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INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS IN TERMS OF DOUBLE INTEGRAL MEANS

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Abstract. In this paper we establish some inequalities for the k-g-fractional integrals of various subclasses of Lebesgue integrable functions in terms of double integral means. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ and for general exponential fractional integrals are also given.

1. Introduction

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b] \tag{1.1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b). \tag{1.2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I^{\alpha}_{a+,g}f(x),\ a<x\leq b\end{aligned} \tag{1.3}$$

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and

$$S_{k,g,b-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)f(t)\,dt \tag{1.4}$$
$$=:I_{b-,g}^{\alpha}f(x),\ \ a\le x<b,$$

which are as defined in [24, p. 100].

For $g(t)=t$ in (1.4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [24, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.5}$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.8}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)f(t)\,dt, \tag{1.9}$$

for $a<x\le b$ and

$$E_{b-,\beta}^{\alpha}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)f(t)\,dt, \tag{1.10}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.1) and (1.2), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)f(t)\,dt,\ \ x\in(a,b] \tag{1.11}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ \ x\in[a,b). \tag{1.12}$$

In [27], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x):=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ |x|<R,\ \text{with}\ R>0 \tag{1.13}$$

for $\rho, \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(x-t)^{\rho}) f(t)\,dt,\ x > a \tag{1.14}$$

where $\rho,\ \lambda > 0,\ w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda}(w(t-x)^{\rho}) f(t)\,dt,\ x < b \tag{1.15}$$

where $\rho,\ \lambda > 0,\ w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.14) and (1.15) from (1.11) and (1.12).

In [25], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\,dt,\ x > a \tag{1.16}$$

and

$$\mathcal{T}^{\alpha}_{b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\,dt,\ x < b \tag{1.17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right),\ t \in \mathbb{R}$ we re-obtain the definitions of (1.16) and (1.17) from (1.11) and (1.12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\} g'(t) f(t)\,dt,\ x > a \tag{1.18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\} g'(t) f(t)\,dt,\ x < b \tag{1.19}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x (g(x)-g(t))^{\alpha-1} \ln(g(x)-g(t))\, g'(t) f(t)\,dt, \tag{1.20}$$

for $0 < a < x \le b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t)-g(x))^{\alpha-1} \ln(g(t)-g(x))\, g'(t) f(t)\,dt, \tag{1.21}$$

for $0 < a \le x < b$, where $\alpha > 0$. These are obtained from (1.11) and (1.12) for the kernel $k(t) = t^{\alpha-1}\ln t,\ t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x)-g(t))\, g'(t) f(t)\,dt,\ 0 < a < x \le b \tag{1.22}$$

and

$$\mathcal{L}_{g,b-}f(x) := \int_x^b \ln(g(t)-g(x))\, g'(t)\, f(t)\, dt,\ 0 < a \le x < b. \tag{1.23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t)\, f(t)\, dt,\ 0 < a < x \le b, \tag{1.24}$$

$$\mathcal{L}_{b-}^{\alpha} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x)\, f(t)\, dt,\ 0 < a \le x < b, \tag{1.25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t)\, f(t)\, dt,\ 0 < a < x \le b \tag{1.26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x)\, f(t)\, dt,\ 0 < a \le x < b. \tag{1.27}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[18], [22]-[35] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} (1.28)\quad & S_{k,g,a+,b-}f(x) \\ & := \frac{1}{2}\left[S_{k,g,a+}f(x) + S_{k,g,b-}f(x)\right] \\ & = \frac{1}{2}\left[\int_a^x k(g(x)-g(t))\, g'(t)\, f(t)\, dt + \int_x^b k(g(t)-g(x))\, g'(t)\, f(t)\, dt\right] \end{aligned}$$

for the Lebesgue integrable function $f : (a,b) \to \mathbb{C}$ and $x \in (a,b)$.

We also define the functions $\mathbf{K}_p : [0,\infty) \to [0,\infty)$ by

$$\mathbf{K}_p(t) := \begin{cases} \left(\int_0^t |k(s)|^p\right)^{1/p} ds \text{ if } 0 < t,\ p \ge 1 \\ \\ 0 \text{ if } t = 0 \end{cases}$$

For $p = 1$ we put

$$\mathbf{K}(t) := \mathbf{K}_1(t) = \begin{cases} \int_0^t |k(s)|\, ds \text{ if } 0 < t, \\ \\ 0 \text{ if } t = 0. \end{cases}$$

Observe that

$$S_{k,g,x+}f(b) = \int_x^b k(g(b)-g(t))\, g'(t)\, f(t)\, dt,\ x \in [a,b) \tag{1.29}$$

and

$$S_{k,g,x-}f(a) = \int_a^x k(g(t)-g(a))\, g'(t)\, f(t)\, dt,\ x \in (a,b]. \tag{1.30}$$

We can define also the mixed operator

$$\begin{aligned}(1.31)\quad & \breve{S}_{k,g,a+,b-}f(x)\\ & := \frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]\\ & = \frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt\right]\end{aligned}$$

for any $x\in(a,b)$.

