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# A TRACE INEQUALITY AS AN ANALOGUE OF A REFINEMENT OF YOUNG'S INEQUALITY

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Abstract. The aim of this paper is to obtain some trace inequalities for positive operators in Hilbert spaces, starting from two refinements of the classical Kittaneh-Manasrah inequalitty. Then several consequances as applications will be presented.

## 1. Introduction

The classical Young's inequality state that:

$$a^{\nu}b^{1-\nu} < \nu a + (1-\nu)b,$$

when $a$ and $b$ are positive numbers, $a \neq b$ and $\nu \in (0,1)$.

We shall consider the following two inequalities, given in [14], which represent improvements of Young's inequality:

**Lemma 1.** *([14]) For $0 < a, b \leq 1$ and $\lambda \in (0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

*where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.*

**Theorem 1.** *([14]) For $a, b \geq 1$ and $\lambda \in (0,1)$ we have*

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda}$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)\log^2\left(\frac{a}{b}\right)$$

*where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.*

We consider the functions:

$$f(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - r(\sqrt{a}-\sqrt{b})^2 - A(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

,

$$g(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - (1-r)(\sqrt{a}-\sqrt{b})^2 - B(\lambda)ab\log^2\left(\frac{a}{b}\right)$$

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$$h(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - r(\sqrt{a}-\sqrt{b})^2 - A(\lambda)\log^2\left(\frac{a}{b}\right)$$

and

$$k(a,b) = \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} - (1-r)(\sqrt{a}-\sqrt{b})^2 - B(\lambda)\log^2\left(\frac{a}{b}\right).$$

The below figures are graphics of this four functions for a particular value of $\lambda$.

Related to such Young type inequalities, often appear the weighted arithmetic mean, geometric mean and harmonic mean defined by $A_\nu(a,b) = (1-\nu)a + \nu b$, $G_\nu(a,b) = a^{1-\nu}b^{\nu}$ and $H_\nu(a,b) = A_\nu^{-1}(a^{-1},b^{-1}) = [(1-\nu)a^{-1} + \nu b^{-1}]^{-1}$, when $a,b>0$ and $\nu \in [0,1]$. We shall consider that $A$ and $B$ are positive operators on a complex Hilbert space $\mathcal{H}$. Then $A\nabla_\nu B = (1-\nu)A + \nu B$, $\nu \in [0,1]$ is the weighted operator arithmetic mean, and $A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\nu}A^{\frac{1}{2}}$, $\nu \in [0,1]$ is the weighted operator geometric mean. The relative operator entropy $\mathcal{S}(A/B)$ was defined in [8], [9] for positive invertible operators $A$ and $B$, by $\mathcal{S}(A/B) = A^{\frac{1}{2}}\left(\ln\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)A^{\frac{1}{2}}$. If $\Phi$ is a continuos function on the interval $J$ of real numbers, $B$ is a selfadjoint operator on the Hilbert space $\mathcal{H}$ , $A$ is a positive invertible operator on $\mathcal{H}$ and $Sp\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) \subset J$ then, in [6], the noncommutative perspective operator is defined by

$$\mathcal{P}_\Phi(B,A) = A^{\frac{1}{2}}\Phi\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}.$$

As in [3], let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}_1(\mathcal{H})$ the trace class operators in $\mathcal{B}(\mathcal{H})$. We define the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ to be $tr(A) = \sum_{i\in I} < Ae_i, e_i >$, where $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$. The main properties of the trace can be fuond in [3] and the references therein.

We establish in this paper in Theorem 2 and Theorem 3 some new trace inequalities via two scalar Young type inequalities presented in [14], using the methods given in [3] . Then will be given several consequences as applications below.

## 2. **Some trace analogue inequalities for two refinements of Young's inequality**

**Theorem 2.** *Let $m$, $M$ be two real munbers with $0 < m < M \leq 1$ and $A, B$ be two positive operators in $\mathcal{B}(\mathcal{H})$ with $Sp(A) \subset [m,M]$, $Sp(B) \subset [m,M]$ and $P, Q \in \mathcal{B}_1(\mathcal{H})$ with $P, Q > 0$. Then for any $\lambda \in [0,1]$ the following inequality takes place:*

$$r[tr(PA)tr(Q) - 2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}}) + tr(P)tr(QB)] + A_1(\lambda)[tr(QB)tr(PA\log^2 A) -$$
$$-2tr(QB\log B)tr(PA\log A) + tr(QB\log^2 B)tr(PA)] \leq$$
$$\leq \lambda tr(PA)tr(Q) + (1-\lambda)tr(QB)tr(P) - tr(QB^{1-\lambda})tr(PA^{\lambda}) \leq$$
$$\leq (1-r)[tr(PA)tr(Q) - 2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}}) + tr(P)tr(QB)] + B_1(\lambda)[tr(QB)tr(PA\log^2 A) -$$
$$-2tr(QB\log B)tr(PA\log A) + tr(QB\log^2 B)tr(PA)],$$

*where* $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ *and* $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ *as in Lemma 1.*

*Proof.* We use the same method as in [3].

