(b-a)}\left|\int_a^b f(t)\, dt + \int_a^b f(a+b-t)\, dt\right|^2 \\
&= \frac{1}{b-a}\left|\int_a^b f(t)\, dt\right|^2,
\end{aligned}$$

which proves the first inequality in (2.3).

If $f \in L_2[a,b]$, then by Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\begin{aligned}
\|f\|_{\smile}^2 &= \int_a^b \breve{f}(t)\overline{f(t)}dt \leq \left(\int_a^b \left|\breve{f}(t)\right|^2 dt\right)^{1/2}\left(\int_a^b |f(t)|^2\, dt\right)^{1/2} \\
&= \|f\|_{\smile}\, \|f\|_2\,,
\end{aligned}$$

which implies that $\|f\|_{\smile} \leq \|f\|_2$ that is equivalent to the second inequality in (2.3).

By the Schwarz inequality for the inner product $\langle\cdot,\cdot\rangle_{\smile}$, namely

$$|\langle f, g\rangle_{\smile}|^2 \leq \|f\|_{\smile}^2\, \|g\|_{\smile}^2\,,$$

which by (2.1) and (2.2) produces the desired result (2.4). □

We have the corresponding result for $L_2^{\sim}[a,b]$.

Theorem 2. *If $f,\ g\in L_2[a,b]$ then $f,\ g\in L_2^{\sim}[a,b]$, we have the representations*

$$\langle f,g\rangle_{\sim}=\frac{1}{2}\left[\int_a^b f(t)\overline{g(t)}-\int_a^b f(a+b-t)\overline{g(t)}dt\right] \tag{2.6}$$
$$=\frac{1}{2}\left[\int_a^b f(t)\overline{g(t)}-\int_a^b f(t)\overline{g(a+b-t)}dt\right]$$
$$=\int_a^b f(t)\overline{\tilde{g}(t)}dt=\int_a^b \tilde{f}(t)\overline{g(t)}dt,$$

$$\|f\|_{\sim}^2=\frac{1}{2}\left[\int_a^b |f(t)|^2-\int_a^b f(t)\overline{f(a+b-t)}dt\right] \tag{2.7}$$
$$=\int_a^b f(t)\overline{\tilde{f}(t)}dt=\int_a^b \tilde{f}(t)\overline{f(t)}dt$$

and the inequalities

$$0\le\frac{1}{2}\left[\frac{1}{b-a}\int_a^b |f(t)|^2-\frac{1}{b-a}\int_a^b f(t)\overline{f(a+b-t)}dt\right] \tag{2.8}$$
$$\le\frac{1}{b-a}\int_a^b |f(t)|^2,$$

$$\left|\int_a^b f(t)\overline{g(t)}-\int_a^b f(a+b-t)\overline{g(t)}dt\right|^2 \tag{2.9}$$
$$\le\left[\int_a^b |f(t)|^2-\int_a^b f(t)\overline{f(a+b-t)}dt\right]$$
$$\times\left[\int_a^b |g(t)|^2-\int_a^b g(t)\overline{g(a+b-t)}dt\right].$$

Proof. If $f\in L_2[a,b]$ then $f\in L_2^{\sim}[a,b]$, we have the representations

$$\langle f,g\rangle_{\smile}=\frac{1}{4}\int_a^b [f(t)-f(a+b-t)]\overline{[g(t)-g(a+b-t)]}dt$$
$$=\frac{1}{4}\int_a^b \Big[f(t)\overline{g(t)}-f(a+b-t)\overline{g(t)}$$
$$-f(t)\overline{g(a+b-t)}+f(a+b-t)\overline{g(a+b-t)}\Big]dt$$
$$=\frac{1}{4}\left[\int_a^b f(t)\overline{g(t)}dt-\int_a^b f(a+b-t)\overline{g(t)}dt\right.$$
$$\left.-\int_a^b f(t)\overline{g(a+b-t)}dt+\int_a^b f(a+b-t)\overline{g(a+b-t)}dt\right]$$
$$=\frac{1}{2}\left[\int_a^b f(t)\overline{g(t)}-\int_a^b f(a+b-t)\overline{g(t)}dt\right]$$

for any $f,\ g\in L_2[a,b]$.

The rest of the equality (2.6) and (2.7) follow from this equality.

As above, we observe that the integral $\int_a^b f(t)\overline{f(a+b-t)}dt$ is a real number for any $f \in L_2[a,b]$.

By Cauchy-Bunyakovsky-Schwarz integral inequality we have for $f \in L_2[a,b]$ that

$$\begin{aligned}\left|\int_a^b f(t)\overline{f(a+b-t)}dt\right| &\le \left(\int_a^b |f(t)|^2 dt\right)^{1/2}\left(\int_a^b \left|\overline{f(a+b-t)}\right|^2 dt\right)^{1/2}\\ &= \int_a^b |f(t)|^2 dt,\end{aligned}$$

namely, since $\int_a^b f(t)\overline{f(a+b-t)}dt$ is real,

$$-\int_a^b |f(t)|^2 dt \le \int_a^b f(t)\overline{f(a+b-t)}dt \le \int_a^b |f(t)|^2 dt,$$

which is equivalent to (2.8).