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [21]:

Lemma 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\begin{aligned}S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right]\\ &\quad+\frac{1}{2}\int_a^x k(g(x)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt\\ &\quad+\frac{1}{2}\int_x^b k(g(t)-g(x))\,g'(t)\,[f(t)-\gamma]\,dt\end{aligned}\tag{1.32}$$

and

$$\begin{aligned}\breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right]\\ &\quad+\frac{1}{2}\int_a^x k(g(t)-g(a))\,g'(t)\,[f(t)-\lambda]\,dt\\ &\quad+\frac{1}{2}\int_x^b k(g(b)-g(t))\,g'(t)\,[f(t)-\gamma]\,dt\end{aligned}\tag{1.33}$$

for $x\in(a,b)$ and for any λ, $\gamma\in\mathbb{C}$.

In the recent paper [20], by using the above representations (1.32) and (1.33) we obtained the following result for functions of bounded variation:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the Ostrowski type inequality*

$$\begin{aligned}(1.34)\quad & \left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|\\ & \le\frac{1}{2}\left[\int_x^b |k(g(t)-g(x))|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x |k(g(x)-g(t))|\bigvee_t^x(f)\,g'(t)\,dt\right]\end{aligned}$$

$$\leq \frac{1}{2}\left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_{x}^{b}\left(f\right)\ \ +\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_{a}^{x}\left(f\right)\right] \tag{1.35}$$

$$\leq \frac{1}{2}\begin{cases} \max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\mathbf{K}^{p}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^{p}\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]\end{cases}$$

and the trapezoid type inequality

$$\left|S_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)f\left(b\right)+K\left(g\left(x\right)-g\left(a\right)\right)f\left(a\right)\right]\right| \tag{1.36}$$

$$\leq \frac{1}{2}\left[\int_{a}^{x}\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|\bigvee_{a}^{t}\left(f\right)g'\left(t\right)dt+\int_{x}^{b}\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|\bigvee_{t}^{b}\left(f\right)g'\left(t\right)dt\right]$$

$$\leq \frac{1}{2}\left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_{x}^{b}\left(f\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_{a}^{x}\left(f\right)\right]$$

$$\leq \frac{1}{2}\begin{cases} \max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\mathbf{K}^{p}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^{p}\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p} \\ \times\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right] \\ \times\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]\end{cases}$$

for any $x\in\left(a,b\right)$, *where* $\bigvee_{c}^{d}\left(f\right)$ *denoted the total variation on the interval* $\left[c,d\right]$.

In this paper we establish some inequalities for the k-g-fractional integrals of Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-}f$ and $\check{S}_{k,g,a+,b-}f$ in terms of the *double integral means*

$$\frac{1}{2}\left[\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{b-x}\int_{x}^{b}f\left(t\right)dt+\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{x-a}\int_{a}^{x}f\left(t\right)dt\right],\ x\in\left(a,b\right).$$

Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2. The Main Results

We use the classical Lebesgue p-norms defined as

$$\left\|h\right\|_{[c,d],\infty}:=\underset{s\in[c,d]}{\operatorname{essup}}\left|h\left(s\right)\right|$$

and

$$\|h\|_{[c,d],p} := \left(\int_c^d |h(s)|^p \, ds \right)^{1/p}, \ p \geq 1.$$

We have

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f : [a,b] \to \mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\left| S_{k,g,a+,b-} f(x) - \frac{1}{2} \left[K(g(b) - g(x)) \frac{1}{b-x} \int_x^b f(t) \, dt + K(g(x) - g(a)) \frac{1}{x-a} \int_a^x f(t) \, dt \right] \right| \tag{2.1}$$

$$\leq \frac{1}{2} \begin{cases} \left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],\infty} \mathbf{K}(g(x) - g(a)) \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],q} \mathbf{K}_p (g(x) - g(a)) \\ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases} + \frac{1}{2} \begin{cases} \left\| f - \frac{1}{b-x} \int_x^b f(s) \, ds \right\|_{[x,b],\infty} \mathbf{K}(g(b) - g(x)) \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{b-x} \int_x^b f(s) \, ds \right\|_{[a,x],q} \mathbf{K}_p (g(b) - g(x)) \\ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases} \tag{2.2}$$

and

$$\left| \breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2} \left[K(g(b) - g(x)) \frac{1}{b-x} \int_x^b f(t) \, dt + K(g(x) - g(a)) \frac{1}{x-a} \int_a^x f(t) \, dt \right] \right| \tag{2.3}$$

$$\leq \frac{1}{2} \begin{cases} \left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],\infty} \mathbf{K}(g(x) - g(a)) \\ \text{if } f \in L_\infty [a,b]; \\ \\ \left\| f - \frac{1}{x-a} \int_a^x f(s) \, ds \right\|_{[a,x],q} \mathbf{K}_p (g(x) - g(a)) \\ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q [a,b] \end{cases} \tag{2.4}$$

$$+\frac{1}{2}\begin{cases} \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\mathbf{K}(g(b)-g(x)) \\ \textit{if } f\in L_\infty[a,b]\,; \\ \\ \left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \ \textit{if } f\in L_q[a,b] \end{cases}$$

for $x\in(a,b)$.

