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SOME INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF CONVEX FUNCTIONS

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Abstract. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some trapezoid and Ostrowski type inequalities for the k-g-fractional integrals of convex functions. Applications for Hermite-Hadamard type inequalities for generalized g-means and examples for Riemann-Liouville and exponential fractional integrals are also given.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

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If we take $k(t) = \frac{1}{\Gamma(\alpha)} t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x [g(x) - g(t)]^{\alpha-1} g'(t) f(t)\, dt \tag{1.3}$$
$$=: I_{a+,g}^{\alpha} f(x), \; a < x \le b$$

and

$$S_{k,g,b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b [g(t) - g(x)]^{\alpha-1} g'(t) f(t)\, dt \tag{1.4}$$
$$=: I_{b-,g}^{\alpha} f(x), \; a \le x < b,$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ as defined in [23, p. 100].

For $g(t) = t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t) = \ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \; 0 \le a < x \le b \tag{1.5}$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \; 0 \le a < x < b. \tag{1.6}$$

One can consider the function $g(t) = -t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{f(t)\, dt}{(x-t)^{1-\alpha} t^{\alpha+1}}, \; 0 \le a < x \le b \tag{1.7}$$

and

$$R_{b-}^{\alpha} f(x) := \frac{x^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{f(t)\, dt}{(t-x)^{1-\alpha} t^{\alpha+1}}, \; 0 \le a < x < b. \tag{1.8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{1.9}$$

for $a < x \le b$ and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{1.10}$$

for $a \le x < b$.

If we take $g(t) = t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt, \; x \in (a, b] \tag{1.11}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt, \; x \in [a, b). \tag{1.12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^{k}, \quad |x| < R, \text{ with } R > 0 \tag{1.13}$$

for ρ, $\lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f(x) := \int_{a}^{x} (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left(w(x-t)^{\rho}\right) f(t)\, dt, \quad x > a \tag{1.14}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} f(x) := \int_{x}^{b} (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}\left(w(t-x)^{\rho}\right) f(t)\, dt, \quad x < b \tag{1.15}$$

where ρ, $\lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}_{a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_{a}^{x} \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\, dt, \quad x > a \tag{1.16}$$

and

$$\mathcal{T}_{b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_{x}^{b} \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\, dt, \quad x < b \tag{1.17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}_{g,a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_{a}^{x} \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\, dt, \quad x > a \tag{1.18}$$

and

$$\mathcal{T}_{g,b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_{x}^{b} \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\, dt, \quad x < b \tag{1.19}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}_{g,a+}^{\alpha} f(x) := \int_{a}^{x} (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\, dt, \tag{1.20}$$

for $0 < a < x \leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha} f(x) := \int_{x}^{b} (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\, dt, \tag{1.21}$$

for $0 < a \le x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t)) g'(t) f(t) \, dt, \; 0 < a < x \le b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x)) g'(t) f(t) \, dt, \; 0 < a \le x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t) \, dt, \; 0 < a < x \le b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t) \, dt, \; 0 < a \le x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t) \, dt, \; 0 < a < x \le b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t) \, dt, \; 0 < a \le x < b. \tag{1.27}$$

Recall the classical Riemann-Liouville fractional integrals defined for $\alpha > 0$ by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt$$

for $a < x \le b$ and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt$$

for $a \le x < b$, where Γ is the *Gamma function*. For $\alpha = 0$, they are defined as

$$J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x) \text{ for } x \in (a,b).$$

In the recent paper [17] we obtained the following results for convex functions and the classical Riemann-Liouville fractional integrals:

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $x \in (a,b)$, then we have the inequalities*

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+2)} \left[f'_+(x)(b-x)^{\alpha+1} - f'_-(x)(x-a)^{\alpha+1} \right] \\ &\le \frac{1}{\Gamma(\alpha+1)} \left[(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b) \right] - J_{a+}^{\alpha} f(x) - J_{b-}^{\alpha} f(x) \\ &\le \frac{1}{\Gamma(\alpha+2)} \left[f'_-(b)(b-x)^{\alpha+1} - f'_+(a)(x-a)^{\alpha+1} \right] \end{aligned} \tag{1.28}$$

and

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha+2)}\left[f'_{+}(x)(b-x)^{\alpha+1}-f'_{-}(x)(x-a)^{\alpha+1}\right] \\
&\leq J^{\alpha}_{x-}f(a)+J^{\alpha}_{x+}f(b)-\frac{1}{\Gamma(\alpha+1)}\left[(x-a)^{\alpha}+(b-x)^{\alpha}\right]f(x) \\
&\leq \frac{1}{\Gamma(\alpha+2)}\left[f'_{-}(b)(b-x)^{\alpha+1}-f'_{+}(a)(x-a)^{\alpha+1}\right],
\end{aligned} \tag{1.29}$$

where $f'_{\pm}(\cdot)$ are the lateral derivatives of f.

