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FURTHER INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some new inequalities for the k-g-fractional integrals of functions of bounded variation.Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

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and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b). \tag{1.2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned} S_{k,g,a+}f(x) &= \frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1} g'(t)\,f(t)\,dt \\ &=: I^{\alpha}_{a+,g}f(x),\ \ a<x\le b \end{aligned} \tag{1.3}$$

and

$$\begin{aligned} S_{k,g,b-}f(x) &= \frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1} g'(t)\,f(t)\,dt \\ &=: I^{\alpha}_{b-,g}f(x),\ \ a\le x<b, \end{aligned} \tag{1.4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.5}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{f(t)\,dt}{(x-t)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.7}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{f(t)\,dt}{(t-x)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x [\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.9}$$

for $a<x\le b$ and

$$E^{\alpha}_{b-,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b [\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)\,f(t)\,dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ x\in[a,b). \tag{1.12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x):=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ |x|<R,\ \text{with } R>0 \tag{1.13}$$

for $\rho,\,\lambda>0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f(x):=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(w(x-t)^{\rho})f(t)\,dt,\ x>a \tag{1.14}$$

where $\rho,\,\lambda>0,\ w\in\mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f(x):=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(w(t-x)^{\rho})f(t)\,dt,\ x<b \tag{1.15}$$

where $\rho,\,\lambda>0,\ w\in\mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t)=t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.17}$$

where $\alpha\in(0,1)$.

We observe that for $k(t)=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right),\ t\in\mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b \tag{1.19}$$

where $\alpha\in(0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha>0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+}f(x):=\int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt, \tag{1.20}$$

for $0<a<x\leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha}f(x):=\int_{x}^{b}(g(t)-g(x))^{\alpha-1}\ln(g(t)-g(x))g'(t)f(t)\,dt, \tag{1.21}$$

for $0<a\leq x<b$, where $\alpha>0$. These are obtained from (1.11) and (1.12) for the kernel $k(t)=t^{\alpha-1}\ln t$, $t>0$.

For $\alpha=1$ we get

$$\mathcal{L}_{g,a+}f(x):=\int_{a}^{x}\ln(g(x)-g(t))g'(t)f(t)\,dt,\ 0<a<x\leq b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-}f(x):=\int_{x}^{b}\ln(g(t)-g(x))g'(t)f(t)\,dt,\ 0<a\leq x<b. \tag{1.23}$$

For $g(t)=t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha}f(x):=\int_{a}^{x}(x-t)^{\alpha-1}\ln(x-t)f(t)\,dt,\ 0<a<x\leq b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha}f(x):=\int_{x}^{b}(t-x)^{\alpha-1}\ln(t-x)f(t)\,dt,\ 0<a\leq x<b, \tag{1.25}$$

$$\mathcal{L}_{a+}f(x):=\int_{a}^{x}\ln(x-t)f(t)\,dt,\ 0<a<x\leq b \tag{1.26}$$

and

$$\mathcal{L}_{b-}f(x):=\int_{x}^{b}\ln(t-x)f(t)\,dt,\ 0<a\leq x<b. \tag{1.27}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [21]-[34] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned}(1.28)\quad & S_{k,g,a+,b-}f(x)\\ &:=\frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right]\\ &=\frac{1}{2}\left[\int_{a}^{x}k(g(x)-g(t))g'(t)f(t)\,dt+\int_{x}^{b}k(g(t)-g(x))g'(t)f(t)\,dt\right]\end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases}\int_0^t|k(s)|\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

In the recent paper [19] we obtained the following result for functions of bounded variation:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on*

(a,b), having a continuous derivative g' on (a,b). Then we have the Ostrowski type inequality