Proof. If we write the equality (1.32) for $\gamma=\frac{1}{b-x}\int_x^b f(s)\,ds$ and $\lambda=\frac{1}{x-a}\int_a^x f(s)\,ds$ we get

$$\begin{aligned}(2.5)\qquad &\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right.\\ &\qquad\left.\left.+K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|\\ &\le\frac{1}{2}\left|\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right]dt\right|\\ &+\frac{1}{2}\left|\int_x^b k(g(t)-g(x))\,g'(t)\left[f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right]dt\right|\\ &\le\frac{1}{2}\int_a^x |k(g(x)-g(t))|\,g'(t)\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|dt\\ &+\frac{1}{2}\int_x^b |k(g(t)-g(x))|\,g'(t)\left|f(t)-\frac{1}{b-x}\int_x^b f(s)\,ds\right|dt\\ &=:B(x)\end{aligned}$$

for $x\in(a,b)$.

Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then by Hölder's integral inequality we have

$$\begin{aligned}(2.6)\qquad &\int_a^x |k(g(x)-g(t))|\,g'(t)\left|f(t)-\frac{1}{x-a}\int_a^x f(s)\,ds\right|dt\\ &\le\begin{cases} \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt \\ \text{if } f\in L_\infty[a,b]\,; \\ \\ \left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\left(\int_a^x |k(g(x)-g(t))|^p\,g'(t)\,dt\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_q[a,b] \end{cases}\end{aligned}$$

and

$$\int_x^b |k(g(t)-g(x))| g'(t) \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| dt \tag{2.7}$$
$$\leq \begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \int_x^b |k(g(t)-g(x))| g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\int_a^x |k(g(t)-g(x))|^p g'(t)\,dt\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}$$

for $x \in (a,b)$.

Observe that, by taking the derivative over t and using the chain rule we have

$$(\mathbf{K}(g(x)-g(t)))' = -\mathbf{K}'(g(x)-g(t))\,g'(t) = -|k(g(x)-g(t))|\,g'(t)$$

for $t \in (a,x)$ and

$$(\mathbf{K}(g(t)-g(x)))' = \mathbf{K}'(g(t)-g(x))\,g'(t) = |k(g(t)-g(x))|\,g'(t)$$

for $t \in (x,b)$.

Then

$$\int_a^x |k(g(x)-g(t))|\,g'(t)\,dt = -\int_a^x (\mathbf{K}(g(x)-g(t)))'\,dt = \mathbf{K}(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))|\,g'(t)\,dt = \int_x^b (\mathbf{K}(g(t)-g(x)))'\,dt = \mathbf{K}(g(x)-g(x))$$

where $x \in (a,b)$.

We also have for $p>1$

$$\left(\mathbf{K}_p^p(g(x)-g(t))\right)' = -|k(g(x)-g(t))|^p\,g'(t)$$

for $t \in (a,x)$ and

$$\left(\mathbf{K}_p^p(g(t)-g(x))\right)' = |k(g(t)-g(x))|^p\,g'(t)$$

for $t \in (x,b)$.

These give

$$\int_a^x |k(g(x)-g(t))|^p\,g'(t)\,dt = -\int_a^x \left(\mathbf{K}_p^p(g(x)-g(t))\right)'\,dt = \mathbf{K}_p^p(g(x)-g(a))$$

and

$$\int_x^b |k(g(t)-g(x))|^p\,g'(t)\,dt = \int_x^b \left(\mathbf{K}_p^p(g(t)-g(x))\right)'\,dt = \mathbf{K}_p^p(g(b)-g(x)),$$

which provide

$$\left(\int_a^x |k(g(x)-g(t))|^p\,g'(t)\,dt\right)^{1/p} = \mathbf{K}_p(g(x)-g(a))$$

and

$$\left(\int_x^b |k(g(t)-g(x))|^p\,g'(t)\,dt\right)^{1/p} = \mathbf{K}_p(g(b)-g(x))$$

for $x \in (a,b)$.

By making use of (2.6) and (2.7) we get

$$B(x) \leq \begin{cases} \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} \mathbf{K}(g(x)-g(a)) \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}$$

$$+ \begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \mathbf{K}(g(x)-g(x)) \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}$$

and by (2.5) we get (2.1).

Further on, by utilising the identity (1.33) for $\gamma = \frac{1}{b-x}\int_x^b f(s)\,ds$ and $\lambda = \frac{1}{x-a}\int_a^x f(s)\,ds$ we get

$$\begin{aligned}
(2.8)\qquad & \left| \breve{S}_{k,g,a+,b-}f(x) - \frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(s)\,ds \right.\right. \\
& \left.\left. + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(s)\,ds \right] \right| \\
& \leq \frac{1}{2}\int_a^x |k(g(t)-g(a))|\, g'(t) \left| f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right| dt \\
& + \frac{1}{2}\int_x^b |k(g(b)-g(t))|\, g'(t) \left| f(t) - \frac{1}{b-x}\int_x^b f(s)\,ds \right| dt \\
& \leq \frac{1}{2}\begin{cases} \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} \int_a^x |k(g(t)-g(a))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\int_a^x |k(g(t)-g(a))|^p g'(t)\,dt \right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases} \\
& + \frac{1}{2}\begin{cases} \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} \int_x^b |k(g(b)-g(t))|\, g'(t)\,dt \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\int_x^b |k(g(b)-g(t))|^p g'(t)\,dt \right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b] \end{cases}
\end{aligned}$$

for $x \in (a,b)$.