By hypothesis we find that $A\log A,\ B\log B,\ A\log^2 A,\ B\log^2 B\in\mathcal{B}(\mathcal{H})$ and because $P,Q\in\mathcal{B}_1(\mathcal{H})$ with $P,Q>0$. we obtain by some properties of the trace, see [3], that $PA\log A,\ QB\log B,\ PA\log^2 A,\ QB\log^2 B\in\mathcal{B}_1(\mathcal{H})$ and $tr(P^{\frac{1}{2}}A\log AP^{\frac{1}{2}})=tr(PA\log A)$, $tr(Q^{\frac{1}{2}}B\log BQ^{\frac{1}{2}})=tr(QB\log B)$,
$tr(P^{\frac{1}{2}}A\log^2 AP^{\frac{1}{2}})=tr(PA\log^2 A)$, $tr(Q^{\frac{1}{2}}B\log^2 BQ^{\frac{1}{2}})=tr(QB\log^2 B)$.

We fix $M>b>m>0$ and then using the functional calculus for the operator $A$ we get from inequality of Lemma 1 that:

$$r\left(<Ax,x>-2\sqrt{b}<A^{\frac{1}{2}}x,x>+b<x.x>\right)+$$
$$+A_1(\lambda)\left(b<A\log^2 Ax,x>-2b\log b<A\log Ax,x>+b\log^2 b<Ax,x>\right)\leq$$
$$\leq\lambda<Ax,x>+(1-\lambda)b<x,x>-b^{1-\lambda}<A^{\lambda}x,x>\leq$$
$$\leq(1-r)\left(<Ax,x>-2\sqrt{b}<A^{\frac{1}{2}}x,x>+b<x.x>\right)+$$
$$+B_1(\lambda)\left(b<A\log^2 Ax,x>-2b\log b<A\log Ax,x>+b\log^2 b<Ax,x>\right),$$

for any $x\in\mathcal{H}$, if we denote there $A(\lambda)$ by $A_!(\lambda)$ and $B(\lambda)$ by $B_!(\lambda)$.

Now, we fix $x\in\mathcal{H}-\{0\}$ and then use the functional calculus for the operator $B$ for previous inequality. We have,

$$r\left(<Ax,x>||y||^2-2<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}y,y>+<By,y>||x||^2\right)+$$
$$+A_1(\lambda)(<By,y><A\log^2 Ax,x>-2<B\log By,y><A\log Ax,x>+$$
$$+<B\log^2 By,y><Ax,x>)\leq$$
$$\leq\lambda<Ax,x>||y||^2+(1-\lambda)<By,y>||x||^2-<B^{1-\lambda}y,y><A^{\lambda}x,x>\leq$$
$$\leq(1-r)\left(<Ax,x>||y||^2-2<A^{\frac{1}{2}}x,x><B^{\frac{1}{2}}y,y>+<By,y>||x||^2\right)+$$
$$+B_1(\lambda)(<By,y><A\log^2 Ax,x>-2<B\log By,y><A\log Ax,x>+$$
$$+<B\log^2 By,y><Ax,x>),$$

for any $x,y\in\mathcal{H}$.

We consider now, $x=P^{\frac{1}{2}}e$, $y=Q^{\frac{1}{2}}f$ where $e,\ f\in\mathcal{H}$. By the above inequality we have,

$$r(<P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e><Qf,f>-2<P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e,e><Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f,f>+$$
$$+<Pe,e><Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f>)+A_1(\lambda)(<Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}A\log^2 AP^{\frac{1}{2}}e,e>-$$
$$-2<Q^{\frac{1}{2}}B\log BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}A\log AP^{\frac{1}{2}}e,e>+<Q^{\frac{1}{2}}B\log^2 BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e>)\leq$$
$$\leq\lambda<P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e><Qf,f>+(1-\lambda)<Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f><Pe,e>-$$
$$-<Q^{\frac{1}{2}}B^{1-\lambda}Q^{\frac{1}{2}}f,f><P^{\frac{1}{2}}A^{\lambda}P^{\frac{1}{2}}e,e>\leq$$
$$\leq(1-r)(<P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e><Qf,f>-2<P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e,e><Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f,f>+$$
$$+<Pe,e><Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f>)+B_1(\lambda)(<Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}A\log^2 AP^{\frac{1}{2}}e,e>-$$
$$-2<Q^{\frac{1}{2}}B\log BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}A\log AP^{\frac{1}{2}}e,e>+<Q^{\frac{1}{2}}B\log^2 BQ^{\frac{1}{2}}f,f><P^{\frac{1}{2}}AP^{\frac{1}{2}}e,e>),$$

for any $e,f\in\mathcal{H}$.

Now, let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of $\mathcal{H}$. We take in previous inequality $e = e_i,\ i \in I$ and $f = f_j,\ j \in J$ and then summing over $i \in I$and $j \in J$, we obtain the following inequality:

$$r(\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_j, f_j > -2\sum_{i\in I} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f_j, f_j > +$$

$$+\sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j >)+$$

$$+A_1(\lambda)(\sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}A\log^2 AP^{\frac{1}{2}}e_i, e_i > -$$

$$-2\sum_{j\in J} < Q^{\frac{1}{2}}B\log BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}A\log AP^{\frac{1}{2}}e_i, e_i > +$$

$$+\sum_{j\in J} < Q^{\frac{1}{2}}B\log^2 BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i >) \leq$$

$$\leq \lambda\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_j, f_j > +(1-\lambda)\sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < Pe_i, e_i > -$$

$$-\sum_{j\in J} < Q^{\frac{1}{2}}B^{1-\lambda}Q^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}A^{\lambda}P^{\frac{1}{2}}e_i, e_i >\leq$$

$$\leq (1-r)(\sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Qf_j, f_j > -$$

$$-2\sum_{i\in I} < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f_j, f_j > +$$

$$+\sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j >)+$$

$$+B_1(\lambda)(\sum_{j\in J} < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}A\log^2 AP^{\frac{1}{2}}e_i, e_i > -$$

$$-2\sum_{j\in J} < Q^{\frac{1}{2}}B\log BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}A\log AP^{\frac{1}{2}}e_i, e_i > +$$

$$+\sum_{j\in J} < Q^{\frac{1}{2}}B\log^2 BQ^{\frac{1}{2}}f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}}AP^{\frac{1}{2}}e_i, e_i >).$$

By the properties of the trace we find the that

$$r[tr(PA)tr(Q)-2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}})+tr(P)tr(QB)]+A_1(\lambda)[tr(QB)tr(PA\log^2 A)-$$

$$-2tr(QB\log B)tr(PA\log A) + tr(QB\log^2 B)tr(PA)] \leq$$

$$\leq \lambda tr(PA)tr(Q) + (1-\lambda)tr(QB)tr(P) - tr(QB^{1-\lambda})tr(PA^{\lambda}) \leq$$

$$\leq (1-r)[tr(PA)tr(Q)-2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}})+tr(P)tr(QB)]+B_1(\lambda)[tr(QB)tr(PA\log^2 A)-$$

$$-2tr(QB\log B)tr(PA\log A) + tr(QB\log^2 B)tr(PA)].$$

∎

If we take instead of $B$, $A$ and instead of $Q$, $P$ then with the same conditions as in Theorem 2, we have the following result:

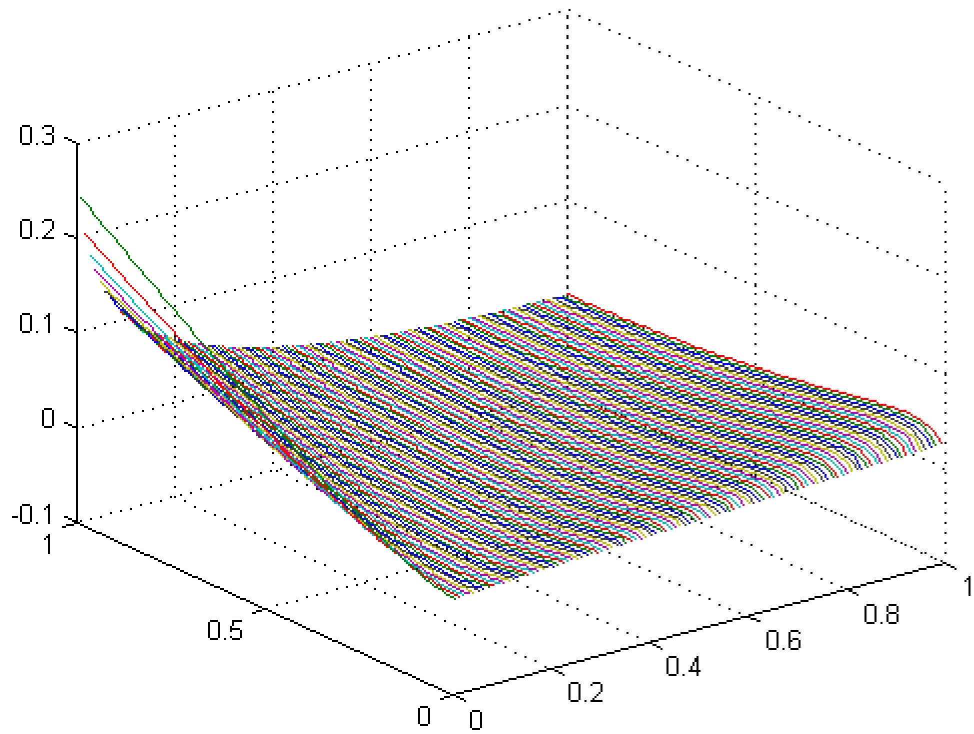


FIGURE 1. Fig. 1. The function f(a,b) on $[0,1]\times[0,1]$ when $\lambda=\frac{1}{6}$

**Corollary 1.** *Let $m,\ M$ be two real munbers with $0<m<M\leq 1$ and $A$ be a positive operator in $\mathcal{B}(\mathcal{H})$ with $Sp(A)\subset[m,M]$ and $P\in\mathcal{B}_1(\mathcal{H})$ with $P>0$. Then for any $\lambda\in[0,1]$ we obtain:*

$$2r\left[\frac{tr(PA)}{tr(P)}-\left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right]+2A_1(\lambda)\left[\frac{tr(PA)}{tr(P)}\frac{tr(PA\log^2 A)}{tr(P)}-\left(\frac{tr(PA\log A)}{tr(P)}\right)^2\right]\leq$$

$$\leq\frac{tr(PA)}{tr(P)}-\frac{tr(PA^{\lambda})}{tr(P)}\frac{tr(PA^{1-\lambda})}{tr(P)}\leq$$

$$\leq 2(1-r)\left[\frac{tr(PA)}{tr(P)}-\left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(PA)}{tr(P)}\frac{tr(PA\log^2 A)}{tr(P)}-\left(\frac{tr(PA\log A)}{tr(P)}\right)^2\right]$$

*where $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ as in Lemma 1.*

**Corollary 2.** *If $P,\ Q$ are two positive invertible operators with $P,Q\in\mathcal{B}_1(\mathcal{H})$ and $Sp(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})\subset[m,M]$, where $m,\ M$ be two real munbers with $0<m<M\leq 1$.*

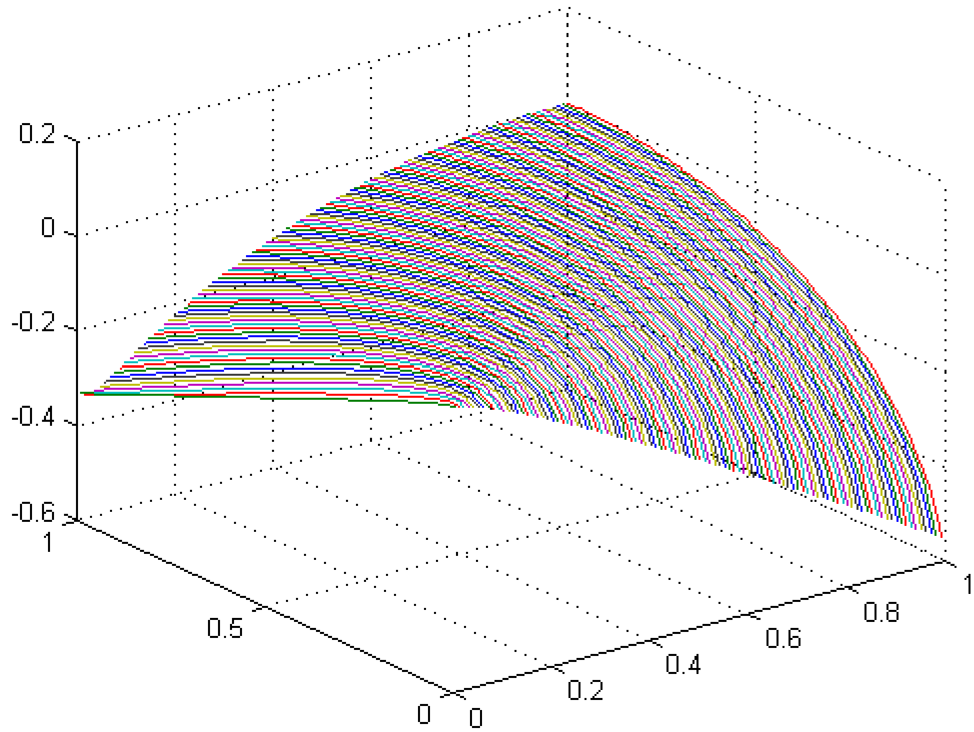


FIGURE 2. Fig. 2. The function g(a,b) on $[0,1]\times[0,1]$ when $\lambda=\frac{1}{6}$

*Then we have,*

$$2r\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\sharp Q)}{tr(P)}\right)^2\right]+2A_1(\lambda)\left[\frac{tr(Q)}{tr(P)}\frac{tr(QP^{-1}\mathcal{P}_{\log^2}(QP))}{tr(P)}-\left(\frac{tr(QP^{-1}\mathcal{S}(P/Q))}{tr(P)}\right)^2\right]\leq$$

$$\leq\frac{tr(Q)}{tr(P)}-\frac{tr(P\sharp_{1-\lambda}Q)}{tr(P)}\frac{tr(P\sharp_\lambda Q)}{tr(P)}\leq$$

$$\leq 2(1-r)\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\sharp Q)}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(Q)}{tr(P)}\frac{tr(QP^{-1}\mathcal{P}_{\log^2}(QP))}{tr(P)}-\left(\frac{tr(QP^{-1}\mathcal{S}(P/Q))}{tr(P)}\right)^2\right],$$

*where* $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ *and* $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ *as in Lemma 1.*

*Proof.* We take in Corollary 1, $A=P^{-\frac{1}{2}}QP^{-\frac{1}{2}}$ and use the definition of the weighted geometric mean, relative operator entropy and noncommutative perspective. ∎

**Theorem 3.** *Let $m$, $M$ be two real munbers with $1 < m < M$ and $A, B$ be two positive operators in $\mathcal{B}(\mathcal{H})$ with $Sp(A) \subset [m, M]$, $Sp(B) \subset [m, M]$ and $P, Q \in \mathcal{B}_1(\mathcal{H})$ with $P, Q > 0$. Then for any $\lambda \in [0,1]$ the following inequality takes place:*

$$r[tr(PA)tr(Q) - 2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}}) + tr(P)tr(QB)] + A_1(\lambda)[tr(Q)tr(P\log^2 A)-$$
$$-2tr(Q\log B)tr(P\log A) + tr(Q\log^2 B)tr(P)] \leq$$
$$\leq \lambda tr(PA)tr(Q) + (1-\lambda)tr(QB)tr(P) - tr(QB^{1-\lambda})tr(PA^{\lambda}) \leq$$
$$\leq (1-r)[tr(PA)tr(Q) - 2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}}) + tr(P)tr(QB)] + B_1(\lambda)[tr(Q)tr(P\log^2 A)-$$
$$-2tr(Q\log B)tr(P\log A) + tr(Q\log^2 B)tr(P)],$$

*where $r = \min\{\lambda, 1-\lambda\}$, $A_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B_1(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$ as in Lemma 1.*

*Proof.* We take into account the inequality from Theorem 1, which holds for any $a, b \geq m \geq 1$ and using the functional calculus for the operator $A$ when $1 \leq m \leq b \leq M$ is fixed, we get

$$r\left(< Ax, x > -2\sqrt{b} < A^{\frac{1}{2}}x, x > +b < x.x >\right) +$$
$$+A_1(\lambda)\left(< \log^2 Ax, x > -2\log b < \log Ax, x > + \log^2 b < x, x >\right) \leq$$
$$\leq \lambda < Ax, x > +(1-\lambda)b < x, x > -b^{1-\lambda} < A^{\lambda}x, x > \leq$$
$$\leq (1-r)\left(< Ax, x > -2\sqrt{b} < A^{\frac{1}{2}}x, x > +b < x.x >\right) +$$
$$+B_1(\lambda)\left(< \log^2 Ax, x > -2\log b < \log Ax, x > + \log^2 b < x, x >\right),$$

for any $x \in \mathcal{H}$, if we denote $A(\lambda)$ by $A_!(\lambda)$ and $B(\lambda)$ by $B_!(\lambda)$.

We fix $x \in \mathcal{H} - \{0\}$ and then by the functional calculus for the operator $B$ for previous inequality, we have,

$$r\left(< Ax, x > ||y||^2 - 2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y > + < By, y > ||x||^2\right) +$$
$$+A_1(\lambda)(||y||^2 < \log^2 Ax, x > -2 < \log By, y >< \log Ax, x > +$$
$$+ < \log^2 By, y > ||x||^2) \leq$$
$$\leq \lambda < Ax, x > ||y||^2 + (1-\lambda) < By, y > ||x||^2 - < B^{1-\lambda}y, y >< A^{\lambda}x, x > \leq$$
$$\leq (1-r)\left(< Ax, x > ||y||^2 - 2 < A^{\frac{1}{2}}x, x >< B^{\frac{1}{2}}y, y > + < By, y > ||x||^2\right) +$$
$$+B_1(\lambda)(||y||^2 < \log^2 Ax, x > -2 < \log By, y >< \log Ax, x > +$$
$$+ < \log^2 By, y > ||x||^2),$$

for any $x, y \in \mathcal{H}$.

We put now, $x = P^{\frac{1}{2}}e$, $y = Q^{\frac{1}{2}}f$ where $e$, $f \in \mathcal{H}$ and by the above inequality we obtain,

$$r(< P^{\frac{1}{2}}AP^{\frac{1}{2}}e, e >< Qf, f > -2 < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e, e >< Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f, f > +$$
$$+ < Pe, e >< Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f, f >) + A_1(\lambda)(< Qf, f >< P^{\frac{1}{2}}\log^2 AP^{\frac{1}{2}}e, e > -$$
$$-2 < Q^{\frac{1}{2}}\log BQ^{\frac{1}{2}}f, f >< P^{\frac{1}{2}}\log AP^{\frac{1}{2}}e, e > + < Q^{\frac{1}{2}}\log^2 BQ^{\frac{1}{2}}f, f >< Pe, e >) \leq$$
$$\leq \lambda < P^{\frac{1}{2}}AP^{\frac{1}{2}}e, e >< Qf, f > +(1-\lambda) < Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f, f >< Pe, e > -$$
$$- < Q^{\frac{1}{2}}B^{1-\lambda}Q^{\frac{1}{2}}f, f >< P^{\frac{1}{2}}A^{\lambda}P^{\frac{1}{2}}e, e > \leq$$
$$\leq (1-r)(< P^{\frac{1}{2}}AP^{\frac{1}{2}}e, e >< Qf, f > -2 < P^{\frac{1}{2}}A^{\frac{1}{2}}P^{\frac{1}{2}}e, e >< Q^{\frac{1}{2}}B^{\frac{1}{2}}Q^{\frac{1}{2}}f, f > +$$
$$+ < Pe, e >< Q^{\frac{1}{2}}BQ^{\frac{1}{2}}f, f >) + B_1(\lambda)(< Qf, f >< P^{\frac{1}{2}}\log^2 AP^{\frac{1}{2}}e, e > -$$

$-2 < Q^{\frac{1}{2}} \log B Q^{\frac{1}{2}} f, f >< P^{\frac{1}{2}} \log A P^{\frac{1}{2}} e, e > + < Q^{\frac{1}{2}} \log^2 B Q^{\frac{1}{2}} f, f >< Pe, e >),$

for any $e, f \in \mathcal{H}$.

Let $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ be two orthonormal bases of $\mathcal{H}$. We take in previous inequality $e = e_i,\ i \in I$ and $f = f_j,\ j \in J$ and then summing over $i \in I$and $j \in J$, we get the following:

$$r(\sum_{i\in I} < P^{\frac{1}{2}} A P^{\frac{1}{2}} e_i, e_i > \sum_{j\in J} < Qf_j, f_j > -2\sum_{i\in I} < P^{\frac{1}{2}} A^{\frac{1}{2}} P^{\frac{1}{2}} e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}} B^{\frac{1}{2}} Q^{\frac{1}{2}} f_j, f_j > +$$
$$+\sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}} B Q^{\frac{1}{2}} f_j, f_j >)+$$
$$+A_1(\lambda)(\sum_{j\in J} < Qf_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}} \log^2 A P^{\frac{1}{2}} e_i, e_i > -$$
$$-2\sum_{j\in J} < Q^{\frac{1}{2}} \log B Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}} \log A P^{\frac{1}{2}} e_i, e_i > +$$
$$+\sum_{j\in J} < Q^{\frac{1}{2}} \log^2 B Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < Pe_i, e_i >) \leq$$
$$\leq \lambda \sum_{i\in I} < P^{\frac{1}{2}} A P^{\frac{1}{2}} e_i, e_i > \sum_{j\in J} < Qf_j, f_j > +(1-\lambda) \sum_{j\in J} < Q^{\frac{1}{2}} B Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < Pe_i, e_i > -$$
$$-\sum_{j\in J} < Q^{\frac{1}{2}} B^{1-\lambda} Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}} A^{\lambda} P^{\frac{1}{2}} e_i, e_i >\leq$$
$$\leq (1-r)(\sum_{i\in I} < P^{\frac{1}{2}} A P^{\frac{1}{2}} e_i, e_i > \sum_{j\in J} < Qf_j, f_j > -$$
$$-2\sum_{i\in I} < P^{\frac{1}{2}} A^{\frac{1}{2}} P^{\frac{1}{2}} e_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}} B^{\frac{1}{2}} Q^{\frac{1}{2}} f_j, f_j > +$$
$$+\sum_{i\in I} < Pe_i, e_i > \sum_{j\in J} < Q^{\frac{1}{2}} B Q^{\frac{1}{2}} f_j, f_j >)+$$
$$+B_1(\lambda)(\sum_{j\in J} < Qf_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}} \log^2 A P^{\frac{1}{2}} e_i, e_i > -$$
$$-2\sum_{j\in J} < Q^{\frac{1}{2}} \log B Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < P^{\frac{1}{2}} \log A P^{\frac{1}{2}} e_i, e_i > +$$
$$+\sum_{j\in J} < Q^{\frac{1}{2}} \log^2 B Q^{\frac{1}{2}} f_j, f_j > \sum_{i\in I} < Pe_i, e_i >).$$

Using the properties of the trace we find the that

$$r[tr(PA)tr(Q) - 2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}}) + tr(P)tr(QB)] + A_1(\lambda)[tr(Q)tr(P\log^2 A)-$$
$$-2tr(Q\log B)tr(P\log A) + tr(Q\log^2 B)tr(P)] \leq$$
$$\leq \lambda tr(PA)tr(Q) + (1-\lambda)tr(QB)tr(P) - tr(QB^{1-\lambda})tr(PA^{\lambda}) \leq$$
$$\leq (1-r)[tr(PA)tr(Q)-2tr(PA^{\frac{1}{2}})tr(QB^{\frac{1}{2}})+tr(P)tr(QB)]+B_1(\lambda)[tr(Q)tr(P\log^2 A)-$$
$$-2tr(Q\log B)tr(P\log A) + tr(Q\log^2 B)tr(P)].$$

∎

Next we take instead of $B$, $A$ and instead of $Q$, $P$ then with the same conditions as in Theorem 3, and we have the following result:

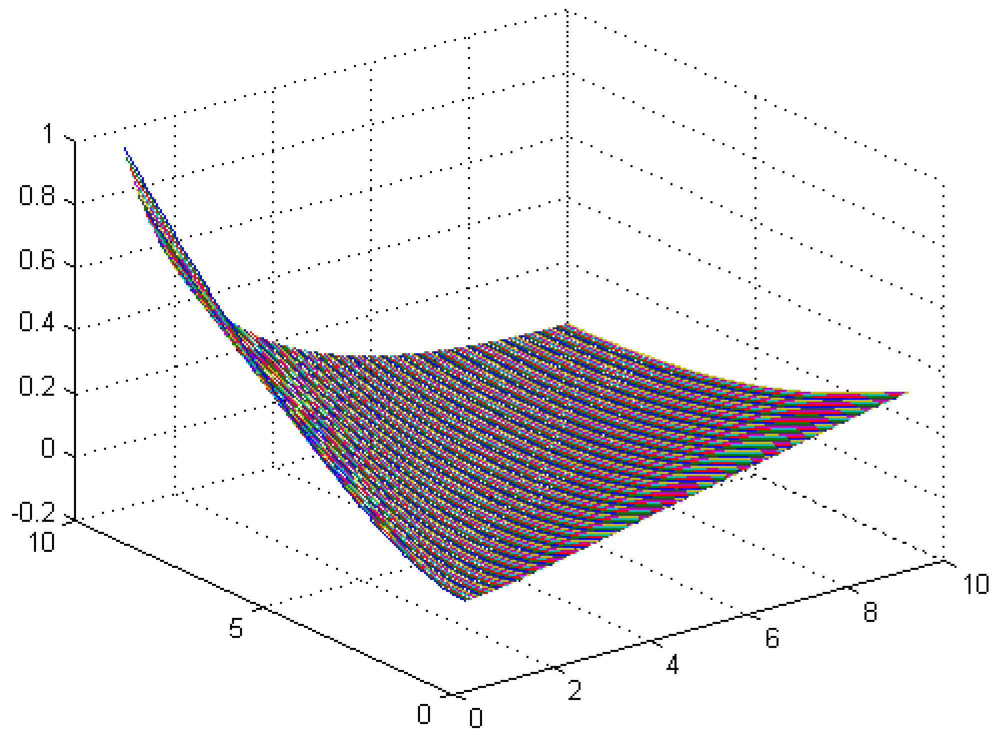


FIGURE 3. Fig. 3. The function h(a,b) on $[1,10]\times[1,10]$ when $\lambda=\frac{1}{6}$

**Corollary 3.** *Let $m$, $M$ be two real munbers with $1<m<M$ and $A$ be a positive operator in $\mathcal{B}(\mathcal{H})$ with $Sp(A)\subset[m,M]$ and $P\in\mathcal{B}_1(\mathcal{H})$ with $P>0$. Then for any $\lambda\in[0,1]$ we obtain:*

$$2r\left[\frac{tr(PA)}{tr(P)}-\left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right]+2A_1(\lambda)\left[\frac{tr(P\log^2 A)}{tr(P)}-\left(\frac{tr(P\log A)}{tr(P)}\right)^2\right]\leq$$

$$\leq\frac{tr(PA)}{tr(P)}-\frac{tr(PA^{\lambda})}{tr(P)}\frac{tr(PA^{1-\lambda})}{tr(P)}\leq$$

$$\leq 2(1-r)\left[\frac{tr(PA)}{tr(P)}-\left(\frac{tr(PA^{\frac{1}{2}})}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(P\log^2 A)}{tr(P)}-\left(\frac{tr(P\log A)}{tr(P)}\right)^2\right].$$

*where $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ as before.*

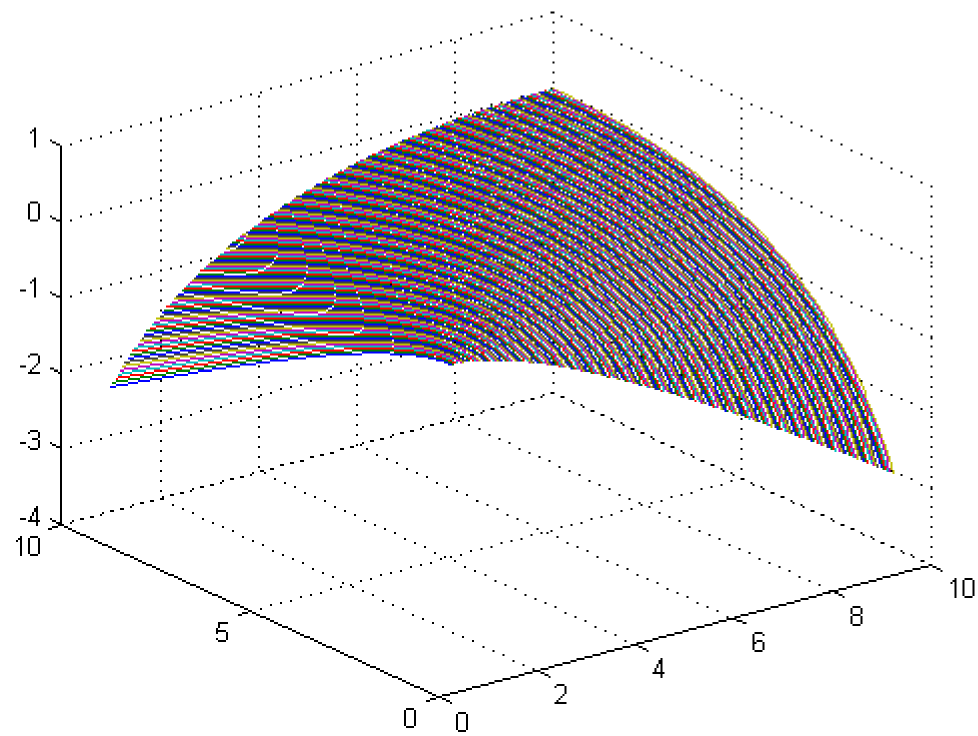


FIGURE 4. Fig. 4. The function k(a,b) on $[1,10]\times[1,10]$ when $\lambda=\frac{1}{6}$

**Corollary 4.** *If $P$, $Q$ are two positive invertible operators with $P,Q\in\mathcal{B}_1(\mathcal{H})$ and $Sp(P^{-\frac{1}{2}}QP^{-\frac{1}{2}})\subset[m,M]$, where $m$, $M$ be two real munbers with $1<m<M$. Then we have,*

$$2r\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\sharp Q)}{tr(P)}\right)^2\right]+2A_1(\lambda)\left[\frac{tr(\mathcal{P}_{\log^2}(QP))}{tr(P)}-\left(\frac{tr(\mathcal{S}(P/Q))}{tr(P)}\right)^2\right]\leq$$

$$\leq\frac{tr(Q)}{tr(P)}-\frac{tr(P\sharp_{1-\lambda}Q)}{tr(P)}\frac{tr(P\sharp_{\lambda}Q)}{tr(P)}\leq$$

$$\leq 2(1-r)\left[\frac{tr(Q)}{tr(P)}-\left(\frac{tr(P\sharp Q)}{tr(P)}\right)^2\right]+2B_1(\lambda)\left[\frac{tr(\mathcal{P}_{\log^2}(QP))}{tr(P)}-\left(\frac{tr(\mathcal{S}(P/Q))}{tr(P)}\right)^2\right],$$

*where $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$ as before.*

If we take $A=B$ and $P=O$ in Theorem 3, see [2], will obtain next inequality below.

**Corollary 5.** *Let $A$ be a positive operator and $P \in \mathcal{B}_1(H)$ with $> 0$. Then for any $\nu \in [0,1]$ we have:*

$$\left(\frac{\nu}{\tau}\right)^n \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k}\tau^k tr(PA^k)tr(PA^{n-k}) - tr(PA^{n\tau})tr(PA^{n(1-\tau)})\right] <$$

$$< \sum_{k=0}^{n} \binom{n}{k} (1-\nu)^{n-k}\nu^k tr(PA^k)tr(PA^{n-k}) - tr(PA^{n\nu})tr(PA^{n(1-\nu)}) <$$

$$< \left(\frac{1-\nu}{1-\tau}\right)^n \left[\sum_{k=0}^{n} \binom{n}{k} (1-\tau)^{n-k}\tau^k tr(PA^k)tr(PA^{n-k}) - tr(PA^{n\tau})tr(PA^{n(1-\tau)})\right].$$

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