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right| \tag{1.29}$$
$$\leq \frac{1}{2}\left[\int_x^b |k(g(t)-g(x))|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x |k(g(x)-g(t))|\bigvee_t^x(f)\,g'(t)\,dt\right]$$
$$\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\ +\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\leq \frac{1}{2}\begin{cases}\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f);\\[2ex] \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

and the trapezoid type inequality

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))f(b)+K(g(x)-g(a))f(a)\right]\right| \tag{1.30}$$
$$\leq \frac{1}{2}\left[\int_a^x |k(g(x)-g(t))|\bigvee_a^t(f)\,g'(t)\,dt+\int_x^b |k(g(t)-g(x))|\bigvee_t^b(f)\,g'(t)\,dt\right]$$
$$\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]$$
$$\leq \frac{1}{2}\begin{cases}\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f);\\[2ex] \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\\ \times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex] \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\\ \times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}$$

for any $x\in(a,b)$, where $\bigvee_c^d(f)$ denoted the total variation on the interval $[c,d]$.

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt,\ x\in[a,b) \tag{1.31}$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt,\ x\in(a,b]. \tag{1.32}$$

We can define also the mixed operator

$$\breve{S}_{k,g,a+,b-}f(x) := \frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right] = \frac{1}{2}\left[\int_x^b k(g(b)-g(t))g'(t)f(t)\,dt+\int_a^x k(g(t)-g(a))g'(t)f(t)\,dt\right] \tag{1.33}$$

for any $x\in(a,b)$.

In this paper we establish some inequalities for the k-g-fractional integrals of functions with bounded variation $f:[a,b]\to\mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-}f$ and $\breve{S}_{k,g,a+,b-}f$ in terms of the *double trapezoid rule*

$$\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right],\ x\in(a,b).$$

Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2. Further Inequalities for Functions of BV

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [20]:

Lemma 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$S_{k,g,a+,b-}f(x)=\frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right] +\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)[f(t)-\lambda]\,dt +\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)[f(t)-\gamma]\,dt \tag{2.1}$$

and

$$\breve{S}_{k,g,a+,b-}f(x)=\frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] +\frac{1}{2}\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]\,dt +\frac{1}{2}\int_x^b k(g(b)-g(t))g'(t)[f(t)-\lambda]\,dt \tag{2.2}$$

for $x\in(a,b)$ and for any $\lambda,\ \gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$[K(g(x)-g(t))]'=K'(g(x)-g(t))(g(x)-g(t))'=-k(g(x)-g(t))g'(t)$$

for $t\in(a,x)$ and

$$[K(g(t)-g(x))]'=K'(g(t)-g(x))(g(t)-g(x))'=k(g(t)-g(x))g'(t)$$

for $t\in(x,b)$.

Therefore, for any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned}(2.3)\quad &\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\lambda\right]dt\\ &=\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt-\lambda\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)dt\\ &=S_{k,g,a+}f\left(x\right)+\lambda\int_a^x\left[K\left(g\left(x\right)-g\left(t\right)\right)\right]'dt\\ &=S_{k,g,a+}f\left(x\right)+\lambda\left[K\left(g\left(x\right)-g\left(t\right)\right)\right]\Big|_a^x=S_{k,g,a+}f\left(x\right)-\lambda K\left(g\left(x\right)-g\left(a\right)\right)\end{aligned}$$

and

$$\begin{aligned}(2.4)\quad &\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\gamma\right]dt\\ &=\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)dt\\ &=S_{k,g,b-}f\left(x\right)-\gamma\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]'dt\\ &=S_{k,g,b-}f\left(x\right)-\gamma\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]\Big|_x^b=S_{k,g,b-}f\left(x\right)-\gamma K\left(g\left(b\right)-g\left(x\right)\right)\end{aligned}$$

for $x\in(a,b)$.

If we add the equalities (2.3) and (2.4) and divide by 2 then we get the desired result (2.1).

Moreover, by taking the derivative over t and using the chain rule, we have that

$$\left[K\left(g\left(b\right)-g\left(t\right)\right)\right]'=K'\left(g\left(b\right)-g\left(t\right)\right)\left(g\left(b\right)-g\left(t\right)\right)'=-k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)$$

for $t\in(x,b)$ and

$$\left[K\left(g\left(t\right)-g\left(a\right)\right)\right]'=K'\left(g\left(t\right)-g\left(a\right)\right)\left(g\left(t\right)-g\left(a\right)\right)'=k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)$$

for $t\in(a,x)$.