Since

$$\int_a^x |k(g(t)-g(a))|\, g'(t)\,dt = \int_a^x (\mathbf{K}(g(t)-g(a)))'\,dt = \mathbf{K}(g(x)-g(a)),$$

$$\int_a^x |k(g(t)-g(a))|^p g'(t)\,dt = \int_a^x \left(\mathbf{K}_p^p(g(t)-g(a))\right)' dt = \mathbf{K}_p^p(g(x)-g(a)),$$

$$\int_x^b |k(g(b)-g(t))|\, g'(t)\,dt = -\int_x^b (\mathbf{K}(g(b)-g(t)))' dt = \mathbf{K}(g(b)-g(x))$$

and

$$\int_x^b |k(g(b)-g(t))|^p g'(t)\,dt = -\int_x^b \left(\mathbf{K}_p^p(g(b)-g(t))\right)' dt = \mathbf{K}_p^p(g(b)-g(x)),$$

where $x \in (a,b)$, then by (2.8) we get the desired result (2.3). □

Remark 1. *We observe that*

$$\mathbf{K}(t) \le t\,\|k\|_{[0,t]} \text{ for } t \ge 0,$$

which implies that

$$\mathbf{K}(g(x)-g(a)) \le (g(x)-g(a))\,\|k\|_{[0,g(x)-g(a)]}$$

and

$$\mathbf{K}(g(b)-g(x)) \le (g(b)-g(x))\,\|k\|_{[0,g(b)-g(x)]}$$

for $x \in (a,b)$.

Therefore by (2.1) and (2.3) we get

$$\begin{aligned}
(2.9)\quad & \left| S_{k,g,a+,b-}f(x) - \frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt \right.\right.\\
&\qquad \left.\left. + K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt \right] \right| \\
&\le \frac{1}{2}\left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} (g(x)-g(a))\,\|k\|_{[0,g(x)-g(a)]} \\
&\quad + \frac{1}{2}\left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} (g(b)-g(x))\,\|k\|_{[0,g(b)-g(x)]} \\
&\le \frac{1}{2}\|k\|_{[0,g(b)-g(a)]}\left[\left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} (g(x)-g(a)) \right.\\
&\qquad \left. + \left\| f - \frac{1}{b-x}\int_x^b f(s)\,ds \right\|_{[x,b],\infty} (g(b)-g(x)) \right]
\end{aligned}$$

and

$$\begin{aligned}
(2.10)\quad & \left|\breve{S}_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right.\\
&\qquad\left.\left.+K(g(x)-g(a))\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|\\
&\leq\frac{1}{2}\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))\|k\|_{[0,g(x)-g(a)]}\\
&\quad+\frac{1}{2}\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))\|k\|_{[0,g(b)-g(x)]}\\
&\leq\frac{1}{2}\|k\|_{[0,g(b)-g(a)]}\left[\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}(g(x)-g(a))\right.\\
&\qquad\left.+\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}(g(b)-g(x))\right]
\end{aligned}$$

for $x\in(a,b)$.

The following result for functions of bounded variation hold [13]:

Lemma 2. *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$. *Then*

$$\|f\|_{[a,b],\infty}\leq\frac{1}{b-a}\left|\int_a^b f(t)\,dt\right|+\bigvee_a^b(f). \tag{2.11}$$

The multiplicative constant 1 *in front of* $\bigvee_a^b(f)$ *cannot be replaced by a smaller quantity.*

Lemma 3. *Let* $f:[a,b]\to\mathbb{R}$ *be a function of bounded variation on* $[a,b]$. *Then for* $p\geq 1$ *one has the inequality*

$$\|f\|_{[a,b],p}\leq\frac{1}{(b-a)^{1-\frac{1}{p}}}\left|\int_a^b f(t)\,dt\right|+\frac{1}{2}\frac{(b-a)^{\frac{1}{p}}\left(2^{p+1}-1\right)^{\frac{1}{p}}}{(p+1)^{\frac{1}{p}}}\bigvee_a^b(f). \tag{2.12}$$

The constant $\frac{1}{2}$ *is best possible in the sense that it cannot be replaced by a smaller quantity.*

The following result may be then stated:

Corollary 1. *Assume that the kernel* k *is defined either on* $(0,\infty)$ *or on* $[0,\infty)$ *with complex values and integrable on any finite subinterval. Let* $f:[a,b]\to\mathbb{C}$ *be a function of bounded variation on* $[a,b]$ *and* g *be a strictly increasing function on*