By the Schwarz inequality for the inner product $\langle\cdot,\cdot\rangle_\sim$, namely

$$|\langle f,g\rangle_\sim|^2 \le \|f\|_\sim^2 \|g\|_\sim^2,$$

for any f, $g \in L_2[a,b]$ and the equalities (2.6) and (2.7) we get the desired result (2.9). □

3. Inequalities for Bounded Functions

Now, for ϕ, $\Phi \in \mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions (see for instance [19])

$$\begin{aligned}&\bar{U}_{[a,b]}(\phi,\Phi)\\ &:= \left\{g : [a,b] \to \mathbb{C} \middle| \operatorname{Re}\left[(\Phi - g(t))\left(\overline{g(t)} - \overline{\phi}\right)\right] \ge 0 \text{ for almost every } t \in [a,b]\right\}\end{aligned}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{g : [a,b] \to \mathbb{C} \middle| \left|g(t) - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi-\phi| \text{ for a.e. } t \in [a,b]\right\}.$$

The following representation result may be stated.

Proposition 1. *For any ϕ, $\Phi \in \mathbb{C}$, $\phi \ne \Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are nonempty, convex and closed sets and*

$$\bar{U}_{[a,b]}(\phi,\Phi) = \bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{3.1}$$

Proof. We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi-\phi|$$

if and only if

$$\operatorname{Re}\left[(\Phi - z)\left(\bar{z} - \overline{\phi}\right)\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Phi-\phi|^2 - \left|z - \frac{\phi+\Phi}{2}\right|^2 = \operatorname{Re}\left[(\Phi - z)\left(\bar{z} - \overline{\phi}\right)\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (3.1) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 1. *For any* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi,$*we have that*

$$\bar{U}_{[a,b]}(\phi,\Phi)=\{g:[a,b]\to\mathbb{C}\mid\ (\operatorname{Re}\Phi-\operatorname{Re}g(t))(\operatorname{Re}g(t)-\operatorname{Re}\phi)+(\operatorname{Im}\Phi-\operatorname{Im}g(t))(\operatorname{Im}g(t)-\operatorname{Im}\phi)\geq 0 \text{ for a.e. } t\in[a,b]\}. \tag{3.2}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\bar{S}_{[a,b]}(\phi,\Phi):=\{g:[a,b]\to\mathbb{C}\mid\ \operatorname{Re}(\Phi)\geq\operatorname{Re}g(t)\geq\operatorname{Re}(\phi) \text{ and } \operatorname{Im}(\Phi)\geq\operatorname{Im}g(t)\geq\operatorname{Im}(\phi)\ \text{ for a.e. } t\in[a,b]\}. \tag{3.3}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\phi,\Phi)\subseteq\bar{U}_{[a,b]}(\phi,\Phi). \tag{3.4}$$

We have the following Grüss' type inequalities:

Theorem 3. *Let* $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$ *and* $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$, $g\in L_2[a,b]$. *Then*

$$\left|\langle f,g\rangle_{\smile}-\frac{\phi+\Phi}{2}\int_a^b\overline{g(t)}dt\right|\leq\frac{1}{2}|\Phi-\phi|\int_a^b|\breve{g}(t)|\,dt\leq\frac{1}{2}|\Phi-\phi|\int_a^b|g(t)|\,dt \tag{3.5}$$

and

$$|\langle f,g\rangle_{\sim}|\leq\frac{1}{2}|\Phi-\phi|\int_a^b|\tilde{g}(t)|\,dt\leq\frac{1}{2}|\Phi-\phi|\int_a^b|g(t)|\,dt. \tag{3.6}$$

We also have

$$\begin{aligned}&\left|\langle f,g\rangle_{\smile}-\frac{1}{b-a}\int_a^b\overline{g(s)}ds\int_a^b f(t)\,dt\right|\\&\leq\frac{1}{2}|\Phi-\phi|\int_a^b\left|\breve{g}(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\&\leq\frac{1}{2}|\Phi-\phi|\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt.\end{aligned} \tag{3.7}$$

Proof. We have by (2.1) that

$$\begin{aligned}\int_a^b\left(f(t)-\frac{\phi+\Phi}{2}\right)\overline{\breve{g}(t)}dt&=\int_a^b f(t)\overline{\breve{g}(t)}dt-\frac{\phi+\Phi}{2}\int_a^b\overline{\breve{g}(t)}dt\\&=\langle f,g\rangle_{\smile}-\frac{\phi+\Phi}{2}\int_a^b\overline{g(t)}dt.\end{aligned} \tag{3.8}$$

Taking the modulus in this equality, we have

$$
\begin{aligned}
\left|\langle f,g\rangle_{\smile}-\frac{\phi+\Phi}{2}\int_a^b\overline{g\left(t\right)}dt\right| &\leq\int_a^b\left|f\left(t\right)-\frac{\phi+\Phi}{2}\right|\left|\breve{g}\left(t\right)\right|dt\\
&\leq\frac{1}{2}\left|\Phi-\phi\right|\int_a^b\left|\breve{g}\left(t\right)\right|dt\\
&=\frac{1}{4}\left|\Phi-\phi\right|\int_a^b\left|\left[g\left(t\right)+g\left(a+b-t\right)\right]\right|dt\\
&\leq\frac{1}{4}\left|\Phi-\phi\right|\int_a^b\left[\left|g\left(t\right)\right|+\left|g\left(a+b-t\right)\right|\right]dt\\
&=\frac{1}{2}\left|\Phi-\phi\right|\int_a^b\left|g\left(t\right)\right|dt
\end{aligned}
$$

and the inequality (3.5) is proved.