For any λ, $\gamma\in\mathbb{C}$ we have

$$\begin{aligned}(2.5)\quad &\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\lambda\right]dt\\ &=\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt-\lambda\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)dt\\ &=S_{k,g,x+}f\left(b\right)+\lambda\int_x^b\left[K\left(g\left(b\right)-g\left(t\right)\right)\right]'dt\\ &=S_{k,g,x+}f\left(b\right)-\lambda K\left(g\left(b\right)-g\left(x\right)\right)\end{aligned}$$

and

$$\int_a^x k(g(t)-g(a))g'(t)[f(t)-\gamma]dt \tag{2.6}$$
$$= \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma\int_a^x k(g(t)-g(a))g'(t)dt$$
$$= \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma\int_a^x [K(g(t)-g(a))]'dt$$
$$= \int_a^x k(g(t)-g(a))g'(t)f(t)dt - \gamma K(g(x)-g(a))$$

for $x \in (a,b)$.

If we add the equalities (2.5) and (2.6) and divide by 2 then we get the desired result (2.2). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$P_{k,g,a+,b-}f := S_{k,g,a+,b-}f(M_g(a,b)) \tag{2.7}$$
$$= \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2} - g(t)\right)g'(t)f(t)dt$$
$$+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t) - \frac{g(a)+g(b)}{2}\right)g'(t)f(t)dt.$$

Using the representation (2.1) we have

$$P_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2} \tag{2.8}$$
$$+ \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2} - g(t)\right)g'(t)[f(t)-\lambda]dt$$
$$+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t) - \frac{g(a)+g(b)}{2}\right)g'(t)[f(t)-\gamma]dt$$

for any λ, $\gamma \in \mathbb{C}$.

Also, if

$$\breve{P}_{k,g,a+,b-}f := \breve{S}_{k,g,a+,b-}f\left(M_g\left(a,b\right)\right) = \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt + \frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt. \tag{2.9}$$

then by (2.2) we get

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g\left(b\right)-g\left(a\right)}{2}\right)\frac{\gamma+\lambda}{2} + \frac{1}{2}\int_{a}^{M_g(a,b)} k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-\gamma\right]dt + \frac{1}{2}\int_{M_g(a,b)}^{b} k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\lambda\right]dt \tag{2.10}$$

for any λ, $\gamma \in \mathbb{C}$.

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the double trapezoid inequalities*

$$\begin{aligned}
&\Big|S_{k,g,a+,b-}f\left(x\right) - \frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right]\Big| \\
&\le \frac{1}{4}\left[\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_a^x\left(f\right)+\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_x^b\left(f\right)\right] \\
&\le \frac{1}{4}\begin{cases}
\max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_a^b\left(f\right); \\[2ex]
\left[\mathbf{K}^p\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^p\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p}\left(\left(\bigvee_a^x\left(f\right)\right)^q+\left(\bigvee_x^b\left(f\right)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\[2ex]
\left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right]\left[\frac{1}{2}\bigvee_a^b\left(f\right)+\frac{1}{2}\left|\bigvee_a^x\left(f\right)-\bigvee_x^b\left(f\right)\right|\right]
\end{cases}
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f(x)\right. \\
&\left.-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\right| \\
&\leq\frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right] \\
&\leq\frac{1}{4}\begin{cases}\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f); \\ \\ [\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ [\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned} \tag{2.12}$$

for $x\in(a,b)$.

Proof. Using the identity (2.1) for $\lambda=\frac{f(a)+f(x)}{2}$ and $\gamma=\frac{f(x)+f(b)}{2}$ we have

$$\begin{aligned}
&S_{k,g,a+,b-}f(x) \\
&=\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right] \\
&+\frac{1}{2}\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-\frac{f(a)+f(x)}{2}\right]dt \\
&+\frac{1}{2}\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-\frac{f(x)+f(b)}{2}\right]dt
\end{aligned} \tag{2.13}$$

for $x\in(a,b)$.