(a,b), having a continuous derivative g' on (a,b). Then

$$\begin{aligned}
(2.13)\quad & \left| S_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x}\int_x^b f(t)\,dt \right.\right. \\
& \left.\left. + K(g(x)-g(a)) \frac{1}{x-a}\int_a^x f(t)\,dt \right] \right| \\
& \leq \frac{1}{2}\begin{cases} \bigvee_a^x (f)\, \mathbf{K}(g(x)-g(a)) \\ \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_a^x (f)\, \mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} \\
& + \frac{1}{2}\begin{cases} \bigvee_x^b (f)\, \mathbf{K}(g(b)-g(x)) \\ \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_x^b (f)\, \mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(2.14)\quad & \left| \breve{S}_{k,g,a+,b-} f(x) - \frac{1}{2}\left[K(g(b)-g(x)) \frac{1}{b-x}\int_x^b f(t)\,dt \right.\right. \\
& \left.\left. + K(g(x)-g(a)) \frac{1}{x-a}\int_a^x f(t)\,dt \right] \right| \\
& \leq \frac{1}{2}\begin{cases} \bigvee_a^x (f)\, \mathbf{K}(g(x)-g(a)) \\ \frac{1}{2}\cdot\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_a^x (f)\, \mathbf{K}_p(g(x)-g(a)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases} \\
& + \frac{1}{2}\begin{cases} \bigvee_x^b (f)\, \mathbf{K}(g(b)-g(x)) \\ \frac{1}{2}\cdot\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_x^b (f)\, \mathbf{K}_p(g(b)-g(x)) \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
\end{aligned}$$

for $x\in(a,b)$.

Proof. By using Lemma 2 we have

$$\begin{aligned}
\left\| f - \frac{1}{x-a}\int_a^x f(s)\,ds \right\|_{[a,x],\infty} & \leq \frac{1}{x-a}\left| \int_a^x \left(f(t) - \frac{1}{x-a}\int_a^x f(s)\,ds \right) dt \right| \\
& + \bigvee_a^x \left(f - \frac{1}{x-a}\int_a^x f(s)\,ds \right) \\
& = \bigvee_a^x (f)
\end{aligned}$$

and

$$\left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],\infty} \le \frac{1}{b-x} \left| \int_x^b \left(f(t) - \frac{1}{b-x} \int_x^b f(s)\,ds \right) dt \right| + \bigvee_x^b \left(f(t) - \frac{1}{b-x} \int_x^b f(s)\,ds \right) = \bigvee_x^b (f)$$

for $x \in (a,b)$.

Also, by using Lemma 3 we have for $q > 1$ that

$$\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} \le \frac{1}{(x-a)^{1-\frac{1}{q}}} \left| \int_a^x \left(f(t) - \frac{1}{x-a} \int_a^x f(s)\,ds \right) dt \right| + \frac{1}{2} \frac{(x-a)^{\frac{1}{q}} \left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_a^x \left(f - \frac{1}{x-a} \int_a^x f(s)\,ds \right) = \frac{1}{2} \frac{(x-a)^{\frac{1}{q}} \left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_a^x (f)$$

and

$$\left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],q} \le \frac{1}{(b-x)^{1-\frac{1}{q}}} \left| \int_a^b \left(f(t) - \frac{1}{b-x} \int_x^b f(s)\,ds \right) dt \right| + \frac{1}{2} \frac{(b-x)^{\frac{1}{q}} \left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_x^b \left(f - \frac{1}{b-x} \int_x^b f(s)\,ds \right) = \frac{1}{2} \frac{(b-x)^{\frac{1}{q}} \left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_x^b (f)$$

for $x \in (a,b)$.

By using Theorem 2 we obtain the desired results (2.13) and (2.14). □

Remark 2. *With the assumptions of Corollary 1 we have*

$$\left| S_{k,g,a+,b-} f(x) - \frac{1}{2} \left[K(g(b)-g(x)) \frac{1}{b-x} \int_x^b f(t)\,dt + K(g(x)-g(a)) \frac{1}{x-a} \int_a^x f(t)\,dt \right] \right| \le \frac{1}{2} \begin{cases} \max\left\{ \mathbf{K}(g(x)-g(a)), \mathbf{K}(g(b)-g(x)) \right\} \bigvee_a^b (f) \\ \\ \left[\mathbf{K}(g(x)-g(a)) + \mathbf{K}(g(b)-g(x)) \right] \max\left\{ \bigvee_a^x (f), \bigvee_x^b (f) \right\} \end{cases} \tag{2.15}$$

and

$$\left|\breve{S}_{k,g,a+,b-}f\left(x\right)-\frac{1}{2}\left[K\left(g\left(b\right)-g\left(x\right)\right)\frac{1}{b-x}\int_{x}^{b}f\left(t\right)dt\right.\right.$$
$$\left.\left.+K\left(g\left(x\right)-g\left(a\right)\right)\frac{1}{x-a}\int_{a}^{x}f\left(t\right)dt\right]\right|$$
$$\leq\frac{1}{2}\left\{\begin{array}{l}\max\left\{\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right),\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\right\}\bigvee_{a}^{b}\left(f\right)\\ \\ \left[\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)+\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\right]\max\left\{\bigvee_{a}^{x}\left(f\right),\bigvee_{x}^{b}\left(f\right)\right\}\end{array}\right. \tag{2.16}$$

for $x\in\left(a,b\right)$.

3. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k\left(t\right)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+}f\left(x\right)=I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{x}\left[g\left(x\right)-g\left(t\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a<x\leq b$ and

$$S_{k,g,b-}f\left(x\right)=I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma\left(\alpha\right)}\int_{x}^{b}\left[g\left(t\right)-g\left(x\right)\right]^{\alpha-1}g'\left(t\right)f\left(t\right)dt$$

for $a\leq x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [24, p. 100].

We consider the mixed operators

$$I_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right] \tag{3.1}$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f\left(b\right)+I_{x-,g}^{\alpha}f(a)\right] \tag{3.2}$$

for $x\in\left(a,b\right)$.

We observe that for $\alpha>0$ we have

$$\mathbf{K}\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma\left(\alpha\right)}=\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)},\ t\geq0$$

and for $\alpha>\frac{p-1}{p}>0$, where $p>1$, we have

$$\mathbf{K}_{p}\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\left(\int_{0}^{t}s^{(\alpha-1)p}ds\right)^{1/p}=\frac{1}{\left(\alpha-1+1/p\right)\Gamma\left(\alpha\right)}t^{\alpha-1+1/p},\ t\geq0.$$

Using Theorem 2 we can state the following inequalities for $\alpha>0$

$$(3.3)\quad \left| I_{g,a+,b-}^{\alpha} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right|$$
$$\leq \frac{1}{2\Gamma(\alpha+1)} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],\infty} (g(x)-g(a))^{\alpha} + \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],\infty} (g(b)-g(x))^{\alpha} \right]$$

and

$$(3.4)\quad \left| \breve{I}_{g,a+,b-}^{\alpha} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right|$$
$$\leq \frac{1}{2\Gamma(\alpha+1)} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],\infty} (g(x)-g(a))^{\alpha} + \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],\infty} (g(b)-g(x))^{\alpha} \right]$$

for $x \in (a,b)$.

If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\alpha > \frac{p-1}{p} = \frac{1}{q} > 0$, then by Theorem 2 we can state the following inequalities as well

$$(3.5)\quad \left| I_{g,a+,b-}^{\alpha} f(x) - \frac{1}{2\Gamma(\alpha+1)} \left[\frac{(g(b)-g(x))^{\alpha}}{b-x} \int_x^b f(t)\,dt + \frac{(g(x)-g(a))^{\alpha}}{x-a} \int_a^x f(t)\,dt \right] \right|$$
$$\leq \frac{1}{2} \frac{1}{(\alpha-1/q)\Gamma(\alpha)} \left[\left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} (g(x)-g(a))^{\alpha-1+1/p} + \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[a,x],q} (g(b)-g(x))^{\alpha-1+1/p} \right]$$

and

$$\begin{aligned}(3.6)\quad &\left|\breve{I}^{\alpha}_{g,a+,b-}f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^{\alpha}}{b-x}\int_x^b f(t)\,dt\right.\right.\\&\left.\left.+\frac{(g(x)-g(a))^{\alpha}}{x-a}\int_a^x f(t)\,dt\right]\right|\\&\le\frac{1}{2}\frac{1}{(\alpha-1/q)\Gamma(\alpha)}\left[\left\|f-\frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}(g(x)-g(a))^{\alpha-1+1/p}\right.\\&\left.+\left\|f-\frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}(g(b)-g(x))^{\alpha-1+1/p}\right]\end{aligned}$$

for $x\in(a,b)$.

If we assume that $f:[a,b]\to\mathbb{C}$ is of bounded variation, then by Corollary 1 we have for $\alpha>0$ that

$$\begin{aligned}(3.7)\quad &\left|I^{\alpha}_{g,a+,b-}f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^{\alpha}}{b-x}\int_x^b f(t)\,dt\right.\right.\\&\left.\left.+\frac{(g(x)-g(a))^{\alpha}}{x-a}\int_a^x f(t)\,dt\right]\right|\\&\le\frac{1}{2\Gamma(\alpha+1)}\left[\bigvee_a^x(f)(g(x)-g(a))^{\alpha}+\bigvee_x^b(f)(g(b)-g(x))^{\alpha}\right]\end{aligned}$$

and

$$\begin{aligned}(3.8)\quad &\left|\breve{I}^{\alpha}_{g,a+,b-}f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^{\alpha}}{b-x}\int_x^b f(t)\,dt\right.\right.\\&\left.\left.+\frac{(g(x)-g(a))^{\alpha}}{x-a}\int_a^x f(t)\,dt\right]\right|\\&\le\frac{1}{2\Gamma(\alpha+1)}\left[\bigvee_a^x(f)(g(x)-g(a))^{\alpha}+\bigvee_x^b(f)(g(b)-g(x))^{\alpha}\right]\end{aligned}$$

for $x\in(a,b)$.