We have by (2.6) that

$$
\begin{aligned}
\int_a^b\left(f\left(t\right)-\frac{\phi+\Phi}{2}\right)\overline{\tilde{g}\left(t\right)}dt &=\int_a^b f\left(t\right)\overline{\tilde{g}\left(t\right)}dt-\frac{\phi+\Phi}{2}\int_a^b\overline{\tilde{g}\left(t\right)}dt\\
&=\int_a^b f\left(t\right)\overline{\tilde{g}\left(t\right)}dt=\langle f,g\rangle_{\sim}\,.
\end{aligned}
\tag{3.9}
$$

Taking the modulus in this equality we have

$$
\begin{aligned}
\left|\int_a^b f\left(t\right)\overline{\tilde{g}\left(t\right)}dt\right| &\leq\int_a^b\left|f\left(t\right)-\frac{\phi+\Phi}{2}\right|\left|\tilde{g}\left(t\right)\right|dt\\
&\leq\frac{1}{2}\left|\Phi-\phi\right|\int_a^b\left|\tilde{g}\left(t\right)\right|dt\\
&=\frac{1}{4}\left|\Phi-\phi\right|\int_a^b\left|\left[g\left(t\right)-g\left(a+b-t\right)\right]\right|dt\\
&\leq\frac{1}{4}\left|\Phi-\phi\right|\int_a^b\left[\left|g\left(t\right)\right|+\left|g\left(a+b-t\right)\right|\right]dt\\
&=\frac{1}{2}\left|\Phi-\phi\right|\int_a^b\left|g\left(t\right)\right|dt
\end{aligned}
$$

and the inequality (3.6) is obtained.

We also have

$$
\begin{aligned}
&\int_a^b\left(f\left(t\right)-\frac{\phi+\Phi}{2}\right)\left(\overline{\breve{g}\left(t\right)}-\frac{1}{b-a}\int_a^b\overline{g\left(s\right)}ds\right)dt\\
&=\int_a^b f\left(t\right)\left(\overline{\breve{g}\left(t\right)}-\frac{1}{b-a}\int_a^b\overline{g\left(s\right)}ds\right)dt\\
&-\frac{\phi+\Phi}{2}\int_a^b\left(\overline{\breve{g}\left(t\right)}-\frac{1}{b-a}\int_a^b\overline{g\left(s\right)}ds\right)dt
\end{aligned}
$$

$$\begin{aligned}
&= \int_a^b f(t)\,\overline{\breve{g}(t)}dt - \frac{1}{b-a}\int_a^b \overline{g(s)}ds \int_a^b f(t)\,dt \\
&- \frac{\phi+\Phi}{2}\int_a^b \left(\overline{\breve{g}(t)} - \frac{1}{b-a}\int_a^b \overline{g(s)}ds\right) dt \\
&= \int_a^b f(t)\,\overline{\breve{g}(t)}dt - \frac{1}{b-a}\int_a^b \overline{g(s)}ds \int_a^b f(t)\,dt \\
&= \langle f,g\rangle_{\smile} - \frac{1}{b-a}\int_a^b \overline{g(s)}ds \int_a^b f(t)\,dt,
\end{aligned}$$

which gives, by taking the modulus,

$$\begin{aligned}
&\left|\langle f,g\rangle_{\smile} - \frac{1}{b-a}\int_a^b \overline{g(s)}ds \int_a^b f(t)\,dt\right| \\
&\le \int_a^b \left|f(t) - \frac{\phi+\Phi}{2}\right| \left|\overline{\breve{g}(t)} - \frac{1}{b-a}\int_a^b \overline{g(s)}ds\right| dt \\
&\le \frac{1}{2}|\Phi-\phi| \int_a^b \left|\breve{g}(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \\
&= \frac{1}{2}|\Phi-\phi| \int_a^b \left|\frac{g(t)+g(a+b-t)}{2} - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \\
&\le \frac{1}{2}|\Phi-\phi| \int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt
\end{aligned}$$

and the last inequality (3.7) is proved. □

We have:

Theorem 4. *Let ϕ, $\Phi \in \mathbb{C}$, $\phi \neq \Phi$ and $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$. If $\breve{\psi}$, $\breve{\Psi} \in \mathbb{C}$, $\breve{\psi} \neq \breve{\Psi}$ and $\breve{g} \in \bar{\Delta}_{[a,b]}\left(\breve{\psi},\breve{\Psi}\right)$, then*

$$\begin{aligned}
\text{(3.10)}\quad &\left|\langle f,g\rangle_{\smile} - \frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\int_a^b f(t)\,dt - \frac{\phi+\Phi}{2}\int_a^b \overline{g(t)}dt \right.\\
&\qquad \left. + \left(\frac{\phi+\Phi}{2}\right)\left(\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\right)(b-a)\right| \\
&\qquad\qquad \le \frac{1}{4}|\Phi-\phi|\left|\breve{\Psi}-\breve{\psi}\right|(b-a)
\end{aligned}$$

and

$$\begin{aligned}
\text{(3.11)}\quad &\left|\langle f,g\rangle_{\smile} - \frac{1}{b-a}\int_a^b \overline{g(s)}ds \int_a^b f(t)\,dt\right| \\
&\le \frac{1}{2}|\Phi-\phi|(b-a)\left(\frac{1}{b-a}\int_a^b |\breve{g}(t)|^2 - \left|\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2\right)^{1/2} \\
&\le \frac{1}{4}|\Phi-\phi|\left|\breve{\Psi}-\breve{\psi}\right|(b-a).
\end{aligned}$$