Since f is of bounded variation, then

$$\begin{aligned}
\left|f(t)-\frac{f(a)+f(x)}{2}\right|&=\left|\frac{f(t)-f(a)+f(t)-f(x)}{2}\right| \\
&\leq\frac{1}{2}[|f(t)-f(a)|+|f(x)-f(t)|]\leq\frac{1}{2}\bigvee_a^x(f)
\end{aligned}$$

and

$$\begin{aligned}
\left|f(t)-\frac{f(x)+f(b)}{2}\right|&=\left|\frac{f(t)-f(x)+f(t)-f(b)}{2}\right| \\
&\leq\frac{1}{2}[|f(t)-f(x)|+|f(b)-f(t)|]\leq\frac{1}{2}\bigvee_x^b(f)
\end{aligned}$$

for $x\in(a,b)$.

Using the equality (2.13) we have

$$\begin{aligned}
(2.14)\quad & \Big| S_{k,g,a+,b-}f(x) \\
& -\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Big| \\
& \le \frac{1}{2}\left|\int_a^x k(g(x)-g(t))g'(t)\left[f(t)-\frac{f(a)+f(x)}{2}\right]dt\right| \\
& +\frac{1}{2}\left|\int_x^b k(g(t)-g(x))g'(t)\left[f(t)-\frac{f(x)+f(b)}{2}\right]dt\right| \\
& \le \frac{1}{2}\int_a^x |k(g(x)-g(t))|\left|f(t)-\frac{f(a)+f(x)}{2}\right|g'(t)\,dt \\
& +\frac{1}{2}\int_x^b |k(g(t)-g(x))|\left|f(t)-\frac{f(x)+f(b)}{2}\right|g'(t)\,dt \\
& \le \frac{1}{4}\left[\bigvee_a^x(f)\int_a^x |k(g(x)-g(t))|g'(t)\,dt+\bigvee_x^b(f)\int_x^b |k(g(t)-g(x))|g'(t)\,dt\right] \\
& =: B(x)
\end{aligned}$$

for $x\in(a,b)$.

We have, by taking the derivative over t and using the chain rule, that

$$[\mathbf{K}(g(x)-g(t))]'=\mathbf{K}'(g(x)-g(t))(g(x)-g(t))'=-|k(g(x)-g(t))|g'(t)$$

for $t\in(a,x)$ and

$$[\mathbf{K}(g(t)-g(x))]'=\mathbf{K}'(g(t)-g(x))(g(t)-g(x))'=|k(g(t)-g(x))|g'(t)$$

for $t\in(x,b)$.

Then

$$\int_a^x |k(g(x)-g(t))|g'(t)\,dt=-\int_a^x [\mathbf{K}(g(x)-g(t))]'\,dt=\mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))|g'(t)\,dt=\int_x^b [\mathbf{K}(g(t)-g(x))]'\,dt=\mathbf{K}(g(b)-g(x)).$$

Therefore

$$\begin{aligned}
B(x) &= \frac{1}{4}\left[\bigvee_a^x(f)\int_a^x |k(g(x)-g(t))|g'(t)\,dt+\bigvee_x^b(f)\int_x^b |k(g(t)-g(x))|g'(t)\,dt\right] \\
&= \frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)+\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right].
\end{aligned}$$

The last part of (2.11) is obvious by making use of the elementary Hölder type inequalities for positive real numbers $c,\ d,\ m,\ n\ge 0$

$$mc+nd\le\begin{cases}\max\{m,n\}(c+d);\\[2ex] (m^p+n^p)^{1/p}(c^q+d^q)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}$$