If $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\alpha>\frac{p-1}{p}=\frac{1}{q}>0$,then by Corollary 1 we have

$$\begin{aligned}(3.9)\quad &\left|I^{\alpha}_{g,a+,b-}f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^{\alpha}}{b-x}\int_x^b f(t)\,dt\right.\right.\\&\left.\left.+\frac{(g(x)-g(a))^{\alpha}}{x-a}\int_a^x f(t)\,dt\right]\right|\\&\le\frac{1}{4}\frac{\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}(\alpha-1/q)\Gamma(\alpha)}\left[(x-a)^{\frac{1}{q}}\bigvee_a^x(f)(g(x)-g(a))^{\alpha-1+1/p}\right.\\&\left.+(b-x)^{\frac{1}{q}}\bigvee_x^b(f)(g(b)-g(x))^{\alpha-1+1/p}\right]\end{aligned}$$

and

$$\left|\breve{I}^{\alpha}_{g,a+,b-}f(x)-\frac{1}{2\Gamma(\alpha+1)}\left[\frac{(g(b)-g(x))^{\alpha}}{b-x}\int_x^b f(t)\,dt+\frac{(g(x)-g(a))^{\alpha}}{x-a}\int_a^x f(t)\,dt\right]\right| \tag{3.10}$$
$$\leq\frac{1}{4}\frac{\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}(\alpha-1/q)\Gamma(\alpha)}\left[(x-a)^{\frac{1}{q}}\bigvee_a^x(f)(g(x)-g(a))^{\alpha-1+1/p}+(b-x)^{\frac{1}{q}}\bigvee_x^b(f)(g(b)-g(x))^{\alpha-1+1/p}\right]$$

for $x\in(a,b)$.

4. Example for an Exponential Kernel

For $\alpha\in\mathbb{R}$ we consider the kernel $k(t):=\exp(\alpha t)$, $t\in\mathbb{R}$. We have

$$|k(s)|=\exp(\alpha s)\ \text{ for } s\in\mathbb{R},$$

$$K(t)=\frac{\exp(\alpha t)-1}{\alpha},\ \text{ if } t\in\mathbb{R}$$

and for $p\geq 1$

$$\mathbf{K}_p(t)=\left(\int_0^t\exp(p\alpha s)\right)^{1/p}ds=\left(\frac{\exp(p\alpha t)-1}{p\alpha}\right)^{1/p}$$

for $\alpha\neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Define

$$\mathcal{H}^{\alpha}_{g,a+,b-}f(x)=\frac{1}{2}\int_x^b\exp[\alpha(g(t)-g(x))]\,g'(t)f(t)\,dt+\frac{1}{2}\int_a^x\exp[\alpha(g(x)-g(t))]\,g'(t)f(t)\,dt \tag{4.1}$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}&\kappa^{\alpha}_{h,a+,b-}f(x)\\&:=\mathcal{H}^{\alpha}_{\ln h,a+,b-}f(x)\\&=\frac{1}{2}\left[\int_x^b\left(\frac{h(t)}{h(x)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt+\int_a^x\left(\frac{h(x)}{h(t)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned} \tag{4.2}$$

for $x\in(a,b)$.

Furthermore, let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Also define

$$\begin{aligned}(4.3)\qquad & \breve{\mathcal{H}}^{\alpha}_{g,a+,b-}f(x)\\ & := \frac{1}{2}\int_x^b \exp\left[\alpha\left(g(b)-g(t)\right)\right]g'(t)\,f(t)\,dt\\ & + \frac{1}{2}\int_a^x \exp\left[\alpha\left(g(t)-g(a)\right)\right]g'(t)\,f(t)\,dt\end{aligned}$$

for any $x \in (a,b)$.

If $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}(4.4)\qquad & \breve{\kappa}^{\alpha}_{h,a+,b-}f(x)\\ & := \breve{\mathcal{H}}^{\alpha}_{\ln h,a+,b-}f(x)\\ & = \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha}\frac{h'(t)}{h(t)}f(t)\,dt\right],\end{aligned}$$

for any $x \in (a,b)$.

Using Theorem 2 we have

$$\begin{aligned}(4.5)\qquad & \left|\mathcal{H}^{\alpha}_{g,a+,b-}f(x) - \frac{1}{2}\left[\frac{\exp\left(\alpha\left(g(b)-g(x)\right)\right)-1}{\alpha}\frac{1}{b-x}\int_x^b f(t)\,dt\right.\right.\\ & \left.\left. + \frac{\exp\left(\alpha\left(g(x)-g(a)\right)\right)-1}{\alpha}\frac{1}{x-a}\int_a^x f(t)\,dt\right]\right|\\ & \le \frac{1}{2}\begin{cases}\left\|f - \frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],\infty}\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha} \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\|f - \frac{1}{x-a}\int_a^x f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(x)-g(a)))-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b]\end{cases}\\ & + \frac{1}{2}\begin{cases}\left\|f - \frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[x,b],\infty}\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha} \\ \text{if } f \in L_\infty[a,b]; \\ \\ \left\|f - \frac{1}{b-x}\int_x^b f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(b)-g(x)))-1}{p\alpha}\right)^{1/p} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f \in L_q[a,b]\end{cases}\end{aligned}$$

and

$$\left|\breve{\mathcal{H}}_{g,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\frac{1}{b-x}\int_{x}^{b}f(t)\,dt+\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\frac{1}{x-a}\int_{a}^{x}f(t)\,dt\right]\right| \tag{4.6}$$
$$\leq\frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_{a}^{x}f(s)\,ds\right\|_{[a,x],\infty}\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\\ \text{if } f\in L_{\infty}[a,b]\,;\\ \\ \left\|f-\frac{1}{x-a}\int_{a}^{x}f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(x)-g(a)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\end{cases}$$
$$+\frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_{x}^{b}f(s)\,ds\right\|_{[x,b],\infty}\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\\ \text{if } f\in L_{\infty}[a,b]\,;\\ \\ \left\|f-\frac{1}{b-x}\int_{x}^{b}f(s)\,ds\right\|_{[a,x],q}\left(\frac{\exp(p\alpha(g(b)-g(x)))-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\,,\end{cases}$$

for any $x\in(a,b)$.

If we take in (4.5) and (4.6) $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we have

$$\left|\kappa_{h,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\frac{1}{b-x}\int_{x}^{b}f(t)\,dt+\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\frac{1}{x-a}\int_{a}^{x}f(t)\,dt\right]\right| \tag{4.7}$$
$$\leq\frac{1}{2}\begin{cases}\left\|f-\frac{1}{x-a}\int_{a}^{x}f(s)\,ds\right\|_{[a,x],\infty}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\\ \text{if } f\in L_{\infty}[a,b]\,;\\ \\ \left\|f-\frac{1}{x-a}\int_{a}^{x}f(s)\,ds\right\|_{[a,x],q}\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\end{cases}$$
$$+\frac{1}{2}\begin{cases}\left\|f-\frac{1}{b-x}\int_{x}^{b}f(s)\,ds\right\|_{[x,b],\infty}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\\ \text{if } f\in L_{\infty}[a,b]\,;\\ \\ \left\|f-\frac{1}{b-x}\int_{x}^{b}f(s)\,ds\right\|_{[a,x],q}\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\end{cases}$$

and

$$\begin{aligned}
(4.8)\quad & \left| \breve{\kappa}^{\alpha}_{h,a+,b-} f(x) - \frac{1}{2} \left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \frac{1}{b-x} \int_x^b f(t)\,dt \right.\right. \\
& \qquad \left.\left. + \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \frac{1}{x-a} \int_a^x f(t)\,dt \right] \right| \\
& \leq \frac{1}{2} \begin{cases} \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],\infty} \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \\ \text{if } f \in L_{\infty}[a,b]; \\ \\ \left\| f - \frac{1}{x-a} \int_a^x f(s)\,ds \right\|_{[a,x],q} \left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q[a,b] \end{cases} \\
& \qquad + \frac{1}{2} \begin{cases} \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[x,b],\infty} \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \\ \text{if } f \in L_{\infty}[a,b]; \\ \\ \left\| f - \frac{1}{b-x} \int_x^b f(s)\,ds \right\|_{[a,x],q} \left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q[a,b] \end{cases}
\end{aligned}$$

for any $x \in (a,b)$.

Finally, if we assume that $f : [a,b] \to \mathbb{C}$ is of bounded variation, then by Corollary 1 we have

$$\begin{aligned}
(4.9)\quad & \left| \kappa^{\alpha}_{h,a+,b-} f(x) - \frac{1}{2} \left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \frac{1}{b-x} \int_x^b f(t)\,dt \right.\right. \\
& \qquad \left.\left. + \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \frac{1}{x-a} \int_a^x f(t)\,dt \right] \right| \\
& \leq \frac{1}{2} \begin{cases} \bigvee_a^x(f) \frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha} - 1}{\alpha} \text{ if } f \in L_{\infty}[a,b]; \\ \\ \frac{1}{2} \frac{(x-a)^{\frac{1}{q}} \left(2^{q+1} - 1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_a^x(f) \left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q[a,b] \end{cases} \\
& \qquad + \frac{1}{2} \begin{cases} \bigvee_x^b(f) \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha} - 1}{\alpha} \text{ if } f \in L_{\infty}[a,b]; \\ \\ \frac{1}{2} \frac{(b-x)^{\frac{1}{q}} \left(2^{q+1} - 1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}} \bigvee_x^b(f) \left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha} - 1}{p\alpha} \right)^{1/p} \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1 \text{ if } f \in L_q[a,b] \end{cases}
\end{aligned}$$

and

$$\left|\breve{\kappa}_{h,a+,b-}^{\alpha}f(x)-\frac{1}{2}\left[\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\frac{1}{b-x}\int_{x}^{b}f(t)\,dt+\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\frac{1}{x-a}\int_{a}^{x}f(t)\,dt\right]\right| \tag{4.10}$$
$$\leq\frac{1}{2}\begin{cases}\bigvee_{a}^{x}(f)\,\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_{\infty}[a,b]\,;\\ \frac{1}{2}\frac{(x-a)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_{a}^{x}(f)\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\end{cases}$$
$$+\frac{1}{2}\begin{cases}\bigvee_{x}^{b}(f)\,\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha} \text{ if } f\in L_{\infty}[a,b]\,;\\ \frac{1}{2}\frac{(b-x)^{\frac{1}{q}}\left(2^{q+1}-1\right)^{\frac{1}{q}}}{(q+1)^{\frac{1}{q}}}\bigvee_{x}^{b}(f)\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{p\alpha}-1}{p\alpha}\right)^{1/p}\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \text{ if } f\in L_{q}[a,b]\end{cases}$$

for any $x\in(a,b)$.

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