If $\tilde{\psi},\ \tilde{\Psi}\in\mathbb{C},\ \tilde{\psi}\neq\tilde{\Psi}$ *and* $\tilde{g}\in\bar{\Delta}_{[a,b]}\left(\tilde{\psi},\tilde{\Psi}\right)$, *then*

$$\left|\langle f,g\rangle_{\sim}-\frac{\overline{\tilde{\psi}}+\overline{\tilde{\Psi}}}{2}\int_a^b f(t)\,dt\right|\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\tilde{\Psi}-\tilde{\psi}\right|(b-a).\tag{3.12}$$

Proof. We have by (3.8) that

$$\begin{aligned}
&\int_a^b\left(f(t)-\frac{\phi+\Phi}{2}\right)\overline{\left(\breve{g}(t)-\frac{\breve{\psi}+\breve{\Psi}}{2}\right)}dt\\
&=\int_a^b f(t)\overline{\left(\breve{g}(t)-\frac{\breve{\psi}+\breve{\Psi}}{2}\right)}dt-\frac{\phi+\Phi}{2}\int_a^b\overline{\left(\breve{g}(t)-\frac{\breve{\psi}+\breve{\Psi}}{2}\right)}dt\\
&=\int_a^b f(t)\overline{\breve{g}(t)}dt-\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\int_a^b f(t)\,dt\\
&-\frac{\phi+\Phi}{2}\int_a^b\overline{\breve{g}(t)}dt+\left(\frac{\phi+\Phi}{2}\right)\left(\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\right)\\
&=\langle f,g\rangle_{\smile}-\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\int_a^b f(t)\,dt-\frac{\phi+\Phi}{2}\int_a^b\overline{g(t)}dt\\
&+\left(\frac{\phi+\Phi}{2}\right)\left(\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\right)(b-a).
\end{aligned}$$

Taking the modulus in this equality, we have

$$\begin{aligned}
&\left|\langle f,g\rangle_{\smile}-\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\int_a^b f(t)\,dt-\frac{\phi+\Phi}{2}\int_a^b\overline{g(t)}dt+\left(\frac{\phi+\Phi}{2}\right)\left(\frac{\overline{\breve{\psi}}+\overline{\breve{\Psi}}}{2}\right)\right|\\
&\leq\int_a^b\left|f(t)-\frac{\phi+\Phi}{2}\right|\left|\breve{g}(t)-\frac{\breve{\psi}+\breve{\Psi}}{2}\right|dt\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\breve{\Psi}-\breve{\psi}\right|(b-a),
\end{aligned}$$

which proves (3.10).

By the Schwarz and Grüss' inequalities, see for instance [13], we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left|\breve{g}(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt\\
&=\frac{1}{b-a}\int_a^b\left|\breve{g}(t)-\frac{1}{b-a}\int_a^b\breve{g}(s)\,ds\right|dt\\
&\leq\left(\frac{1}{b-a}\int_a^b\left|\breve{g}(t)-\frac{1}{b-a}\int_a^b\breve{g}(s)\,ds\right|^2dt\right)^{1/2}\\
&=\left(\frac{1}{b-a}\int_a^b\left|\breve{g}(t)\right|^2-\left|\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2\right)^{1/2}\leq\frac{1}{2}\left|\breve{\Psi}-\breve{\psi}\right|
\end{aligned}$$

and by (3.7) we get (3.11).

By (3.9) we have

$$\int_a^b \left(f(t) - \frac{\phi+\Phi}{2}\right)\overline{\left(\tilde{g}(t) - \frac{\tilde{\psi}+\tilde{\Psi}}{2}\right)}dt$$
$$= \int_a^b f(t)\overline{\left(\tilde{g}(t) - \frac{\tilde{\psi}+\tilde{\Psi}}{2}\right)}dt = \langle f,g\rangle_{\sim} - \frac{\overline{\tilde{\psi}}+\overline{\tilde{\Psi}}}{2}\int_a^b f(t).$$

By taking the modulus in this equality, we have

$$\left|\langle f,g\rangle_{\sim} - \frac{\overline{\tilde{\psi}}+\overline{\tilde{\Psi}}}{2}\int_a^b f(t)\,dt\right| \le \int_a^b \left|f(t) - \frac{\phi+\Phi}{2}\right|\left|\tilde{g}(t) - \frac{\tilde{\psi}+\tilde{\Psi}}{2}\right|dt$$
$$\le \frac{1}{4}|\Phi-\phi|\left|\tilde{\Psi}-\tilde{\psi}\right|(b-a)$$

and the inequality (3.12) is proved. □

Remark 1. *We observe that if ϕ, $\Phi \in \mathbb{R}$, $\phi < \Phi$ and f is real-valued function, then $f \in \bar{\Delta}_{[a,b]}(\phi,\Phi)$ is equivalent to*

$$\phi \le f(t) \le \Phi \text{ for a.e. } t \in [a,b].$$

If $\breve{\psi}$, $\breve{\Psi} \in \mathbb{R}$, $\breve{\psi} < \breve{\Psi}$ and g is real valued function, then $\breve{g} \in \bar{\Delta}_{[a,b]}\left(\breve{\psi},\breve{\Psi}\right)$ is equivalent to

$$\breve{\psi} \le \frac{1}{2}[g(t) + g(a+b-t)] \le \breve{\Psi} \text{ for a.e. } t \in [a,b]. \tag{3.13}$$