Using the identity (2.2) for $\lambda=\frac{f(x)+f(b)}{2}$ and $\gamma=\frac{f(x)+f(a)}{2}$ we also have

$$\begin{aligned}&\Bigg|\breve{S}_{k,g,a+,b-}f(x)\\&\qquad-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(x)+f(a)}{2}K(g(x)-g(a))\right]\Bigg|\\&\qquad\qquad\le\frac{1}{2}\int_a^x|k(g(t)-g(a))|\left|f(t)-\frac{f(x)+f(a)}{2}\right|g'(t)\,dt\\&\qquad\qquad+\frac{1}{2}\int_x^b|k(g(b)-g(t))|\left|f(t)-\frac{f(x)+f(b)}{2}\right|g'(t)\,dt\\&\le\frac{1}{4}\bigvee_a^x(f)\int_a^x|k(g(t)-g(a))|g'(t)\,dt+\frac{1}{4}\bigvee_x^b(f)\int_x^b|k(g(b)-g(t))|g'(t)\,dt\\&\qquad\qquad\qquad\qquad=:C(x).\end{aligned}$$

We also have, by taking the derivative over t and using the chain rule, that

$$[\mathbf{K}(g(b)-g(t))]'=\mathbf{K}'(g(b)-g(t))(g(b)-g(t))'=-|k(g(b)-g(t))|g'(t)$$

for $t\in(x,b)$ and

$$[\mathbf{K}(g(t)-g(a))]'=\mathbf{K}'(g(t)-g(a))(g(t)-g(a))'=|k(g(t)-g(a))|g'(t)$$

for $t\in(a,x)$.

Therefore

$$\int_a^x|k(g(t)-g(a))|g'(t)\,dt=\mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b|k(g(b)-g(t))|g'(t)\,dt=\mathbf{K}(g(b)-g(x))$$

giving that

$$C(x)=\frac{1}{4}\bigvee_a^x(f)\,\mathbf{K}(g(x)-g(a))+\frac{1}{4}\bigvee_x^b(f)\,\mathbf{K}(g(b)-g(x))$$

for $x\in(a,b)$, and the inequality (2.12) is thus proved. □

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}&\left|P_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right|\\&\le\frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b(f)\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}&\left|\breve{P}_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right|\\&\le\frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b(f).\end{aligned}\tag{2.16}$$

If we take $x=\frac{a+b}{2}$ in (2.11) and (2.12), then we get

$$\begin{aligned}(2.17)\quad &\left|S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\ &\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\ &\leq\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_{a}^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_{\frac{a+b}{2}}^{b}(f)\right]\\ &\leq\frac{1}{4}\begin{cases}\max\left\{\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right\}\bigvee_{a}^{b}(f);\\ \\ \left[\mathbf{K}^{p}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^{p}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]^{1/p}\\ \left(\left(\bigvee_{a}^{\frac{a+b}{2}}(f)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]\\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^{b}(f)\right|\right]\end{cases}\end{aligned}$$

and

$$\begin{aligned}(2.18)\quad &\left|\breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\ &\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\ &\leq\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_{a}^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_{x}^{b}(f)\right]\\ &\leq\frac{1}{4}\begin{cases}\max\left\{\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right\}\bigvee_{a}^{b}(f);\\ \\ \left[\mathbf{K}^{p}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^{p}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]^{1/p}\\ \left(\left(\bigvee_{a}^{\frac{a+b}{2}}(f)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]\\ \left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^{b}(f)\right|\right]\end{cases}\end{aligned}$$

for $x\in(a,b)$.

We use the classical Lebesgue p-norms defined as

$$\|h\|_{[c,d],\infty}:=\operatorname*{essup}_{s\in[c,d]}|h(s)|$$

and

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(s)|^p \, ds \right)^{1/p}, \ p \geq 1.$$

Using Hölder's integral inequality we have for $t > 0$ that

$$K(t) = \int_0^t |k(s)| \, ds \leq \begin{cases} t \|k\|_{[0,t],\infty} & \text{if } k \in L_\infty [0,t] \\ \\ t^{1/p} \|k\|_{[0,t],q} & \text{if } k \in L_q [0,t], \ p,q > 1, \ \frac{1}{p} + \frac{1}{q} = 1. \end{cases}$$