In particular, we have:

Corollary 1. *Let $f:[a,b]\to\mathbb{R}$ be a convex function, then we have the inequalities*

$$\begin{aligned}
0 &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\leq \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}(b-a)^{\alpha}-J^{\alpha}_{a+}f\left(\frac{a+b}{2}\right)-J^{\alpha}_{b-}f\left(\frac{a+b}{2}\right) \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{\alpha+1},
\end{aligned} \tag{1.30}$$

$$\begin{aligned}
0 &\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{\alpha+1} \\
&\leq J^{\alpha}_{\frac{a+b}{2}-}f(a)+J^{\alpha}_{\frac{a+b}{2}+}f(b)-\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}f\left(\frac{a+b}{2}\right)(b-a)^{\alpha} \\
&\leq \frac{1}{2^{\alpha+1}\Gamma(\alpha+2)}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{\alpha+1},
\end{aligned} \tag{1.31}$$

and

$$\begin{aligned}
0 &\leq \frac{1}{\Gamma(\alpha+1)}\frac{f(b)+f(a)}{2}(b-a)^{\alpha}-\frac{J^{\alpha}_{b-}f(a)+J^{\alpha}_{a+}f(b)}{2} \\
&\leq \frac{2^{\alpha}-1}{2^{\alpha+1}\Gamma(\alpha+2)}\left(f'_{-}(b)-f'_{+}(a)\right)(b-a)^{\alpha+1}.
\end{aligned} \tag{1.32}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[18], [21]-[34] and the references therein.

In this paper we establish some trapezoid and Ostrowski type inequalities for the k-g-fractional integrals of convex functions. Applications for Hermite-Hadamard type inequalities for generalized g-means and examples for Riemann-Liouville and exponential fractional integrals are also given.

2. Some Identities

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned}
&S_{k,g,a+,b-}f(x) \\
&:= \frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right] \\
&= \frac{1}{2}\left[\int_a^x k(g(x)-g(t))g'(t)f(t)\,dt+\int_x^b k(g(t)-g(x))g'(t)f(t)\,dt\right]
\end{aligned} \tag{2.1}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

Observe that

$$S_{k,g,x+}f(b)=\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b) \tag{2.2}$$

and

$$S_{k,g,x-}f(a)=\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt,\ \ x\in(a,b]. \tag{2.3}$$

We can define also the mixed operator

$$\begin{aligned}&\breve{S}_{k,g,a+,b-}f(x)\\&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]\\&=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt\right]\end{aligned} \tag{2.4}$$

for any $x\in(a,b)$.

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [20]:

Lemma 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{C}$ is absolutely continuous on $[a,b]$, then we have for $x\in(a,b)$ that*

$$\begin{aligned}S_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[K(g(x)-g(a))\,f(a)+K(g(b)-g(x))\,f(b)\right]\\&+\frac{1}{2}\lambda\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}\gamma\int_x^b K(g(t)-g(x))\,dt\\&+\frac{1}{2}\int_a^x K(g(x)-g(t))\left[f'(t)-\lambda\right]dt+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[\gamma-f'(t)\right]dt\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x)&=\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\\&+\frac{1}{2}\gamma\int_x^b K(g(b)-g(t))\,dt-\frac{1}{2}\lambda\int_a^x K(g(t)-g(a))\,dt\\&+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_a^x K(g(t)-g(a))\left[\lambda-f'(t)\right]dt\end{aligned} \tag{2.6}$$

for any λ, $\gamma\in\mathbb{C}$.

Proof. Using the integration by parts formula, we have

$$\begin{aligned}
&\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=-\int_a^x\left[K\left(g\left(x\right)-g\left(t\right)\right)\right]'f\left(t\right)dt \\
&=-\left[K\left(g\left(x\right)-g\left(t\right)\right)f\left(t\right)\Big|_a^x-\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)f'\left(t\right)dt\right] \\
&=K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)f'\left(t\right)dt
\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}
&\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]'f\left(t\right)dt \\
&=\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f\left(t\right)\Big|_x^b-\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f'\left(t\right)dt \\
&=\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)-\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]f'\left(t\right)dt
\end{aligned} \tag{2.8}$$

for any $x\in(a,b)$.

From (2.7) and (2.8) we get

$$\begin{aligned}
&\int_a^x k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)+\lambda\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)dt \\
&+\int_a^x K\left(g\left(x\right)-g\left(t\right)\right)\left[f'\left(t\right)-\lambda\right]dt
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
&\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt \\
&=\left[K\left(g\left(b\right)-g\left(x\right)\right)\right]f\left(b\right)-\gamma\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)dt \\
&-\int_x^b K\left(g\left(t\right)-g\left(x\right)\right)\left[f'\left(t\right)-\gamma\right]dt
\end{aligned} \tag{2.10}$$

for any $x\in(a,b)$.

If we add the equalities (2.9) and (2.10) and divide by 2 then we get the desired result (2.5).

Using the integration by parts formula, we have

$$\begin{aligned}
(2.11)\qquad & \int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= -\int_x^b \left[K\left(g\left(b\right)-g\left(t\right)\right)\right]' f\left(t\right)dt \\
&= -\left[K\left(g\left(b\right)-g\left(t\right)\right)f\left(t\right)\Big|_x^b - \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)f'\left(t\right)dt\right] \\
&= K\left(g\left(b\right)-g\left(x\right)\right)f\left(x\right) + \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)f'\left(t\right)dt
\end{aligned}$$

and

$$\begin{aligned}
(2.12)\qquad & \int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= \int_a^x \left[K\left(g\left(t\right)-g\left(a\right)\right)\right]' f\left(t\right)dt \\
&= K\left(g\left(t\right)-g\left(a\right)\right)f\left(t\right)\Big|_a^x - \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt \\
&= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right) - \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)f'\left(t\right)dt
\end{aligned}$$

for any $x \in (a,b)$.