If ψ, Ψ are real numbers so that $\psi \le g(t) \le \Psi$ for a.e. $t \in [a,b]$, then

$$\psi \le \frac{1}{2}[g(t) + g(a+b-t)] \le \Psi \text{ for a.e. } t \in [a,b]. \tag{3.14}$$

One can find examples of functions for which the bounds provided by (3.13) are better than (3.14). For instance, if we consider the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ given by $g(t) = \ln t$, then we have

$$\breve{g}(t) = \frac{1}{2}[\ln t + \ln(a+b-t)],$$
$$(\breve{g}(t))' = \frac{1}{2}\left(\frac{1}{t} - \frac{1}{a+b-t}\right) = \frac{\frac{a+b}{2}-t}{t(a+b-t)},\ t \in (a,b)$$

and

$$(\breve{g}(t))'' = -\frac{1}{2}\left(\frac{1}{t^2} + \frac{1}{(a+b-t)^2}\right),\ t \in (a,b).$$

These shows that $\breve{f}$ is strictly increasing on $\left(a, \frac{a+b}{2}\right)$, strictly decreasing on $\left(\frac{a+b}{2}, b\right)$ and strictly concave on (a,b). Therefore

$$\breve{\psi} := \ln G(a,b) \le \breve{g}(t) \le \ln A(a,b) =: \breve{\Psi} \text{ for any } t \in [a,b], \tag{3.15}$$

where $G(a,b) := \sqrt{ab}$ is the geometric mean and $A(a,b) := \frac{1}{2}(a+b)$ is the arithmetic mean of positive numbers a, b.

Since $\psi := \ln a \le \ln t \le \ln b =: \Psi$, then by (3.14) we get

$$\psi \le \breve{g}(t) \le \Psi \text{ for any } t \in [a,b]. \tag{3.16}$$

We observe that the bounds provided by (3.15) for $\breve{g}$ are better than (3.16).

4. The Case of One Function of Bounded Variation

For a function of bounded variation $f:[a,b]\to\mathbb{C}$ we denote by $\bigvee_a^b(f)$ its total variation on $[a,b]$.

Theorem 5. *Assume that $f:[a,b]\to\mathbb{C}$ is of bounded variation g is integrable on $[a,b]$. Then we have*

$$\left|\langle f,g\rangle_{\smile}-\frac{f(a)+f(b)}{2}\int_a^b\overline{g(t)}dt\right|\le\frac{1}{2}\bigvee_a^b(f)\int_a^b|\breve{g}(t)|\,dt \le\frac{1}{2}\bigvee_a^b(f)\int_a^b|g(t)|\,dt \tag{4.1}$$

and

$$|\langle f,g\rangle_{\sim}|\le\frac{1}{2}\bigvee_a^b(f)\int_a^b|\tilde{g}(t)|\,dt\le\frac{1}{2}\bigvee_a^b(f)\int_a^b|g(t)|\,dt. \tag{4.2}$$

Proof. We have by (2.1) that

$$\begin{aligned}\int_a^b\left(f(t)-\frac{f(a)+f(b)}{2}\right)\overline{\breve{g}(t)}dt&=\int_a^b f(t)\overline{\breve{g}(t)}dt-\frac{f(a)+f(b)}{2}\int_a^b\overline{\breve{g}(t)}dt\\&=\langle f,g\rangle_{\smile}-\frac{f(a)+f(b)}{2}\int_a^b\overline{g(t)}dt.\end{aligned}$$

Taking the modulus in this equality, we get

$$\left|\langle f,g\rangle_{\smile}-\frac{f(a)+f(b)}{2}\int_a^b\overline{g(t)}dt\right|\le\int_a^b\left|f(t)-\frac{f(a)+f(b)}{2}\right||\breve{g}(t)|\,dt. \tag{4.3}$$

Observe that, for any $t\in[a,b]$ we have

$$\begin{aligned}\left|f(t)-\frac{f(a)+f(b)}{2}\right|&=\left|\frac{f(t)-f(a)+f(t)-f(b)}{2}\right|\\&\le\frac{1}{2}\left[|f(t)-f(a)|+|f(b)-f(t)|\right]\le\frac{1}{2}\bigvee_a^b(f)\end{aligned}$$

and by (4.3) we get the first inequality in (4.1).

Since

$$\begin{aligned}\int_a^b|\breve{g}(t)|\,dt&=\frac{1}{2}\int_a^b|g(t)+g(a+b-t)|\,dt\\&\le\frac{1}{2}\int_a^b\left[|g(t)|+|g(a+b-t)|\right]dt=\int_a^b|g(t)|\,dt,\end{aligned}$$

the last part of (4.1) also holds.