Therefore by the first inequality in (2.11) and (2.12) we get for p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$

$$\begin{aligned} & \Big| S_{k,g,a+,b-} f(x) \\ & - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right] \Big| \\ & \leq \frac{1}{4} \bigvee_a^x (f) \begin{cases} (g(x) - g(a)) \|k\|_{[0,g(x)-g(a)],\infty} \\ \\ (g(x) - g(a))^{1/p} \|k\|_{[0,g(x)-g(a)],q} \end{cases} \\ & + \frac{1}{4} \bigvee_x^b (f) \begin{cases} (g(b) - g(x)) \|k\|_{[0,g(b)-g(x)],\infty} \\ \\ (g(b) - g(x))^{1/p} \|k\|_{[0,g(b)-g(x)],q} \end{cases} \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} & \Big| \breve{S}_{k,g,a+,b-} f(x) \\ & - \frac{1}{2} \left[\frac{f(x) + f(b)}{2} K(g(b) - g(x)) + \frac{f(a) + f(x)}{2} K(g(x) - g(a)) \right] \Big| \\ & \leq \frac{1}{4} \bigvee_a^x (f) \begin{cases} (g(x) - g(a)) \|k\|_{[0,g(x)-g(a)],\infty} \\ \\ (g(x) - g(a))^{1/p} \|k\|_{[0,g(x)-g(a)],q} \end{cases} \\ & + \frac{1}{4} \bigvee_x^b (f) \begin{cases} (g(b) - g(x)) \|k\|_{[0,g(b)-g(x)],\infty} \\ \\ (g(b) - g(x))^{1/p} \|k\|_{[0,g(b)-g(x)],q} \end{cases} \end{aligned} \tag{2.20}$$

for $x \in (a,b)$.

From (2.15) and (2.16) we also have for p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned} & \left| P_{k,g,a+,b-} f - \frac{1}{2} K\left(\frac{g(b) - g(a)}{2} \right) \left[f(M_g(a,b)) + \frac{f(a) + f(b)}{2} \right] \right| \\ & \leq \frac{1}{4} \bigvee_a^b (f) \begin{cases} \left(\frac{g(b)-g(a)}{2} \right) \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right],\infty} \\ \\ \left(\frac{g(b)-g(a)}{2} \right)^{1/p} \|k\|_{\left[0, \frac{g(b)-g(a)}{2}\right],q} \end{cases} \end{aligned} \tag{2.21}$$

and

$$\left|\breve{P}_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f\left(M_{g}\left(a,b\right)\right)+\frac{f(a)+f(b)}{2}\right]\right| \le \frac{1}{4}\bigvee_{a}^{b}(f)\begin{cases}\left(\frac{g(b)-g(a)}{2}\right)\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],\infty}\\ \left(\frac{g(b)-g(a)}{2}\right)^{1/p}\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],q}.\end{cases} \tag{2.22}$$

3. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+}f(x)=I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}\left[g(x)-g(t)\right]^{\alpha-1}g'(t)f(t)\,dt$$

for $a<x\le b$ and

$$S_{k,g,b-}f(x)=I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}\left[g(t)-g(x)\right]^{\alpha-1}g'(t)f(t)\,dt$$

for $a\le x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

We consider the mixed operators

$$I_{g,a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right] \tag{3.1}$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f(b)+I_{x-,g}^{\alpha}f(a)\right] \tag{3.2}$$

for $x\in(a,b)$.

We observe that for $\alpha>0$ we have

$$K(t)=\frac{1}{\Gamma(\alpha)}\int_{0}^{t}s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}=\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ t\ge 0.$$

If we use the inequalities (2.11) and (2.12) we get

$$\begin{aligned}(3.3)\quad &\Big|I^{\alpha}_{g,a+,b-}f(x)\\ &-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{f(x)+f(b)}{2}(g(b)-g(x))^{\alpha}+\frac{f(a)+f(x)}{2}(g(x)-g(a))^{\alpha}\right]\Big|\\ &\leq\frac{1}{4\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_{a}^{x}(f)+(g(b)-g(x))^{\alpha}\bigvee_{x}^{b}(f)\right]\\ &\leq\frac{1}{4\Gamma(\alpha+1)}\\ &\times\begin{cases}\left[\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f);\\ \\ [(g(b)-g(x))^{p\alpha}+(g(x)-g(a))^{p\alpha}]^{1/p}\left((\bigvee_{a}^{x}(f))^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ [(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}]\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\end{cases}\end{aligned}$$