From (2.11) and (2.12) we have

$$\begin{aligned}
(2.13)\qquad & \int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= K\left(g\left(b\right)-g\left(x\right)\right)f\left(x\right) + \gamma\int_x^b K\left(g\left(b\right)-g\left(t\right)\right)dt \\
&+ \int_x^b K\left(g\left(b\right)-g\left(t\right)\right)\left[f'\left(t\right)-\gamma\right]dt
\end{aligned}$$

and

$$\begin{aligned}
(2.14)\qquad & \int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt \\
&= K\left(g\left(x\right)-g\left(a\right)\right)f\left(x\right) - \lambda\int_a^x K\left(g\left(t\right)-g\left(a\right)\right)dt \\
&- \int_a^x K\left(g\left(t\right)-g\left(a\right)\right)\left[f'\left(t\right)-\lambda\right]dt
\end{aligned}$$

for any $x \in (a,b)$.

If we add the equalities (2.13) and (2.14) and divide by 2 then we get the desired result (2.6). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b \in I$ as

$$M_g\left(a,b\right) := g^{-1}\left(\frac{g\left(a\right)+g\left(b\right)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$\begin{aligned}P_{k,g,a+,b-}f&:=S_{k,g,a+,b-}f\left(M_g(a,b)\right)\\&=\frac{1}{2}\int_a^{M_g(a,b)}k\left(\frac{g(a)+g(b)}{2}-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(t)-\frac{g(a)+g(b)}{2}\right)g'(t)f(t)\,dt\end{aligned}\tag{2.15}$$

and

$$\begin{aligned}\breve{P}_{k,g,a+,b-}f&:=\breve{S}_{k,g,a+,b-}f\left(M_g(a,b)\right)\\&=\frac{1}{2}\int_{M_g(a,b)}^{b}k\left(g(b)-g(t)\right)g'(t)f(t)\,dt\\&+\frac{1}{2}\int_a^{M_g(a,b)}k\left(g(t)-g(a)\right)g'(t)f(t)\,dt.\end{aligned}\tag{2.16}$$

Corollary 2. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}P_{k,g,a+,b-}f&=K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}\\&+\frac{1}{2}\lambda\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\\&-\frac{1}{2}\gamma\int_{M_g(a,b)}^{b}K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\\&+\frac{1}{2}\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)\left[f'(t)-\lambda\right]dt\\&+\frac{1}{2}\int_{M_g(a,b)}^{b}K\left(g(t)-\frac{g(a)+g(b)}{2}\right)\left[\gamma-f'(t)\right]dt\end{aligned}\tag{2.17}$$

and

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right) f(M_g(a,b)) \tag{2.18}$$
$$+\frac{1}{2}\left[\gamma\int_{M_g(a,b)}^{b} K(g(b)-g(t))\,dt - \lambda\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right]$$
$$+\frac{1}{2}\int_{M_g(a,b)}^{b} K(g(b)-g(t))[f'(t)-\gamma]\,dt$$
$$+\frac{1}{2}\int_a^{M_g(a,b)} K(g(t)-g(a))[\lambda-f'(t)]\,dt$$

for any λ, $\gamma \in \mathbb{C}$.

For $x = \frac{a+b}{2}$ we can consider

$$M_{k,g,a+,b-}f := S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \tag{2.19}$$
$$= \frac{1}{2}\int_a^{\frac{a+b}{2}} k\left(g\left(\frac{a+b}{2}\right)-g(t)\right) g'(t) f(t)\,dt$$
$$+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} k\left(g(t)-g\left(\frac{a+b}{2}\right)\right) g'(t) f(t)\,dt$$

and

$$\breve{M}_{k,g,a+,b-}f := \breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right) \tag{2.20}$$
$$= \frac{1}{2}\int_{\frac{a+b}{2}}^{b} k(g(b)-g(t))\, g'(t) f(t)\,dt$$
$$+\frac{1}{2}\int_a^{\frac{a+b}{2}} k(g(t)-g(a))\, g'(t) f(t)\,dt.$$

We have the mid-point representation as well:

Corollary 3. *With the assumptions of Lemma 1 we have*

$$M_{k,g,a+,b-}f \tag{2.21}$$
$$= \frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right) f(a) + K\left(g(b)-g\left(\frac{a+b}{2}\right)\right) f(b)\right]$$
$$+\frac{1}{2}\left[\lambda\int_a^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right) dt - \gamma\int_{\frac{a+b}{2}}^{b} K\left(g(t)-g\left(\frac{a+b}{2}\right)\right) dt\right]$$
$$+\frac{1}{2}\int_a^{\frac{a+b}{2}} K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)[f'(t)-\lambda]\,dt$$
$$+\frac{1}{2}\int_{\frac{a+b}{2}}^{b} K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)[\gamma-f'(t)]\,dt$$

and

$$\begin{aligned}(2.22)\quad & \breve{M}_{k,g,a+,b-}f \\ & = \frac{1}{2}\left[K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]f\left(\frac{a+b}{2}\right) \\ & \quad + \frac{1}{2}\left[\gamma\int_{\frac{a+b}{2}}^{b}K(g(b)-g(t))\,dt-\lambda\int_{a}^{\frac{a+b}{2}}K(g(t)-g(a))\,dt\right] \\ & + \frac{1}{2}\int_{\frac{a+b}{2}}^{b}K(g(b)-g(t))\left[f'(t)-\gamma\right]dt+\frac{1}{2}\int_{a}^{\frac{a+b}{2}}K(g(t)-g(a))\left[\lambda-f'(t)\right]dt\end{aligned}$$

for any λ, $\gamma \in \mathbb{C}$.