We have by (2.6) that

$$\begin{aligned}\int_a^b\left(f(t)-\frac{f(a)+f(b)}{2}\right)\overline{\tilde{g}(t)}dt&=\int_a^b f(t)\overline{\tilde{g}(t)}dt-\frac{f(a)+f(b)}{2}\int_a^b\overline{\tilde{g}(t)}dt\\&=\int_a^b f(t)\overline{\tilde{g}(t)}dt=\langle f,g\rangle_{\sim}.\end{aligned}$$

Taking the modulus in this equality, we get

$$\begin{aligned}
\left|\langle f,g\rangle_{\sim}\right| &\leq \int_a^b \left|f(t)-\frac{f(a)+f(b)}{2}\right| \left|\tilde{g}(t)\right| dt \leq \frac{1}{2}\bigvee_a^b(f)\int_a^b \left|\tilde{g}(t)\right| dt \\
&= \frac{1}{4}\bigvee_a^b(f)\int_a^b \left|g(t)-g(a+b-t)\right| dt \\
&\leq \frac{1}{4}\bigvee_a^b(f)\int_a^b \left[\left|g(t)\right|+\left|g(a+b-t)\right|\right] dt = \frac{1}{2}\bigvee_a^b(f)\int_a^b \left|g(t)\right| dt
\end{aligned}$$

and the inequality (4.2) is proved. □

We say that the function $h:[a,b]\to\mathbb{R}$ is *H-r-Hölder continuous* with the constant $H>0$ and power $r\in(0,1]$ if

$$|h(t)-h(s)| \leq H|t-s|^r \tag{4.4}$$

for any $t,\ s\in[a,b]$. If $r=1$ we call that h is *L-Lipschitzian* when $H=L>0$.

Corollary 2. *Assume that $f:[a,b]\to\mathbb{C}$ is of bounded variation and g is H-r-Hölder continuous with the constant $H>0$ and power $r\in(0,1]$. Then*

$$\left|\langle f,g\rangle_{\sim}\right| \leq \frac{1}{4(r+1)}H\bigvee_a^b(f)(b-a)^{r+1}. \tag{4.5}$$

In particular, if L-Lipschitzian with $L>0$, then

$$\left|\langle f,g\rangle_{\sim}\right| \leq \frac{1}{8}L\bigvee_a^b(f)(b-a)^2. \tag{4.6}$$

Proof. Since g is H-r-Hölder continuous with the constant $H>0$ and power $r\in(0,1]$, then

$$\begin{aligned}
\left|\tilde{g}(t)\right| &= \frac{1}{2}\left|g(t)-g(a+b-t)\right| \leq \frac{1}{2}H\left|2t-a-b\right|^r \\
&= \frac{1}{2}2^r H\left|t-\frac{a+b}{2}\right|^r = \frac{1}{2^{1-r}}H\left|t-\frac{a+b}{2}\right|^r,
\end{aligned}$$

which implies that

$$\begin{aligned}
\int_a^b \left|\tilde{g}(t)\right| dt &\leq \frac{1}{2^{1-r}}H\int_a^b\left|t-\frac{a+b}{2}\right|^r dt = \frac{1}{2^{1-r}}H\frac{(b-a)^{r+1}}{2^r(r+1)} \\
&= \frac{1}{2(r+1)}H(b-a)^{r+1}
\end{aligned}$$

and the inequality (4.5) is proved. □

5. The Case of One Hölder Continuous Function

We say that the function $h:[a,b]\to\mathbb{C}$ is *K-p-Hölder continuous in the middle* with the constant $K>0$ and power $p>0$ if

$$\left|h(t)-h\left(\frac{a+b}{2}\right)\right| \leq K\left|t-\frac{a+b}{2}\right|^p \tag{5.1}$$

for any $t \in [a,b]$. We observe that if $h : [a,b] \to \mathbb{C}$ is *H-r-Hölder continuous* with the constant $H > 0$ and power $r \in (0,1]$, then is Hölder continuous in the middle with the same constants.

We define the following Lebesgue norms for a measurable function $h : [a,b] \to \mathbb{C}$

$$\|h\|_\infty := \operatorname*{essup}_{t\in[a,b]} |h(t)| < \infty \text{ if } h \in L_\infty [a,b]$$

and, for $\beta \geq 1$,

$$\|h\|_\beta := \left(\int_a^b |h(t)|^\beta \, dt\right)^{1/\beta} < \infty \text{ if } h \in L_\beta [a,b].$$

Theorem 6. *Assume that $f : [a,b] \to \mathbb{C}$ is K-p-Hölder continuous in the middle with the constant $K > 0$ and power $p > 0$, and g is integrable on $[a,b]$. Then we have*

$$\left|\langle f, g\rangle_{\smile} - f\left(\frac{a+b}{2}\right)\int_a^b \overline{g(t)}dt\right| \leq K \int_a^b \left|t - \frac{a+b}{2}\right|^p |\breve{g}(t)| \, dtt \tag{5.2}$$
$$\leq K \begin{cases} \frac{1}{2^p}(b-a)^p \|\breve{g}\|_1, \\ \\ \frac{1}{2^p(p\alpha+1)^{1/\alpha}}(b-a)^{p+1/\alpha}\|\breve{g}\|_\beta \\ \textit{where } \alpha, \beta > 1 \textit{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2^p(p+1)}(b-a)^{p+1}\|\breve{g}\|_\infty, \end{cases}$$
$$\leq K \begin{cases} \frac{1}{2^p}(b-a)^p \|g\|_1, \\ \\ \frac{1}{2^p(p\alpha+1)^{1/\alpha}}(b-a)^{p+1/\alpha}\|g\|_\beta \\ \textit{where } \alpha, \beta > 1 \textit{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2^p(p+1)}(b-a)^{p+1}\|g\|_\infty, \end{cases}$$