and

$$\begin{aligned}(3.4)\quad &\Big|\breve{I}^{\alpha}_{g,a+,b-}f(x)\\ &-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{f(x)+f(b)}{2}(g(b)-g(x))^{\alpha}+\frac{f(a)+f(x)}{2}(g(x)-g(a))^{\alpha}\right]\Big|\\ &\leq\frac{1}{4\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}\bigvee_{a}^{x}(f)+(g(b)-g(x))^{\alpha}\bigvee_{x}^{b}(f)\right]\\ &\leq\frac{1}{4\Gamma(\alpha+1)}\\ &\times\begin{cases}\left[\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}(f);\\ \\ [(g(b)-g(x))^{p\alpha}+(g(x)-g(a))^{p\alpha}]^{1/p}\left((\bigvee_{a}^{x}(f))^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ [(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}]\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]\end{cases}\end{aligned}$$

for $x\in(a,b)$.

From (2.15) and (2.16) we get

$$\begin{aligned}(3.5)\quad &\left|I^{\alpha}_{g,a+,b-}f(M_g(a,b))-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right|\\ &\leq\frac{1}{2^{\alpha+2}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_{a}^{b}(f)\end{aligned}$$

and

$$\left|\breve{I}_{g,a+,b-}^{\alpha} f\left(M_g(a,b)\right)-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f\left(M_g(a,b)\right)+\frac{f(a)+f(b)}{2}\right]\right| \leq \frac{1}{2^{\alpha+2}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f). \tag{3.6}$$

4. Example for an Exponential Kernel

For $\alpha,\ \beta\in\mathbb{R}$ we consider the kernel $k(t):=\exp[(\alpha+\beta i)t]$, $t\in\mathbb{R}$. We have

$$K(t)=\frac{\exp[(\alpha+\beta i)t]-1}{(\alpha+\beta i)},\text{ if } t\in\mathbb{R}$$

for $\alpha,\ \beta\neq 0$.

Also, we have

$$|k(s)|:=|\exp[(\alpha+\beta i)s]|=\exp(\alpha s)\text{ for } s\in\mathbb{R}$$

and

$$\mathbf{K}(t)=\int_0^t \exp(\alpha s)\,ds=\frac{\exp(\alpha t)-1}{\alpha}\text{ if } 0<t,$$

for $\alpha\neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We consider the operator

$$\mathcal{H}_{g,a+,b-}^{\alpha+\beta i} f(x):=\frac{1}{2}\int_a^x \exp[(\alpha+\beta i)(g(x)-g(t))]\,g'(t)\,f(t)\,dt +\frac{1}{2}\int_x^b \exp[(\alpha+\beta i)(g(t)-g(x))]\,g'(t)\,f(t)\,dt \tag{4.1}$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}\kappa_{h,a+,b-}^{\alpha+\beta i} f(x) &:=\mathcal{H}_{\ln h,a+,b-}^{\alpha+\beta i} f(x)\\ &=\frac{1}{2}\left[\int_a^x\left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt+\int_x^b\left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned} \tag{4.2}$$

for $x\in(a,b)$.

Using the inequality (2.11) we have for $x\in(a,b)$

$$
\begin{aligned}
(4.3)\quad &\left|\mathcal{H}_{g,a+,b-}^{\alpha+\beta i}f(x)-\frac{1}{2}\frac{f(x)+f(b)}{2}\frac{\exp[(\alpha+\beta i)(g(b)-g(x))]-1}{(\alpha+\beta i)}\right.\\
&\qquad\left.-\frac{f(a)+f(x)}{2}\frac{\exp[(\alpha+\beta i)(g(x)-g(a))]-1}{(\alpha+\beta i)}\right|\\
&\le\frac{1}{4}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\bigvee_a^x(f)+\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\bigvee_x^b(f)\right]\\
&\le\frac{1}{4}\begin{cases}\max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha},\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]
\left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p\right]^{1/p}\\
\times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right]\\
\times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}
$$

and if we take $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$
\begin{aligned}
(4.4)\quad &\left|\kappa_{h,a+,b-}^{\alpha+\beta i}f(x)-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right.\right.\\
&\qquad\left.\left.-\frac{f(a)+f(x)}{2}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right]\right|\\
&\le\frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\bigvee_a^x(f)+\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\bigvee_x^b(f)\right]\\
&\le\frac{1}{4}\begin{cases}\max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha},\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]
\left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^p+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}
\end{aligned}
$$

If we take if we take $x_h := h^{-1}\left(\sqrt{h(a)h(b)}\right) = h^{-1}(G(h(a),h(b))) \in (a,b)$, where G is the geometric mean, then from (4.4) we get

$$\left|\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}} - 1}{2(\alpha+\beta i)}\left[f\left(h^{-1}(G(h(a),h(b)))\right) + \frac{f(a)+f(b)}{2}\right]\right| \leq \frac{1}{4}\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}} - 1}{\alpha}\bigvee_a^b (f), \tag{4.5}$$

where $\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i} f = \kappa_{h,a+,b-}^{\alpha+\beta i} f(x_h)$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Also define

$$\begin{aligned} &\breve{\mathcal{H}}_{g,a+,b-}^{\alpha} f(x) \\ &:= \frac{1}{2}\int_x^b \exp[\alpha(g(b)-g(t))] g'(t) f(t)\, dt \\ &+ \frac{1}{2}\int_a^x \exp[\alpha(g(t)-g(a))] g'(t) f(t)\, dt \end{aligned} \tag{4.6}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned} &\breve{\kappa}_{h,a+,b-}^{\alpha} f(x) \\ &:= \breve{\mathcal{H}}_{\ln h,a+,b-}^{\alpha} f(x) \\ &= \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\, dt\right], \end{aligned} \tag{4.7}$$

for any $x \in (a,b)$.

Using the inequality (2.12) we have for $x\in(a,b)$ that

$$\begin{aligned}
&\left|\breve{\mathcal{H}}_{g,a+,b-}^{\alpha+\beta i}f(x)-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\exp\left[(\alpha+\beta i)(g(b)-g(x))\right]-1}{(\alpha+\beta i)}\right.\right.\\
&\qquad\left.\left.-\frac{f(a)+f(x)}{2}\frac{\exp\left[(\alpha+\beta i)(g(x)-g(a))\right]-1}{(\alpha+\beta i)}\right]\right|\\
&\leq\frac{1}{4}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\bigvee_a^x(f)+\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\bigvee_x^b(f)\right]\\
&\leq\frac{1}{4}\begin{cases}\max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha},\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]
\left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p\right]^{1/p}\\
\times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right]\\
\times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}
\end{aligned}\tag{4.8}$$

and if we take $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$\begin{aligned}
&\left|\breve{\kappa}_{h,a+,b-}^{\alpha+\beta i}f(x)-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right.\right.\\
&\qquad\left.\left.-\frac{f(a)+f(x)}{2}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right]\right|\\
&\leq\frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\bigvee_a^x(f)+\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\bigvee_x^b(f)\right]\\
&\leq\frac{1}{4}\begin{cases}\max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha},\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]
\left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^p+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}
\end{aligned}\tag{4.9}$$

If we take if we take $x_h = h^{-1}(G(h(a), h(b))) \in (a,b)$, where G is the geometric mean, then from (4.4) we get

$$\left| \bar{\ell}^{\alpha+\beta i}_{h,a+,b-} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}} - 1}{2(\alpha+\beta i)} \left[f\left(h^{-1}(G(h(a), h(b)))\right) + \frac{f(a)+f(b)}{2} \right] \right| \leq \frac{1}{4} \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}} - 1}{\alpha} \bigvee_a^b (f), \tag{4.10}$$

where $\bar{\ell}^{\alpha+\beta i}_{h,a+,b-} f = \breve{\kappa}^{\alpha+\beta i}_{h,a+,b-} f(x_h)$.

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