3. Trapezoid Type Inequality for Convex Functions

We have the following trapezoid type inequality for convex functions:

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then we have*

$$\begin{aligned}(3.1)\quad & \frac{1}{2}\left[f'_{+}(x)\int_{x}^{b}K(g(t)-g(x))\,dt-f'_{-}(x)\int_{a}^{x}K(g(x)-g(t))\,dt\right] \\ & \le \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]-S_{k,g,a+,b-}f(x) \\ & \quad \le \frac{1}{2}\left[f'_{-}(b)\int_{x}^{b}K(g(t)-g(x))\,dt-f'_{+}(a)\int_{a}^{x}K(g(x)-g(t))\,dt\right]\end{aligned}$$

for any $x\in(a,b)$.

Proof. Since $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then the lateral derivatives $f'_{\pm}$ exist on (a,b) and they are equal except at most a countably subset of (a,b). Also $f'_{+}(a)$ and $f'_{-}(b)$ exist and we have $f'_{+}(a)\le f'_{-}(t)\le f'_{+}(t)\le f'_{-}(b)$ for any $t\in(a,b)$.

Observe that by the positivity of the kernel k we have $K(g(x)-g(t))\ge 0$ for $t\in(a,x)$ and $K(g(t)-g(x))\ge 0$ for $t\in(x,b)$.

If we use the equality (2.5) for $\lambda=f'_{+}(a)$ and $\gamma=f'_{-}(b)$, then we have for $x\in(a,b)$ that

$$
\begin{aligned}
S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right] \\
&+\frac{1}{2}f'_{+}(a)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{-}(b)\int_x^b K(g(t)-g(x))\,dt \\
+\frac{1}{2}\int_a^x & K(g(x)-g(t))\left[f'(t)-f'_{+}(a)\right]dt+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[f'_{-}(b)-f'(t)\right]dt \\
&\geq \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right] \\
&+\frac{1}{2}f'_{+}(a)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{-}(b)\int_x^b K(g(t)-g(x))\,dt,
\end{aligned}
$$

which proves the second part of (3.1).

If we use the equality (2.5) for $\lambda = f'_{-}(x)$ and $\gamma = f'_{+}(x)$, then we have for $x \in (a,b)$ that

$$
\begin{aligned}
S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right] \\
&+\frac{1}{2}f'_{-}(x)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{+}(x)\int_x^b K(g(t)-g(x))\,dt \\
+\frac{1}{2}\int_a^x & K(g(x)-g(t))\left[f'(t)-f'_{-}(x)\right]dt+\frac{1}{2}\int_x^b K(g(t)-g(x))\left[f'_{+}(x)-f'(t)\right]dt \\
&\leq \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right] \\
&+\frac{1}{2}f'_{-}(x)\int_a^x K(g(x)-g(t))\,dt-\frac{1}{2}f'_{+}(x)\int_x^b K(g(t)-g(x))\,dt,
\end{aligned}
$$

which proves the first part of (3.1). □

Remark 1. *If the functions is differentiable convex on (a,b), then the first inequality in (3.1) becomes*

$$
\begin{aligned}
\frac{1}{2}&\left[\int_x^b K(g(t)-g(x))\,dt-\int_a^x K(g(x)-g(t))\,dt\right]f'(x) \\
&\leq \frac{1}{2}\left[K(g(x)-g(a))f(a)+K(g(b)-g(x))f(b)\right]-S_{k,g,a+,b-}f(x)
\end{aligned} \tag{3.2}
$$

for any $x \in (a,b)$.

Corollary 4. *With the assumptions of Theorem 2 we have the Hermite-Hadamard type inequality for the g-mean $M_g(a,b)$*

$$\begin{aligned}(3.3)\quad & \frac{1}{2}\left[f'_+(M_g(a,b))\int_{M_g(a,b)}^{b}K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\right.\\ &\left.-f'_-(M_g(a,b))\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right]\\ &\leq K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}-P_{k,g,a+,b-}f\\ &\leq\frac{1}{2}\left[f'_-(b)\int_{M_g(a,b)}^{b}K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt\right.\\ &\left.-f'_+(a)\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right].\end{aligned}$$

In particular, if f is differentiable in $M_g(a,b)$, then we have the simpler inequality

$$\begin{aligned}(3.4)\quad & \frac{1}{2}f'_-(M_g(a,b))\\ &\times\left[\int_{M_g(a,b)}^{b}K\left(g(t)-\frac{g(a)+g(b)}{2}\right)dt-\int_a^{M_g(a,b)}K\left(\frac{g(a)+g(b)}{2}-g(t)\right)dt\right]\\ &\leq K\left(\frac{g(b)-g(a)}{2}\right)\frac{f(a)+f(b)}{2}-P_{k,g,a+,b-}f.\end{aligned}$$

We also have:

Corollary 5. *With the assumptions of Theorem 2 we have*

$$\begin{aligned}(3.5)\quad & \frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt\right.\\ &\left.-f'_-\left(\frac{a+b}{2}\right)\int_a^{\frac{a+b}{2}}K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right]\\ &\leq\frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)\right]-M_{k,g,a+,b-}f\\ &\leq\frac{1}{2}\left[f'_-(b)\int_{\frac{a+b}{2}}^{b}K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt\right.\\ &\left.-f'_+(a)\int_a^{\frac{a+b}{2}}K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right].\end{aligned}$$

In particular, if f is differentiable in $\frac{a+b}{2}$, then we have the simpler inequality

$$\begin{aligned}(3.6)\quad & \frac{1}{2}f'\left(\frac{a+b}{2}\right) \\ & \times\left[\int_{\frac{a+b}{2}}^{b}K\left(g(t)-g\left(\frac{a+b}{2}\right)\right)dt-\int_{a}^{\frac{a+b}{2}}K\left(g\left(\frac{a+b}{2}\right)-g(t)\right)dt\right] \\ \leq{} & \frac{1}{2}\left[K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)f(a)+K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)f(b)\right]-M_{k,g,a+,b-}f.\end{aligned}$$

4. Ostrowski Type Inequalities for Convex Functions

We also have:

Theorem 3. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then we have*

$$\begin{aligned}(4.1)\quad & \frac{1}{2}\left[f'_{+}(x)\int_{x}^{b}K(g(b)-g(t))\,dt-f'_{-}(x)\int_{a}^{x}K(g(t)-g(a))\,dt\right] \\ & \leq \breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ & \leq \frac{1}{2}\left[f'_{-}(b)\int_{x}^{b}K(g(b)-g(t))\,dt-f'_{+}(a)\int_{a}^{x}K(g(t)-g(a))\,dt\right]\end{aligned}$$

for $x\in(a,b)$.

Proof. Observe that by the positivity of the kernel k we have $K(g(b)-g(t))\geq 0$ for $t\in(x,b)$ and $K(g(t)-g(a))\geq 0$ for $t\in(a,x)$.

Using the identity (2.6), we have for $\gamma=f'_{+}(x)$ and $\lambda=f'_{-}(x)$ that

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &\quad+\frac{1}{2}f'_{+}(x)\int_{x}^{b}K(g(b)-g(t))\,dt-\frac{1}{2}f'_{-}(x)\int_{a}^{x}K(g(t)-g(a))\,dt \\ &\quad+\frac{1}{2}\int_{x}^{b}K(g(b)-g(t))\left[f'(t)-f'_{+}(x)\right]dt+\frac{1}{2}\int_{a}^{x}K(g(t)-g(a))\left[f'_{-}(x)-f'(t)\right]dt \\ &\geq \frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x) \\ &\quad+\frac{1}{2}f'_{+}(x)\int_{x}^{b}K(g(b)-g(t))\,dt-\frac{1}{2}f'_{-}(x)\int_{a}^{x}K(g(t)-g(a))\,dt,\end{aligned}$$

which proves the first inequality in (4.1).

Using the identity (2.6), we have for $\gamma=f'_{-}(b)$ and $\lambda=f'_{+}(a)$ that

$$\breve{S}_{k,g,a+,b-}f(x) = \frac{1}{2}\left[K(g(b)-g(x)) + K(g(x)-g(a))\right] f(x)$$
$$+\frac{1}{2}f'_{-}(b)\int_x^b K(g(b)-g(t))\,dt - \frac{1}{2}f'_{+}(a)\int_a^x K(g(t)-g(a))\,dt$$
$$+\frac{1}{2}\int_x^b K(g(b)-g(t))\left[f'(t)-f'_{-}(b)\right]dt + \frac{1}{2}\int_a^x K(g(t)-g(a))\left[f'_{+}(a)-f'(t)\right]dt$$
$$\leq \frac{1}{2}\left[K(g(b)-g(x)) + K(g(x)-g(a))\right] f(x)$$
$$+\frac{1}{2}f'_{-}(b)\int_x^b K(g(b)-g(t))\,dt - \frac{1}{2}f'_{+}(a)\int_a^x K(g(t)-g(a))\,dt,$$

which proves the second inequality in (4.1). □

Remark 2. *If the function is differentiable convex on (a,b), then the first inequality in (4.1) becomes*

$$\frac{1}{2}\left[\int_x^b K(g(b)-g(t))\,dt - \int_a^x K(g(t)-g(a))\,dt\right] f'(x) \tag{4.2}$$
$$\leq \breve{S}_{k,g,a+,b-}f(x) - \frac{1}{2}\left[K(g(b)-g(x)) + K(g(x)-g(a))\right] f(x)$$

for any $x \in (a,b)$.

Corollary 6. *With the assumptions of Theorem 3 we have the Hermite-Hadamard type inequality for the g-mean $M_g(a,b)$*

$$\frac{1}{2}\left[f'_{+}(M_g(a,b))\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt \right. \tag{4.3}$$
$$\left. - f'_{-}(M_g(a,b))\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right]$$
$$\leq \breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right) f(M_g(a,b))$$
$$\leq \frac{1}{2}\left[f'_{-}(b)\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt - f'_{+}(a)\int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right].$$

In particular, if f is differentiable in $M_g(a,b)$, then we have the simpler inequality

$$\frac{1}{2}f'(M_g(a,b))\left[\int_{M_g(a,b)}^b K(g(b)-g(t))\,dt - \int_a^{M_g(a,b)} K(g(t)-g(a))\,dt\right] \tag{4.4}$$
$$\leq \breve{P}_{k,g,a+,b-}f - K\left(\frac{g(b)-g(a)}{2}\right) f(M_g(a,b)).$$

We also have:

Corollary 7. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}(4.5)\quad & \frac{1}{2}\left[f_+'\left(\frac{a+b}{2}\right)\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt\right.\\ &\qquad\left.-f_-'\left(\frac{a+b}{2}\right)\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right]\\ &\leq \breve{M}_{k,g,a+,b-}f-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right)\\ &\leq \frac{1}{2}\left[f_-'\left(b\right)\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-f_+'\left(a\right)\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right].\end{aligned}$$

In particular, if f is differentiable in $\frac{a+b}{2}$, then we have the simpler inequality

$$\begin{aligned}(4.6)\quad & \frac{1}{2}f'\left(\frac{a+b}{2}\right)\left[\int_{\frac{a+b}{2}}^{b}K\left(g\left(b\right)-g\left(t\right)\right)dt-\int_{a}^{\frac{a+b}{2}}K\left(g\left(t\right)-g\left(a\right)\right)dt\right]\\ &\leq \breve{M}_{k,g,a+,b-}f\\ &\quad-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(\frac{a+b}{2}\right)\right)+K\left(g\left(\frac{a+b}{2}\right)-g\left(a\right)\right)\right]f\left(\frac{a+b}{2}\right).\end{aligned}$$

5. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k\left(t\right)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+}f\left(x\right)=I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left[g\left(x\right)-g\left(t\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a<x\leq b$ and

$$S_{k,g,b-}f\left(x\right)=I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left[g\left(t\right)-g\left(x\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a\leq x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

We consider the mixed operators

$$(5.1)\qquad I_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right]$$

and

$$(5.2)\qquad \breve{I}_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f\left(b\right)+I_{x-,g}^{\alpha}f(a)\right]$$

for $x\in\left(a,b\right)$.

We observe that for $\alpha>0$ we have

$$K\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma\left(\alpha\right)}=\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)},\ t\geq0.$$

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then by Theorem 2 we have the trapezoid type inequalities

$$
\begin{aligned}
(5.3)\quad & \frac{1}{2\Gamma(\alpha+1)}\left[f'_{+}(x)\int_{x}^{b}(g(t)-g(x))^{\alpha}dt-f'_{-}(x)\int_{a}^{x}(g(x)-g(t))^{\alpha}dt\right]\\
&\leq\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]-I^{\alpha}_{g,a+,b-}f(x)\\
&\leq\frac{1}{2\Gamma(\alpha+1)}\left[f'_{-}(b)\int_{x}^{b}(g(t)-g(x))^{\alpha}dt-f'_{+}(a)\int_{a}^{x}(g(x)-g(t))^{\alpha}dt\right]
\end{aligned}
$$

for $x\in(a,b)$.

In particular, if f is differentiable convex on (a,b), then by the first inequality in (5.3) we have

$$
\begin{aligned}
(5.4)\quad & \frac{1}{2\Gamma(\alpha+1)}\left[\int_{x}^{b}(g(t)-g(x))^{\alpha}dt-\int_{a}^{x}(g(x)-g(t))^{\alpha}dt\right]f'(x)\\
&\leq\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha}f(a)+(g(b)-g(x))^{\alpha}f(b)\right]-I^{\alpha}_{g,a+,b-}f(x)
\end{aligned}
$$

for $x\in(a,b)$.

If we take in (5.3) and (5.4) $x=M_g(a,b)$, then we get

$$
\begin{aligned}
(5.5)\quad & \frac{1}{2\Gamma(\alpha+1)}f'(M_g(a,b))\\
&\times\left[\int_{M_g(a,b)}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha}dt-\int_{a}^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha}dt\right]\\
&\leq\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\frac{f(a)+f(b)}{2}-I^{\alpha}_{g,a+,b-}f(M_g(a,b))\\
&\leq\frac{1}{2\Gamma(\alpha+1)}\left[f'_{-}(b)\int_{M_g(a,b)}^{b}\left(g(t)-\frac{g(a)+g(b)}{2}\right)^{\alpha}dt\right.\\
&\qquad\left.-f'_{+}(a)\int_{a}^{M_g(a,b)}\left(\frac{g(a)+g(b)}{2}-g(t)\right)^{\alpha}dt\right].
\end{aligned}
$$

If we take in (5.3) and (5.4) $x=\frac{a+b}{2}$, then we also get

$$
\begin{aligned}
(5.6)\quad & \frac{1}{2\Gamma(\alpha+1)} f'\left(\frac{a+b}{2}\right) \\
& \times\left[\int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} dt-\int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} dt\right] \\
& \leq \frac{1}{2\Gamma(\alpha+1)} \\
& \times\left[\left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha} f(a)+\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} f(b)\right] \\
& -I_{g,a+,b-}^{\alpha} f\left(\frac{a+b}{2}\right) \\
& \leq \frac{1}{2\Gamma(\alpha+1)}\left[f_{-}'(b) \int_{\frac{a+b}{2}}^{b}\left(g(t)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} dt\right. \\
& \left.-f_{+}'(a) \int_{a}^{\frac{a+b}{2}}\left(g\left(\frac{a+b}{2}\right)-g(t)\right)^{\alpha} dt\right].
\end{aligned}
$$

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a continuous convex function on $[a,b]$, then on making use of Theorem 3 we can state the following Ostrowski type inequality

$$
\begin{aligned}
(5.7)\quad & \frac{1}{2\Gamma(\alpha+1)}\left[f_{+}'(x) \int_{x}^{b}(g(b)-g(t))^{\alpha} dt-f_{-}'(x) \int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right] \\
& \leq \breve{I}_{g,a+,b-}^{\alpha} f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right] f(x) \\
& \leq \frac{1}{2\Gamma(\alpha+1)}\left[f_{-}'(b) \int_{x}^{b}(g(b)-g(t))^{\alpha} dt-f_{+}'(a) \int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right]
\end{aligned}
$$

for $x\in(a,b)$.

In particular, if f is differentiable convex on (a,b), then by the first inequality in (5.7) we have

$$
\begin{aligned}
(5.8)\quad & \frac{1}{2\Gamma(\alpha+1)}\left[\int_{x}^{b}(g(b)-g(t))^{\alpha} dt-\int_{a}^{x}(g(t)-g(a))^{\alpha} dt\right] f'(x) \\
& \leq \breve{I}_{g,a+,b-}^{\alpha} f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right] f(x)
\end{aligned}
$$

for $x\in(a,b)$.

If we take in (5.7) and (5.8) $x = M_g(a,b)$, then we get

$$\begin{aligned}
(5.9)\quad & \frac{1}{2\Gamma(\alpha+1)} f'(M_g(a,b)) \\
&\times \left[\int_{M_g(a,b)}^{b} (g(b)-g(t))^{\alpha} dt - \int_{a}^{M_g(a,b)} (g(t)-g(a))^{\alpha} dt\right] \\
&\le \breve{I}^{\alpha}_{g,a+,b-} f(M_g(a,b)) - \frac{(g(b)-g(a))^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)} f(M_g(a,b)) \\
&\le \frac{1}{2\Gamma(\alpha+1)} \left[f'_-(b)\int_{M_g(a,b)}^{b} (g(b)-g(t))^{\alpha} dt - f'_+(a)\int_{a}^{M_g(a,b)} (g(t)-g(a))^{\alpha} dt\right].
\end{aligned}$$

If we take in (5.7) and (5.8) $x = \frac{a+b}{2}$, then we also get

$$\begin{aligned}
(5.10)\quad & \frac{1}{2\Gamma(\alpha+1)} f'\left(\frac{a+b}{2}\right) \\
&\times \left[\int_{\frac{a+b}{2}}^{b} (g(b)-g(t))^{\alpha} dt - \int_{a}^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} dt\right] \\
&\le \breve{I}^{\alpha}_{g,a+,b-} f\left(\frac{a+b}{2}\right) \\
&- \frac{1}{2\Gamma(\alpha+1)} \left[\left(g(b)-g\left(\frac{a+b}{2}\right)\right)^{\alpha} + \left(g\left(\frac{a+b}{2}\right)-g(a)\right)^{\alpha}\right] f\left(\frac{a+b}{2}\right) \\
&\le \frac{1}{2\Gamma(\alpha+1)} \left[f'_-(b)\int_{\frac{a+b}{2}}^{b} (g(b)-g(t))^{\alpha} dt - f'_+(a)\int_{a}^{\frac{a+b}{2}} (g(t)-g(a))^{\alpha} dt\right].
\end{aligned}$$

If we take in these inequalities $g(t) = t$, we recapture the results for the classical Riemann-Liouville fractional integrals outlined in Introduction.

6. Example for an Exponential Kernel

For $\alpha \in \mathbb{R}$ we consider the kernel $k(t) := \exp(\alpha t)$, $t \in \mathbb{R}$. We have

$$|k(s)| = \exp(\alpha s) \text{ for } s \in \mathbb{R}$$

and

$$K(t) = \frac{\exp(\alpha t) - 1}{\alpha}, \text{ if } t \in \mathbb{R}$$

for $\alpha \neq 0$.

Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Define

$$\begin{aligned}
(6.1)\quad \mathcal{H}^{\alpha}_{g,a+,b-} f(x) &= \frac{1}{2}\int_{x}^{b} \exp[\alpha(g(t)-g(x))] g'(t) f(t) dt \\
&+ \frac{1}{2}\int_{a}^{x} \exp[\alpha(g(x)-g(t))] g'(t) f(t) dt
\end{aligned}$$

for $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}
(6.2)\qquad & \kappa^{\alpha}_{h,a+,b-}f(x) \\
& := \mathcal{H}^{\alpha}_{\ln h,a+,b-}f(x) \\
& = \frac{1}{2}\left[\int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt + \int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt\right],
\end{aligned}$$

for $x \in (a,b)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is differentiable convex function on (a,b), then by Theorem 2 we have the trapezoid type inequalities

$$\begin{aligned}
(6.3)\qquad & \frac{1}{2}f'(x) \\
& \times\left(\int_x^b \frac{\exp(\alpha(g(t)-g(x)))-1}{\alpha}dt - \int_a^x \frac{\exp(\alpha(g(x)-g(t)))-1}{\alpha}dt\right) \\
& \le \frac{1}{2}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}f(a) + \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}f(b)\right] \\
& \qquad - \mathcal{H}^{\alpha}_{g,a+,b-}f(x) \\
& \le \frac{1}{2}\left[f'_-(b)\int_x^b \frac{\exp(\alpha(g(t)-g(x)))-1}{\alpha}dt - f'_+(a)\int_a^x \frac{\exp(\alpha(g(x)-g(t)))-1}{\alpha}dt\right]
\end{aligned}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then by (6.3) we get

$$\begin{aligned}
(6.4)\qquad & \frac{1}{2}f'(x)\left(\int_x^b \frac{\left(\frac{h(t)}{h(x)}\right)^{\alpha}-1}{\alpha}dt - \int_a^x \frac{\left(\frac{h(x)}{h(t)}\right)^{\alpha}-1}{\alpha}dt\right) \\
& \le \frac{1}{2}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}f(a) + \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}f(b)\right] - \kappa^{\alpha}_{h,a+,b-}f(x) \\
& \le \frac{1}{2}\left[f'_-(b)\int_x^b \frac{\left(\frac{h(t)}{h(x)}\right)^{\alpha}-1}{\alpha}dt - f'_+(a)\int_a^x \frac{\left(\frac{h(x)}{h(t)}\right)^{\alpha}-1}{\alpha}dt\right]
\end{aligned}$$

for any $x \in (a,b)$.

Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Also define

$$\begin{aligned}
(6.5)\qquad & \breve{\mathcal{H}}^{\alpha}_{g,a+,b-}f(x) \\
& := \frac{1}{2}\int_x^b \exp[\alpha(g(b)-g(t))]\,g'(t)\,f(t)\,dt \\
& + \frac{1}{2}\int_a^x \exp[\alpha(g(t)-g(a))]\,g'(t)\,f(t)\,dt
\end{aligned}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$
\begin{aligned}
(6.6)\qquad & \breve{\kappa}^{\alpha}_{h,a+,b-} f(x) \\
& := \breve{\mathcal{H}}^{\alpha}_{\ln h,a+,b-} f(x) \\
& = \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt\right],
\end{aligned}
$$

for any $x \in (a,b)$.

If $f : [a,b] \to \mathbb{R}$ is differentiable convex function on (a,b), then by Theorem 3 we have the Ostrowski type inequalities

$$
\begin{aligned}
(6.7)\qquad & \frac{1}{2} f'(x)\left[\int_x^b \exp[\alpha(g(b)-g(t))]\,dt - \int_a^x \exp[\alpha(g(t)-g(a))]\,dt\right] \\
& \leq \breve{\mathcal{H}}^{\alpha}_{g,a+,b-} f(x) \\
& - \frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x))) + \exp(\alpha(g(x)-g(a))) - 2}{\alpha}\right] f(x) \\
& \leq \frac{1}{2}\left[f'_{-}(b)\int_x^b \exp[\alpha(g(b)-g(t))]\,dt - f'_{+}(a)\int_a^x \exp[\alpha(g(t)-g(a))]\,dt\right]
\end{aligned}
$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then by (6.7) we get

$$
\begin{aligned}
(6.8)\qquad & \frac{1}{2} f'(x)\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right] \\
& \leq \breve{\kappa}^{\alpha}_{h,a+,b-} f(x) - \frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} + \left(\frac{h(x)}{h(a)}\right)^{\alpha} - 2}{\alpha}\right] f(x) \\
& \leq \frac{1}{2}\left[f'_{-}(b)\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - f'_{+}(a)\int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right]
\end{aligned}
$$

for any $x \in (a,b)$.

Finally, if we take $x_h := h^{-1}\left(\sqrt{h(a)\,h(b)}\right) = h^{-1}\left(G\left(h(a), h(b)\right)\right) \in (a,b)$, where G is the geometric mean, in (6.4) and (6.9), then we get

$$
\begin{aligned}
(6.9)\quad & \frac{1}{2} f'(x_h)\left(\int_{x_h}^{b} \frac{\left(\frac{h(t)}{G(h(a),h(b))}\right)^{\alpha}-1}{\alpha} dt - \int_{a}^{x_h} \frac{\left(\frac{G(h(a),h(b))}{h(t)}\right)^{\alpha}-1}{\alpha} dt\right) \\
& \leq \frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha/2}-1}{\alpha}\,\frac{f(a)+f(b)}{2} - \kappa_{h,a+,b-}^{\alpha} f(x_h) \\
& \leq \frac{1}{2}\left[f'_{-}(b)\int_{x_h}^{b} \frac{\left(\frac{h(t)}{G(h(a),h(b))}\right)^{\alpha}-1}{\alpha} dt - f'_{+}(a)\int_{a}^{x_h} \frac{\left(\frac{G(h(a),h(b))}{h(t)}\right)^{\alpha}-1}{\alpha} dt\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(6.10)\quad & \frac{1}{2} f'(x_h)\left[\int_{x_h}^{b}\left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - \int_{a}^{x_h}\left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right] \\
& \leq \breve{\kappa}_{h,a+,b-}^{\alpha} f(x_h) - \frac{\left(\frac{h(b)}{h(a)}\right)^{\alpha/2}-1}{\alpha} f(x_h) \\
& \leq \frac{1}{2}\left[f'_{-}(b)\int_{x_h}^{b}\left(\frac{h(b)}{h(t)}\right)^{\alpha} dt - f'_{+}(a)\int_{a}^{x_h}\left(\frac{h(t)}{h(a)}\right)^{\alpha} dt\right].
\end{aligned}
$$

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