and

$$|\langle f, g\rangle_\sim| \leq K \int_a^b \left|t - \frac{a+b}{2}\right|^p |\tilde{g}(t)| \, dt \tag{5.3}$$
$$\leq K \begin{cases} \frac{1}{2^p}(b-a)^p \|\tilde{g}\|_1, \\ \\ \frac{1}{2^p(p\alpha+1)^{1/\alpha}}(b-a)^{p+1/\alpha}\|\tilde{g}\|_\beta \\ \textit{where } \alpha, \beta > 1 \textit{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2^p(p+1)}(b-a)^{p+1}\|\tilde{g}\|_\infty, \end{cases}$$
$$\leq K \begin{cases} \frac{1}{2^p}(b-a)^p \|g\|_1, \\ \\ \frac{1}{2^p(p\alpha+1)^{1/\alpha}}(b-a)^{p+1/\alpha}\|g\|_\beta \\ \textit{where } \alpha, \beta > 1 \textit{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \\ \frac{1}{2^p(p+1)}(b-a)^{p+1}\|g\|_\infty. \end{cases}$$

Proof. We have by (2.1) that

$$\begin{aligned}\int_a^b \left(f(t)-f\left(\frac{a+b}{2}\right)\right)\overline{\breve{g}(t)}dt &= \int_a^b f(t)\overline{\breve{g}(t)}dt - f\left(\frac{a+b}{2}\right)\int_a^b \overline{\breve{g}(t)}dt \\ &= \langle f,g\rangle_{\smile} - f\left(\frac{a+b}{2}\right)\int_a^b \overline{g(t)}dt.\end{aligned}$$

Taking the modulus in this equality, we get

$$\begin{aligned}\left|\langle f,g\rangle_{\smile} - f\left(\frac{a+b}{2}\right)\int_a^b \overline{g(t)}dt\right| &\leq \int_a^b \left|f(t)-f\left(\frac{a+b}{2}\right)\right| |\breve{g}(t)|\,dt \\ &\leq K\int_a^b \left|t-\frac{a+b}{2}\right|^p |\breve{g}(t)|\,dt.\end{aligned}$$

By the Hölder's integral inequality we have

$$\begin{aligned}\int_a^b \left|t-\frac{a+b}{2}\right|^p |\breve{g}(t)|\,dt &\leq \begin{cases} \max_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|^p \int_a^b |\breve{g}(t)|\,dt, \\ \\ \left(\int_a^b \left|t-\frac{a+b}{2}\right|^{p\alpha} dt\right)^{1/\alpha}\left(\int_a^b |\breve{g}(t)|^\beta dt\right)^{1/\beta} \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \int_a^b \left|t-\frac{a+b}{2}\right|^p dt\,\operatorname{essup}_{t\in[a,b]}|\breve{g}(t)| \end{cases} \\ &= \begin{cases} \frac{1}{2^p}(b-a)^p \|\breve{g}\|_1, \\ \\ \frac{1}{2^p(p\alpha+1)^{1/\alpha}}(b-a)^{p+1/\alpha}\|\breve{g}\|_\beta \\ \text{where } \alpha,\beta>1 \text{ with } \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2^p(p+1)}(b-a)^{p+1}\|\breve{g}\|_\infty, \end{cases}\end{aligned}$$

which proves the second inequality in (5.2).

By the triangle inequality for the Lebesgue norms we have

$$\|\breve{g}\|_\beta = \frac{1}{2}\|g+g(a+b-\cdot)\|_\beta \leq \frac{1}{2}\left[\|g\|_\beta + \|g(a+b-\cdot)\|_\beta\right] = \|g\|_\beta,$$

which proves the last part of (5.2).

We have by (2.6) that

$$\begin{aligned}\int_a^b \left(f(t)-f\left(\frac{a+b}{2}\right)\right)\overline{\tilde{g}(t)}dt &= \int_a^b f(t)\overline{\tilde{g}(t)}dt - f\left(\frac{a+b}{2}\right)\int_a^b \overline{\tilde{g}(t)}dt \\ &= \int_a^b f(t)\overline{\tilde{g}(t)}dt = \langle f,g\rangle_{\sim}.\end{aligned}$$

Taking the modulus in this equality, we get

$$|\langle f,g\rangle_{\sim}| \leq \int_a^b \left|f(t)-f\left(\frac{a+b}{2}\right)\right| |\tilde{g}(t)|\,dt \leq K\int_a^b \left|t-\frac{a+b}{2}\right|^p |\tilde{g}(t)|\,dt,$$

which proves the second inequality in (5.3).

The rest follows in a similar manner and the details are omitted. □

Corollary 3. *Assume that $f : [a,b] \to \mathbb{C}$ is K-p-Hölder continuous in the middle with the constant $K > 0$ and power $p > 0$, and g is H-r-Hölder continuous with the constant $H > 0$ and power $r \in (0,1]$. Then*

$$|\langle f,g\rangle_{\sim}| \le \frac{1}{2^{p+1}(p+r+1)} HK(b-a)^{p+r+1}. \tag{5.4}$$

In particular, if L-Lipschitzian with $L > 0$, then

$$|\langle f,g\rangle_{\sim}| \le \frac{1}{2^{p+1}(p+2)} LK(b-a)^{p+2}. \tag{5.5}$$

Proof. From the first inequality in (5.3) we have

$$\begin{aligned} |\langle f,g\rangle_{\sim}| &\le K\int_a^b \left|t-\frac{a+b}{2}\right|^p |\tilde{g}(t)|\,dt \le \frac{1}{2^{1-r}} HK \int_a^b \left|t-\frac{a+b}{2}\right|^{p+r} dt \\ &= \frac{1}{2^{1-r}} HK \frac{(b-a)^{p+r+1}}{2^{p+r}(p+r+1)} = \frac{1}{2^{p+1}(p+r+1)} HK(b-a)^{p+r+1}, \end{aligned}$$

which proves (5.4). □

References

[1] A. M. Acu, Improvement of Grüss and Ostrowski type inequalities. *Filomat* **29** (2015), no. 9, 2027–2035.

[2] A. M. Acu, H. Gonska and I. Raşa, Grüss-type and Ostrowski-type inequalities in approximation theory. *Ukrainian Math. J.* **63** (2011), no. 6, 843–864.

[3] A. M. Acu, F. Sofonea and C. V. Muraru, Grüss and Ostrowski type inequalities and their applications. *Sci. Stud. Res. Ser. Math. Inform.* **23** (2013), no. 1, 5–14.

[4] M. W. Alomari, New inequalities of Grüss-Lupaş type and applications for selfadjoint operators. *Armen. J. Math.* **8** (2016), no. 1, 25–37.

[5] M. W. Alomari, New Grüss type inequalities for double integrals. *Appl. Math. Comput.* **228** (2014), 102–107.

[6] M. W. Alomari, Some Grüss type inequalities for Riemann-Stieltjes integral and applications. *Acta Math. Univ. Comenian.* (N.S.) **81** (2012), no. 2, 211–220.

[7] G. A. Anastassiou, Fractional Ostrowski and Grüss type inequalities involving several functions. *PanAmer. Math. J.* **24** (2014), no. 3, 1–14.

[8] G. A. Anastassiou, Further interpretation of some fractional Ostrowski and Grüss type inequalities. *J. Appl. Funct. Anal.* **9** (2014), no. 3-4, 392–403.

[9] G. A. Anastassiou, Basic and s-convexity Ostrowski and Grüss type inequalities involving several functions. *Commun. Appl. Anal.* **17** (2013), no. 2, 189–212.

[10] G. A. Anastassiou, General Grüss and Ostrowski type inequalities involving s-convexity. *Bull. Allahabad Math. Soc.* **28** (2013), no. 1, 101–129.880

[11] P.; Cerone and S. S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications. *J. Math. Inequal.* **8** (2014), no. 1, 159–170.

[12] X.-K. Chai and Y. Miao, Note on the Grüss inequality. *Gen. Math.* **19** (2011), no. 3, 93–99.

[13] S. S. Dragomir, A generalization of Grüss's inequality in inner product spaces and applications. *J. Math. Anal. Appl.* **237** (1999), no. 1, 74–82.

[14] S. S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.

[15] S. S. Dragomir, Bounds for convex functions of Čebyšev functional via Sonin's identity with applications. *Commun. Math.* **22** (2014), no. 2, 107–132.

[16] S. S. Dragomir, Some Grüss-type results via Pompeiu's-like inequalities. *Arab. J. Math.* (Springer) **4** (2015), no. 3, 159–170.

[17] S. S. Dragomir, Bounding the Čebyšev functional for a differentiable function whose derivative is h or λ-convex in absolute value and applications. *Matematiche* (Catania) **71** (2016), no. 1, 173–202.

[18] S. S. Dragomir, Bounding the Čebyšev functional for a function that is convex in absolute value and applications. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 1, 33–54.

[19] S. S. Dragomir, M. S. Moslehian and Y. J. Cho, Some reverses of the Cauchy-Schwarz inequality for complex functions of self-adjoint operators in Hilbert spaces. *Math. Inequal. Appl.* **17** (2014), no. 4, 1365–1373. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 84. [Online http://rgmia.org/papers/v14/v14a84.pdf].

[20] Q. Feng and F. Meng, Some generalized Ostrowski-Grüss type integral inequalities. *Comput. Math. Appl.* **63** (2012), no. 3, 652–659.

[21] B. Gavrea, Improvement of some inequalities of Chebysev-Grüss type. *Comput. Math. Appl.* **64** (2012), no. 6, 2003–2010.

[22] A. G. Ghazanfari, A Grüss type inequality for vector-valued functions in Hilbert C^*-modules. *J. Inequal. Appl.* **2014**, 2014:16, 10 pp.

[23] S.Hussain and A. Qayyum, A generalized Ostrowski-Grüss type inequality for bounded differentiable mappings and its applications. *J. Inequal. Appl.* **2013**, 2013:1, 7 pp.

[24] N. Minculete and L. Ciurdariu, A generalized form of Grüss type inequality and other integral inequalities. *J. Inequal. Appl.* **2014**, 2014:119, 18 pp.

[25] A. Qayyum and S. Hussain, A new generalized Ostrowski Grüss type inequality and applications. *Appl. Math. Lett.* **25** (2012), no. 11, 1875–1880

[26] M. Z. Sarikaya and H. Budak, An inequality of Grüss like via variant of Pompeiu's mean value theorem. *Konuralp J. Math.* **3** (2015), no. 1, 29–35.

[27] M.-D. Rusu, On Grüss-type inequalities for positive linear operators. *Stud. Univ. Babeş-Bolyai Math.* **56** (2011), no. 2, 551–565.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa