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INTEGRAL GRUSS' TYPE INEQUALITIES FOR COMPLEX-VALUED FUNCTIONS

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Abstract. In this paper we provide several upper bounds for the modulus of the *complex Čebyšev functional*

$$C(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,dt$$

under various assumptions for the integrable functions $f,\ g:[a,b]\to\mathbb{C}$. Some particular cases via Wirtinger and Alzer inequalities for complex-valued functions are also given.

1. Introduction

For two Lebesgue integrable real-valued functions $f,\ g:[a,b]\to\mathbb{R}$, in order to compare the integral mean of the product with the product of the integral means, in 1934, G. Grüss [12] showed that

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty < m \le f \le M < \infty, \quad -\infty < n \le g \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

In order to extend this inequality for complex-valued functions we need the following preparations.

For $\phi,\ \Phi\in\mathbb{C}$ and $[a,b]$ an interval of real numbers, define the sets of complex-valued functions (see [7], [8] and [11])

$$\bar{U}_{[a,b]}(\phi,\Phi) := \left\{g:[a,b]\to\mathbb{C} \mid \operatorname{Re}\left[(\Phi-g(t))\left(\overline{g(t)}-\overline{\phi}\right)\right]\ge 0 \text{ for almost every } t\in[a,b]\right\}$$

and

$$\bar{\Delta}_{[a,b]}(\phi,\Phi) := \left\{g:[a,b]\to\mathbb{C} \mid \left|g(t)-\frac{\phi+\Phi}{2}\right|\le\frac{1}{2}|\Phi-\phi| \text{ for a.e. } t\in[a,b]\right\}.$$

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For any $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$, we have that $\bar{U}_{[a,b]}(\phi,\Phi)$ and $\bar{\Delta}_{[a,b]}(\phi,\Phi)$ are non-empty, convex and closed sets and

$$\bar{U}_{[a,b]}(\phi,\Phi)=\bar{\Delta}_{[a,b]}(\phi,\Phi). \tag{1.3}$$

We observe that for any $z\in\mathbb{C}$ we have the equivalence

$$\left|z-\frac{\phi+\Phi}{2}\right|\leq\frac{1}{2}\left|\Phi-\phi\right|$$

if and only if

$$\operatorname{Re}\left[(\Phi-z)\left(\bar{z}-\overline{\phi}\right)\right]\geq 0.$$

This follows by the equality

$$\frac{1}{4}\left|\Phi-\phi\right|^{2}-\left|z-\frac{\phi+\Phi}{2}\right|^{2}=\operatorname{Re}\left[(\Phi-z)\left(\bar{z}-\overline{\phi}\right)\right]$$

that holds for any $z\in\mathbb{C}$.

The equality (1.3) is thus a simple consequence of this fact.

For any $\phi,\ \Phi\in\mathbb{C},\ \phi\neq\Phi$,we also have that

$$\begin{aligned}\bar{U}_{[a,b]}(\phi,\Phi)=\{g:[a,b]\to\mathbb{C}\mid\ &(\operatorname{Re}\Phi-\operatorname{Re}g(t))(\operatorname{Re}g(t)-\operatorname{Re}\phi)\\&+(\operatorname{Im}\Phi-\operatorname{Im}g(t))(\operatorname{Im}g(t)-\operatorname{Im}\phi)\geq 0\text{ for a.e. }t\in[a,b]\}.\end{aligned} \tag{1.4}$$

Now, if we assume that $\operatorname{Re}(\Phi)\geq\operatorname{Re}(\phi)$ and $\operatorname{Im}(\Phi)\geq\operatorname{Im}(\phi)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\phi,\Phi):=\{g:[a,b]\to\mathbb{C}\mid\ &\operatorname{Re}(\Phi)\geq\operatorname{Re}g(t)\geq\operatorname{Re}(\phi)\\&\text{and }\operatorname{Im}(\Phi)\geq\operatorname{Im}g(t)\geq\operatorname{Im}(\phi)\ \text{ for a.e. }t\in[a,b]\}.\end{aligned} \tag{1.5}$$

One can easily observe that $\bar{S}_{[a,b]}(\phi,\Phi)$ is closed, convex and

$$\emptyset\neq\bar{S}_{[a,b]}(\phi,\Phi)\subseteq\bar{U}_{[a,b]}(\phi,\Phi). \tag{1.6}$$

This fact provides also numerous example of complex functions belonging to the class $\bar{\Delta}_{[a,b]}(\phi,\Phi)$.

For Lebesgue integrable functions $f,\ g:[a,b]\to\mathbb{C}$ we consider the *complex Čebyšev functional*

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt.$$

In [7] we obtained the following complex version of Gruss' inequality:

$$\left|C\left(f,\overline{g}\right)\right|\leq\frac{1}{4}\left|\Phi-\phi\right|\left|\Psi-\psi\right| \tag{1.7}$$

provided $f\in\bar{\Delta}_{[a,b]}(\phi,\Phi)$ and $g\in\bar{\Delta}_{[a,b]}(\psi,\Psi)$, where $\overline{g}$ denotes the complex conjugate function of g.

We denote *the variance* of the complex-valued function $f:[a,b]\to\mathbb{C}$ by $D(f)$ and defined as

$$D(f)=\left[C\left(f,\bar{f}\right)\right]^{1/2}=\left[\frac{1}{b-a}\int_a^b\left|f(t)\right|^{2}dt-\left|\frac{1}{b-a}\int_a^b f(t)\,dt\right|^{2}\right]^{1/2},$$

where $\bar{f}$ denotes the complex conjugate function of f.

If we apply the inequality (1.7) for $g = f$, then we get

$$D(f) \leq \frac{1}{2} |\Phi - \phi| . \tag{1.8}$$

We observe that, if $g \in \bar{\Delta}_{[a,b]}(\psi, \Psi)$, then $\left| g(t) - \frac{\psi+\Psi}{2} \right| \leq \frac{1}{2} |\Psi - \psi|$ for a.e. $t \in [a,b]$ that is equivalent to $\left| \overline{g(t)} - \frac{\overline{\psi}+\overline{\Psi}}{2} \right| \leq \frac{1}{2} \left| \overline{\Psi} - \overline{\psi} \right|$ meaning that $\overline{g} \in \bar{\Delta}_{[a,b]}\left(\overline{\psi}, \overline{\Psi}\right)$ and by 1.7, for $\overline{g}$ instead of g we also have

$$|C(f,g)| \leq \frac{1}{4} |\Phi - \phi| |\Psi - \psi| \tag{1.9}$$

provided $f \in \bar{\Delta}_{[a,b]}(\phi, \Phi)$ and $g \in \bar{\Delta}_{[a,b]}(\psi, \Psi)$.

We can also consider the following quantity associated with a complex-valued function $f : [a,b] \to \mathbb{C}$,

$$E(f) := |C(f,f)|^{1/2} = \left| \frac{1}{b-a} \int_a^b f^2(t)\, dt - \left(\frac{1}{b-a} \int_a^b f(t)\, dt \right)^2 \right|^{1/2} .$$

By using (1.9) we also have

$$E(f) \leq \frac{1}{2} |\Phi - \phi| . \tag{1.10}$$

For an integrable function $f : [a,b] \to \mathbb{C}$, consider the *mean deviation* of f defined by

$$R(f) := \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\, ds \right| dt.$$

The following result holds (see [10] or the more extensive preprint version [9]).

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (f)\, R(g) \leq \frac{1}{2} \bigvee_a^b (f)\, D(g), \tag{1.11}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.11).

Corollary 1. *If $f,\ g : [a,b] \to \mathbb{C}$ are of bounded variation on $[a,b]$, then*

$$|C(f,g)| \leq \frac{1}{2} \bigvee_a^b (f)\, R(g) \leq \frac{1}{2} \bigvee_a^b (f)\, D(g) \leq \frac{1}{4} \bigvee_a^b (f) \bigvee_a^b (g) . \tag{1.12}$$

The constant $\frac{1}{4}$ is best possible in (1.12).

We also have

$$D(f) \leq \frac{1}{2} \bigvee_a^b (f), \tag{1.13}$$

and the constant $\frac{1}{2}$ is best possible in (1.13).

Utilising the above results we can state, for a function of bounded variation $f:[a,b]\to\mathbb{C}$, that

$$E^2(f)\le\frac{1}{2}\bigvee_a^b(f)\,R(f)\le\frac{1}{2}\bigvee_a^b(f)\,D(f)\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2. \tag{1.14}$$

Moreover, define

$$\begin{aligned}G(f)&:=|C(f,|f|)|^{1/2}\\&=\left|\frac{1}{b-a}\int_a^b f(t)\,|f(t)|\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b|f(t)|\,dt\right|^{1/2},\end{aligned}$$

then we also have, for a function of bounded variation $f:[a,b]\to\mathbb{C}$, that

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(f)\,R(|f|)\le\frac{1}{2}\bigvee_a^b(f)\,D(|f|)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\\&\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}G^2(f)&\le\frac{1}{2}\bigvee_a^b(|f|)\,R(f)\le\frac{1}{2}\bigvee_a^b(|f|)\,D(f)\le\frac{1}{4}\bigvee_a^b(f)\bigvee_a^b(|f|)\\&\le\frac{1}{4}\left[\bigvee_a^b(f)\right]^2.\end{aligned} \tag{1.16}$$

Motivated by the above results, in this paper we provide several upper bounds for the modulus of the complex Čebyšev functional $C(f,g)$ under various assumptions for the integrable functions $f,\ g:[a,b]\to\mathbb{C}$. Some particular cases via Wirtinger and Alzer inequalities for complex-valued functions are also given.

2. Main Results

We have the following inequality for the complex Čebyšev functional that extends naturally the real case:

Lemma 1. *If $f,\ g:[a,b]\to\mathbb{C}$ are Lebesgue integrable on $[a,b]$, then*

$$|C(f,g)|\le D(f)\,D(g) \tag{2.1}$$

and

$$|C(f,\overline{g})|\le D(f)\,D(g). \tag{2.2}$$

In particular

$$E(f)\le D(f)\ \text{ and } G(f)\le\sqrt{D(f)\,D(|f|)}. \tag{2.3}$$

Proof. As in the real case, we have Korkine's identity

$$C(f,g):=\frac{1}{2(b-a)^2}\int_a^b\int_a^b(f(t)-f(s))(g(t)-g(s))\,dtds,$$

that can be proved directly by doing the calculations in the right hand side.

Using the Cauchy-Bunyakovsky-Schwarz integral inequality for complex functions, we have

$$\left|\frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))(g(t)-g(s))\,dtds\right|^2 \tag{2.4}$$
$$\leq \frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|^2\,dtds$$
$$\times \frac{1}{2(b-a)^2}\int_a^b\int_a^b |g(t)-g(s)|^2\,dtds$$

and since

$$\frac{1}{2(b-a)^2}\int_a^b\int_a^b |f(t)-f(s)|^2\,dtds$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))\overline{(f(t)-f(s))}dtds$$
$$= \frac{1}{2(b-a)^2}\int_a^b\int_a^b (f(t)-f(s))\left(\overline{f(t)}-\overline{f(s)}\right)dtds$$
$$= C\left(f,\overline{f}\right) = D^2(f),$$

and a similar equality for g, hence we get from (2.4) the desired inequality (2.1).

Since $D(\overline{g}) = D(g)$ the inequality (2.2) follows by (2.1). Also, by (2.1) we have

$$E^2(f) := |C(f,f)| \leq D(f)D(f) = D^2(f),$$

which produces the first inequality in (2.3). Similarly, by (2.1) we have

$$G^2(f) := |C(f,|f|)| \leq D(f)D(|f|),$$

which proves the second part of (2.3). □

We define the following *Lebesgue norms* for a measurable function $f:[a,b]\to\mathbb{C}$

$$\|f\|_\infty := \operatorname*{essup}_{t\in[a,b]}|f(t)| < \infty \text{ if } f\in L_\infty[a,b]$$

and, for $\beta \geq 1$,

$$\|f\|_\beta := \left(\int_a^b |f(t)|^\beta\,dt\right)^{1/\beta} < \infty \text{ if } f\in L_\beta[a,b].$$

For real-valued functions h, k that are absolutely continuous on $[a,b]$ and for which the derivatives h', $k' \in L_2[a,b]$ we have *Lupaş's inequality*

$$|C(h,k)| \leq \frac{1}{\pi^2}(b-a)\|h'\|_2\|k'\|_2, \tag{2.5}$$

in which the constant $\frac{1}{\pi^2}$ is best possible. For $k=h$ we have from (2.5) that

$$D^2(h) \leq \frac{1}{\pi^2}(b-a)\|h'\|_2^2. \tag{2.6}$$

The following version for complex-valued functions also holds:

Theorem 2. *Assume that the functions f, $g : [a, b] \to \mathbb{C}$ are absolutely continuous on $[a, b]$ with the derivatives f', $g' \in L_2 [a, b]$. Then*

$$|C(f, g)| \le \frac{1}{\pi^2} (b - a) \|f'\|_2 \|g'\|_2 . \tag{2.7}$$

The constant $\frac{1}{\pi^2}$ is best possible.

Proof. Let $f = \operatorname{Re} f + i \operatorname{Im} f$. If we write (2.6) for $\operatorname{Re} f$ and $\operatorname{Im} f$, then we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (\operatorname{Re} f(t))^2 dt - \frac{1}{(b-a)^2}\left(\int_a^b \operatorname{Re} f(t)\, dt\right)^2\\ &\le \frac{1}{\pi^2}(b-a)\int_a^b (\operatorname{Re} f'(t))^2 dt\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{b-a}\int_a^b (\operatorname{Im} f(t))^2 dt - \frac{1}{(b-a)^2}\left(\int_a^b \operatorname{Im} f(t)\, dt\right)^2\\ &\le \frac{1}{\pi^2}(b-a)\int_a^b (\operatorname{Im} f'(t))^2 dt.\end{aligned}$$

If we add these inequalities we get

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \left[(\operatorname{Re} f(t))^2 + (\operatorname{Im} f(t))^2\right] dt\\ &- \frac{1}{(b-a)^2}\left[\left(\operatorname{Re}\int_a^b f(t)\, dt\right)^2 + \left(\operatorname{Im}\int_a^b f(t)\, dt\right)^2\right]\\ &\le \frac{1}{\pi^2}(b-a)\left[\int_a^b \left[(\operatorname{Re} f'(t))^2 + (\operatorname{Im} f'(t))^2\right] dt\right],\end{aligned}$$

namely

$$\frac{1}{b-a}\int_a^b |f(t)|^2 dt - \frac{1}{(b-a)^2}\left|\int_a^b f(t)\, dt\right|^2 \le \frac{1}{\pi^2}(b-a)\int_a^b |f'(t)|^2 dt,$$

which can be written as

$$D^2(f) \le \frac{1}{\pi^2}(b-a)\|f'\|_2^2 . \tag{2.8}$$

If we use the inequality (2.1), then we get the desired result (2.7). □

Another lesser known inequality for $C(f, g)$ was derived in 1882 by Čebyšev [5] under the assumption that f', g' exist and are continuous on $[a, b]$, and is given by

$$|C(f, g)| \le \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^2 , \tag{2.9}$$

where $\|f'\|_\infty := \max_{t\in[a,b]} |f'(t)| < \infty$. The constant $\frac{1}{12}$ cannot be improved in general in (1.5).

We have the following version for complex functions:

Theorem 3. *Assume that the complex-valued functions $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$ with the derivatives $f',\ g' \in L_\infty [a,b]$. Then*

$$|C(f,g)| \le \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{2.10}$$

The constant $\frac{1}{12}$ is best possible.

Proof. Since f is absolutely continuous on $[a,b]$ with the derivative $f' \in L_\infty [a,b]$, we have

$$|f(t) - f(s)| = \left| \int_s^t f'(u)\,du \right| \le |t-s| \operatorname*{essup}_{u \in [t,s]([s,t])} |f'(u)| \le \|f'\|_\infty |t-s|$$

for any $t,\ s \in [a,b]$.

This implies that

$$\begin{aligned} D^2(f) &= \frac{1}{2(b-a)^2} \int_a^b \int_a^b |f(t) - f(s)|^2 \,dtds \\ &\le \frac{1}{2(b-a)^2} \|f'\|_\infty^2 \int_a^b \int_a^b (t-s)^2 \,dtds \\ &= \|f'\|_\infty^2 \left[\frac{1}{b-a} \int_a^b t^2 dt - \left(\frac{1}{b-a} \int_a^b t dt \right)^2 \right] \\ &= \|f'\|_\infty^2 \left(\frac{b^2 + ba + a^2}{3} - \frac{b^2 + 2ab + a^2}{4} \right) = \frac{1}{12}(b-a)^2 \|f'\|_\infty^2 \end{aligned}$$

and similarly,

$$D^2(g) \le \frac{1}{12}(b-a)^2 \|g'\|_\infty^2 .$$

By using (2.1) we get (2.10). □

In [3], P. Cerone and S. S. Dragomir proved the following inequalities for real-valued functions:

$$|C(f,g)| \le \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \\ \text{if } g \in L_\infty [a,b],\ f \in L_1 [a,b] \\ \\ \inf_{\gamma \in \mathbb{R}} \|g - \gamma\|_q \cdot \frac{1}{b-a} \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \text{if } g \in L_q [a,b],\ f \in L_p [a,b], \text{ where } p > 1,\ 1/p + 1/q = 1. \end{cases} \tag{2.11}$$

For $\gamma = 0$, we get from the first inequality in (1.12)

$$|C(f,g)| \le \|g\|_\infty \cdot \frac{1}{b-a} \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \tag{2.12}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le g \le M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.12) we can deduce the following result obtained by Cheng and

Sun [6] by a more complicated technique

$$|C(f,g)| \leq \frac{1}{2}(M-m)\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{2.13}$$

The constant $\frac{1}{2}$ is best in (2.13) as shown by Cerone and Dragomir in [4] where a general version for Lebesgue integral and measurable spaces was also given.

For a complex-valued function $f:[a,b]\to\mathbb{C}$ we define the *p-mean deviations* of f by

$$R_p(f) := \left(\frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}}$$

where $p \geq 1$ and $f \in L_p[a,b]$. For $p=\infty$ we define

$$R_\infty(f) := \operatorname*{essup}_{t\in[a,b]} \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|$$

if $f \in L_\infty[a,b]$.

For $p=2$ we obviously have $R_2(f) = D(f)$ and $R_p(\overline{f}) = R_p(f)$ for any $p\in[1,\infty]$. We denote $R(f)$ for $R_1(f)$.

By utilising a simpler technique than the one employed in [3] we can prove the following result for complex-valued functions.

Theorem 4. *Let $f,\ g:[a,b]\to\mathbb{C}$ be measurable on $[a,b]$. Then*

$$|C(f,g)| \leq \begin{cases} \inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_\infty R(f) \ \ \text{if } g\in L_\infty[a,b] \ \text{ and } f\in L[a,b], \\ \\ \frac{1}{(b-a)^{1/q}}\inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_q R_p(f),\ \ g\in L_q[a,b],\ \ f\in L_p[a,b], \\ \text{and } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{b-a}\inf_{\gamma\in\mathbb{C}} \|g-\gamma\|_1 R_\infty(f) \ \ \text{if } g\in L[a,b] \ \text{ and } f\in L_\infty[a,b]. \end{cases} \tag{2.14}$$

Proof. We use the following version of Sonin's identity for complex-valued functions

$$C(f,g) = \frac{1}{b-a}\int_a^b \left(f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right)(g(t)-\gamma)\,dt$$

provided $f,\ g:[a,b]\to\mathbb{C}$ are integrable on $[a,b]$ and $\gamma\in\mathbb{C}$. This can be easily proved by performing the calculation in the right hand side of the equality.

We have for $g\in L_\infty[a,b]$ and $f\in L[a,b]$ that

$$\begin{aligned} |C(f,g)| &\leq \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| |g(t)-\gamma|\,dt \\ &\leq \operatorname*{essup}_{t\in[a,b]} |g(t)-\gamma| \frac{1}{b-a}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \\ &= \|g-\gamma\|_\infty R(f), \end{aligned}$$

which proves the first part of (2.14).

By Hölder's integral inequality we have for $g \in L_q[a,b]$ and $f \in L_p[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}
|C(f,g)| &\leq \frac{1}{b-a}\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| |g(t)-\gamma|\,dt \\
&\leq \frac{1}{b-a}\left(\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b |g(t)-\gamma|^q\,dt\right)^{1/q} \\
&= \|g-\gamma\|_q \frac{1}{(b-a)^{1/q}}\left(\frac{1}{b-a}\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{1/p} \\
&= \frac{1}{(b-a)^{1/q}}\|g-\gamma\|_q R_p(f),
\end{aligned}$$

which proves the second part of (2.14).

We have for $g \in L[a,b]$ and $f \in L_\infty[a,b]$ that

$$\begin{aligned}
|C(f,g)| &\leq \frac{1}{b-a}\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| |g(t)-\gamma|\,dt \\
&\leq \frac{1}{b-a}\operatorname*{essup}_{t\in[a,b]}\left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| \int_a^b |g(t)-\gamma|\,dt \\
&= \frac{1}{b-a}\|g-\gamma\|_1 R_\infty(f),
\end{aligned}$$

which proves the last part of (2.14). □

An obvious particular case of interest is:

Corollary 2. *Let $f,\ g:[a,b]\to\mathbb{C}$ be measurable on $[a,b]$. Then*

$$|C(f,g)| \leq \begin{cases} \|g\|_\infty R(f) \ \text{ if } g\in L_\infty[a,b] \text{ and } f\in L[a,b], \\[2ex] \frac{1}{(b-a)^{1/q}}\|g\|_q R_p(f),\ g\in L_q[a,b],\ f\in L_p[a,b], \\ \text{and } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\[2ex] \frac{1}{b-a}\|g\|_1 R_\infty(f) \ \text{ if } g\in L[a,b] \text{ and } f\in L_\infty[a,b] \end{cases} \tag{2.15}$$

and

$$|C(f,g)| \leq \begin{cases} R_\infty(g) R(f) \ \text{ if } g\in L_\infty[a,b] \text{ and } f\in L[a,b], \\[2ex] R_q(g) R_p(f),\ g\in L_q[a,b],\ f\in L_p[a,b], \\ \text{and } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1, \\[2ex] R(g) R_\infty(f) \ \text{ if } g\in L[a,b] \text{ and } f\in L_\infty[a,b]. \end{cases} \tag{2.16}$$

3. Some General Examples

We have the following result:

Proposition 1. *Assume that $g : [a,b] \to \mathbb{C}$ is measurable on $[a,b]$ and $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$ for some distinct complex numbers ψ, Ψ. Then*

$$|C(f,g)| \le \begin{cases} \frac{1}{2}|\Psi-\psi| R(f) & \text{if } f \in L[a,b], \\ \\ \frac{1}{2}|\Psi-\psi| R_p(f) & \text{if } f \in L_p[a,b],\ p>1, \\ \\ \frac{1}{2}|\Psi-\psi| R_\infty(f) & \text{if } f \in L_\infty[a,b]. \end{cases} \tag{3.1}$$

Proof. If $g \in \bar{\Delta}_{[a,b]}(\psi,\Psi)$, then $\left|g(t) - \frac{\phi+\Phi}{2}\right| \le \frac{1}{2}|\Phi-\phi|$ for a.e. $t \in [a,b]$, which implies that

$$\left\| g - \frac{\phi+\Phi}{2} \right\|_\infty \le \frac{1}{2}|\Phi-\phi|,$$

$$\left\| g - \frac{\phi+\Phi}{2} \right\|_q = \left(\int_a^b \left| g(t) - \frac{\phi+\Phi}{2} \right|^q dt \right)^{1/q} \le \left(\int_a^b \left(\frac{1}{2}|\Phi-\phi| \right)^q dt \right)^{1/q}$$
$$= \frac{1}{2}|\Phi-\phi|(b-a)^{1/q}$$

and

$$\left\| g - \frac{\phi+\Phi}{2} \right\|_1 = \int_a^b \left| g(t) - \frac{\phi+\Phi}{2} \right| dt \le \frac{1}{2}|\Phi-\phi|(b-a).$$

By making use of (2.14) for $\gamma = \frac{\phi+\Phi}{2}$ we deduce (3.1). $\square$

Remark 1. *If $f \in L_\infty[a,b]$, then $f \in L_p[a,b]$ for $p \ge 1$ and by Hölder's inequality we have*

$$R(f) \le R_p(f) \le R_\infty(f),$$

which shows that the first inequality in (3.1) is better than the second that is better than the third.

If we assume that the following more general condition holds

$$\left\| g - \frac{\phi+\Phi}{2} \right\|_q \le \frac{1}{2}|\Phi-\phi|(b-a)^{1/q},\ q>1 \tag{3.2}$$

for some distinct complex numbers ψ, Ψ, then the second inequality in (3.1) also holds. Moreover, if the inequality (3.2) holds for $q=1$, then the third inequality in (3.1) is valid as well.

Proposition 2. *Assume that $g : [a,b] \to \mathbb{C}$ is of bounded variation on $[a,b]$. Then*

$$|C(f,g)| \le \begin{cases} \frac{1}{2}\bigvee_a^b(g) R(f) & \text{if } f \in L[a,b], \\ \\ \frac{1}{2}\bigvee_a^b(g) R_p(f) & \text{if } f \in L_p[a,b],\ p>1, \\ \\ \frac{1}{2}\bigvee_a^b(g) R_\infty(f) & \text{if } f \in L_\infty[a,b]. \end{cases} \tag{3.3}$$

Proof. For any $t \in [a,b]$ we have

$$\left| g(t) - \frac{g(a)+g(b)}{2} \right| = \left| \frac{g(t)-g(a)+g(t)-g(b)}{2} \right|$$
$$\leq \frac{1}{2}\left[|g(t)-g(a)| + |g(b)-g(t)| \right] \leq \frac{1}{2} \bigvee_a^b (g).$$

Using this inequality, we then have

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_\infty \leq \frac{1}{2} \bigvee_a^b (g),$$
$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_q \leq \frac{1}{2} \bigvee_a^b (g)(b-a)^{1/q}$$

and

$$\left\| g - \frac{g(a)+g(b)}{2} \right\|_1 \leq \frac{1}{2} \bigvee_a^b (g)(b-a).$$

By making use of (2.14) for $\gamma = \frac{g(a)+g(b)}{2}$ we deduce (3.3). □

We say that the function $h : [a,b] \to \mathbb{R}$ is *H-r-Hölder continuous* with the constant $H > 0$ and power $r \in (0,1]$ if

$$|h(t) - h(s)| \leq H |t-s|^r \tag{3.5}$$

for any $t, s \in [a,b]$. If $r = 1$ we call that h is *L-Lipschitzian* when $H = L > 0$.

Proposition 3. *Assume that $g : [a,b] \to \mathbb{C}$ is H-r-Hölder continuous with the constant $H > 0$ and power $r \in (0,1]$ on $[a,b]$. Then*

$$|C(f,g)| \leq \begin{cases} \frac{1}{2^r} H (b-a)^r R(f) \ \text{ if } f \in L[a,b], \\[2ex] \frac{1}{2^r (qr+1)^{1/q}} H (b-a)^r R_p(f) \ \text{ if } f \in L_p[a,b], \ p,q>1, \\ \text{and } \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{2^r(r+1)} H (b-a)^r R_\infty(f) \ \text{ if } f \in L_\infty[a,b]. \end{cases} \tag{3.6}$$

In particular, if $g : [a,b] \to \mathbb{C}$ is L-Lipschitzian on $[a,b]$, then

$$|C(f,g)| \leq \begin{cases} \frac{1}{2} L (b-a) R(f) \ \text{ if } f \in L[a,b], \\[2ex] \frac{1}{2(q+1)^{1/q}} L (b-a) R_p(f) \ \text{ if } f \in L_p[a,b], \ p,q>1, \\ \text{and } \frac{1}{p}+\frac{1}{q}=1; \\[2ex] \frac{1}{4} L (b-a) R_\infty(f) \ \text{ if } f \in L_\infty[a,b]. \end{cases} \tag{3.7}$$

Proof. For any $t \in [a,b]$ we have

$$\left| g(t) - g\left(\frac{a+b}{2}\right) \right| \leq H \left| t - \frac{a+b}{2} \right|^r.$$

This implies that

$$\left\| g(t) - g\left(\frac{a+b}{2}\right) \right\|_\infty \leq H \sup_{t\in[a,b]} \left| t - \frac{a+b}{2} \right|^r = \frac{1}{2^r} H (b-a)^r,$$

$$\begin{aligned}\left\|g(t)-g\left(\frac{a+b}{2}\right)\right\|_q &= \left(\int_a^b \left|g(t)-g\left(\frac{a+b}{2}\right)\right|^q dt\right)^{1/q}\\ &\le H\left(\int_a^b \left|t-\frac{a+b}{2}\right|^{qr} dt\right)^{1/q}\\ &= H\left(\frac{(b-a)^{qr+1}}{2^{qr}(qr+1)}\right)^{1/q} = \frac{1}{2^r(qr+1)^{1/q}}H(b-a)^{r+1/q}\end{aligned}$$

and

$$\left\|g(t)-g\left(\frac{a+b}{2}\right)\right\|_1 \le \frac{1}{2^r(r+1)}H(b-a)^{r+1}.$$

By making use of (2.14) for $\gamma = g\left(\frac{a+b}{2}\right)$ we deduce

$$|C(f,g)| \le \begin{cases} \frac{1}{2^r}H(b-a)^r R(f) \text{ if } g\in L_\infty[a,b] \text{ and } f\in L[a,b],\\ \\ \frac{1}{(b-a)^{1/q}}\frac{1}{2^r(qr+1)^{1/q}}H(b-a)^{r+1/q}R_p(f),\ g\in L_q[a,b],\ f\in L_p[a,b],\\ \text{and } p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \\ \frac{1}{b-a}\frac{1}{2^r(r+1)}H(b-a)^{r+1}R_\infty(f) \text{ if } g\in L[a,b] \text{ and } f\in L_\infty[a,b],\end{cases}$$

and the desired inequality (3.6) is proved. □

We say that the function $h:[a,b]\to\mathbb{C}$ is *K-s-Hölder continuous in the middle* with the constant $K>0$ and power $s>0$ if

$$\left|h(t)-h\left(\frac{a+b}{2}\right)\right| \le K\left|t-\frac{a+b}{2}\right|^s \tag{3.8}$$

for any $t\in[a,b]$. We observe that if $h:[a,b]\to\mathbb{C}$ is *H-r-Hölder continuous* with the constant $H>0$ and power $r\in(0,1]$, then is Hölder continuous in the middle with the same constants.

Remark 2. *Assume that $g:[a,b]\to\mathbb{C}$ is K-s-Hölder continuous in the middle with the constant $K>0$ and power $s>0$. Using a similar argument as above, we get*

$$|C(f,g)| \le \begin{cases} \frac{1}{2^s}K(b-a)^s R(f) \text{ if } f\in L[a,b],\\ \\ \frac{1}{2^s(qs+1)^{1/q}}K(b-a)^s R_p(f) \text{ if } f\in L_p[a,b],\ p,\ q>1,\\ \text{and } \frac{1}{p}+\frac{1}{q}=1;\\ \\ \frac{1}{2^s(s+1)}K(b-a)^s R_\infty(f) \text{ if } f\in L_\infty[a,b].\end{cases} \tag{3.9}$$

4. Examples Via Wirtinger's Inequality

In 1916 a remarkable result of W. Wirtinger that compares the integral of a square of a function with that of the square of its first derivative was published in W. Blaschke's book *"Kreis und Kugel"*, [2, p. 105]:

Let f be a real-valued function with period 2π and $\int_0^{2\pi} f(t)\,dt = 0$. If $f' \in L_2[0,2\pi]$, then

$$\int_0^{2\pi} [f(t)]^2\,dt \le \int_0^{2\pi} [f'(t)]^2\,dt \tag{4.1}$$

with equality holding if and only if

$$f(t) = A\cos t + B\sin t,\ A,\ B \in \mathbb{R}.$$

The following version for complex functions holds:

Lemma 2. *Let f be a complex-valued function with period 2π and $\int_0^{2\pi} f(t)\,dt = 0$. If $f' \in L_2[0,2\pi]$, then*

$$\int_0^{2\pi} |f(t)|^2\,dt \le \int_0^{2\pi} |f'(t)|^2\,dt. \tag{4.2}$$

The inequality is sharp.

Proof. Let $f = \operatorname{Re} f + i\operatorname{Im} f$. Since f is periodical with the period 2π and $\int_0^{2\pi} f(t)\,dt = 0$ it follows that $\operatorname{Re} f$ and $\operatorname{Im} f$ have the same properties and by (4.1) we get

$$\int_0^{2\pi} [\operatorname{Re} f(t)]^2\,dt \le \int_0^{2\pi} [\operatorname{Re} f'(t)]^2\,dt$$

and

$$\int_0^{2\pi} [\operatorname{Im} f(t)]^2\,dt \le \int_0^{2\pi} [\operatorname{Im} f'(t)]^2\,dt.$$

If we add these inequalities we get (4.2). □

For a complex-valued function $h : [0,2\pi] \to \mathbb{C}$, consider the dispersion

$$D_{[0,2\pi]}(h) := \left[\frac{1}{2\pi}\int_0^{2\pi} |h(t)|^2\,dt - \left|\frac{1}{2\pi}\int_0^{2\pi} h(t)\,dt\right|^2\right]^{1/2}.$$

We have:

Lemma 3. *Let h be a complex-valued function with period 2π. If $h' \in L_2[0,2\pi]$, then*

$$D^2_{[0,2\pi]}(h) \le \frac{1}{2\pi}\int_0^{2\pi} |h'(t)|^2\,dt. \tag{4.3}$$

The inequality is sharp.

Proof. Let $f := h - \frac{1}{2\pi}\int_0^{2\pi} h(s)\,ds$. Then f has the period 2π and $\int_0^{2\pi} f(t)\,dt = 0$. Then by (4.2) we get

$$\frac{1}{2\pi}\int_0^{2\pi} \left|h(t) - \frac{1}{2\pi}\int_0^{2\pi} h(s)\,ds\right|^2 dt \le \frac{1}{2\pi}\int_0^{2\pi} |h'(t)|^2\,dt. \tag{4.4}$$

Since

$$\frac{1}{2\pi}\int_0^{2\pi} \left|h(t) - \frac{1}{2\pi}\int_0^{2\pi} h(s)\,ds\right|^2 dt = D^2_{[0,2\pi]}(h),$$

then for (4.4). □

Remark 3. *By Lupaş's inequality (2.8) we have for $a = 0$, $b = 2\pi$ that*

$$D^2_{[0,2\pi]}(h) \le \frac{2}{\pi}\int_0^{2\pi} |h'(t)|^2 \, dt, \tag{4.5}$$

provided $h' \in L_2[0, 2\pi]$. In this inequality no periodicity condition for the function h is postulated. However, if the periodicity is assumed, then the inequality (4.3) holds and this provides a better upper bound for $D^2_{[0,2\pi]}(h)$ than (4.5).

Proposition 4. *Let f be a complex-valued function with period 2π and $f' \in L_2[0, 2\pi]$, then for $g \in L_2[0, 2\pi]$,*

$$\begin{aligned}\left|C_{[0,2\pi]}(f, g)\right| &\le \frac{1}{\sqrt{2\pi}} \inf_{\gamma\in\mathbb{C}} \|g - \gamma\|_{[0,2\pi],2} D_{[0,2\pi]}(f) \\ &\le \frac{1}{2\pi} \inf_{\gamma\in\mathbb{C}} \|g - \gamma\|_{[0,2\pi],2} \|f'\|_{[0,2\pi],2},\end{aligned} \tag{4.6}$$

where

$$C_{[0,2\pi]}(f, g) = \frac{1}{2\pi}\int_0^{2\pi} f(t) g(t) \, dt - \frac{1}{2\pi}\int_0^{2\pi} f(t) \, dt \frac{1}{2\pi}\int_0^{2\pi} g(t) \, dt.$$

In particular,

$$\left|C_{[0,2\pi]}(f, g)\right| \le \frac{1}{\sqrt{2\pi}} \|g\|_{[0,2\pi],2} D_{[0,2\pi]}(f) \le \frac{1}{2\pi} \|g\|_{[0,2\pi],2} \|f'\|_{[0,2\pi],2}. \tag{4.7}$$

Proof follows by (2.14) for $p = q = 2$ and $a = 0$, $b = 2\pi$.

Corollary 3. *Let f, g be a complex-valued functions with period 2π and f', $g' \in L_2[0, 2\pi]$, then*

$$\left|C_{[0,2\pi]}(f, g)\right| \le D_{[0,2\pi]}(g) D_{[0,2\pi]}(f) \le \frac{1}{2\pi} \|g'\|_{[0,2\pi],2} \|f'\|_{[0,2\pi],2}. \tag{4.8}$$

We also have:

Proposition 5. *Assume that $g : [0, 2\pi] \to \mathbb{C}$ is measurable on $[0, 2\pi]$ and $g \in \bar{\Delta}_{[0,2\pi]}(\psi, \Psi)$ for some distinct complex numbers ψ, Ψ. Let f be a complex-valued function with period 2π and $f' \in L_2[0, 2\pi]$, then*

$$\left|C_{[0,2\pi]}(f, g)\right| \le \frac{1}{2} |\Psi - \psi| D_{[0,2\pi]}(f) \le \frac{1}{2\sqrt{2\pi}} |\Psi - \psi| \|f'\|_{[0,2\pi],2}. \tag{4.9}$$

Proof follows by (3.1) for $p = 2$ and $a = 0$, $b = 2\pi$.

Proposition 6. *Assume that $g : [0, 2\pi] \to \mathbb{C}$ is of bounded variation on $[0, 2\pi]$. Let f be a complex-valued function with period 2π and $f' \in L_2[0, 2\pi]$, then*

$$\left|C_{[0,2\pi]}(f, g)\right| \le \frac{1}{2} \bigvee_0^{2\pi}(g) D_{[0,2\pi]}(f) \le \frac{1}{2\sqrt{2\pi}} \bigvee_0^{2\pi}(g) \|f'\|_{[0,2\pi],2}. \tag{4.10}$$

Proof follows by (3.3) for $p = 2$ and $a = 0$, $b = 2\pi$.

Proposition 7. *Assume that $g : [0, 2\pi] \to \mathbb{C}$ is H-r-Hölder continuous with the constant $H > 0$ and power $r \in (0, 1]$ on $[0, 2\pi]$. Let f be a complex-valued function with period 2π and $f' \in L_2[0, 2\pi]$, then*

$$\left|C_{[0,2\pi]}(f, g)\right| \le \frac{\pi^r}{\sqrt{2r+1}} H D_{[0,2\pi]}(f) \le \frac{\pi^{r-1/2}}{\sqrt{2(2r+1)}} H \|f'\|_{[0,2\pi],2}. \tag{4.11}$$

In particular, if $g:[0,2\pi]\to\mathbb{C}$ *is* L*-Lipschitzian on* $[0,2\pi]$, *then*

$$\left|C_{[0,2\pi]}(f,g)\right| \le \frac{\pi}{\sqrt{3}} L D_{[0,2\pi]}(f) \le \sqrt{\frac{\pi}{6}} L \left\|f'\right\|_{[0,2\pi],2}. \tag{4.12}$$

5. Examples Via Alzer's Inequality

In 1992, H. Alzer [1] obtained the following variant of Wirtinger's inequality:

$$\max_{t\in[0,2\pi]} [h(t)]^2 \le \frac{\pi}{6}\int_0^{2\pi} [h'(t)]^2\, dt, \tag{5.1}$$

provided that h is a real-valued continuously differentiable function with period 2π and $\int_0^{2\pi} h(t)\,dt = 0$. Equality holds in (5.1) if and only if

$$h(t) = C\left[3\left(\frac{t-\pi}{\pi}\right)^2 - 1\right],\ t\in[0,2\pi].$$

The following version for complex functions holds:

Lemma 4. *Let* f *be a continuously differentiable complex-valued function with period* 2π *and* $\int_0^{2\pi} f(t)\,dt = 0$. *Then*

$$\max_{t\in[0,2\pi]} |f(t)|^2 \le \frac{\pi}{6}\int_0^{2\pi} |f'(t)|^2\, dt. \tag{5.2}$$

The inequality is sharp.

Proof. Let $f = \operatorname{Re} f + i \operatorname{Im} f$. Since f is continuously differentiable function with period 2π and $\int_0^{2\pi} f(t)\,dt = 0$, it follows that $\operatorname{Re} f$ and $\operatorname{Im} f$ have the same properties and by (1.3) we get

$$\max_{t\in[0,2\pi]} [\operatorname{Re} f(t)]^2 \le \frac{\pi}{6}\int_0^{2\pi} [\operatorname{Re} f'(t)]^2\, dt,$$

and

$$\max_{t\in[0,2\pi]} [\operatorname{Im} f(t)]^2 \le \frac{\pi}{6}\int_0^{2\pi} [\operatorname{Im} f'(t)]^2\, dt.$$

If we add these inequalities we get

$$\max_{t\in[0,2\pi]} [\operatorname{Re} f(t)]^2 + \max_{t\in[0,2\pi]} [\operatorname{Im} f(t)]^2 \le \frac{\pi}{6}\int_0^{2\pi} |f'(t)|^2\, dt. \tag{5.3}$$

By the properties of maximum, we also have

$$\begin{aligned}\max_{t\in[0,2\pi]} |f(t)|^2 &= \max_{t\in[0,2\pi]} \left([\operatorname{Re} f(t)]^2 + [\operatorname{Im} f(t)]^2\right) \\ &\le \max_{t\in[0,2\pi]} [\operatorname{Re} f(t)]^2 + \max_{t\in[0,2\pi]} [\operatorname{Im} f(t)]^2.\end{aligned} \tag{5.4}$$

On utilising the inequalities (5.3) and (5.4) we get the desired result (5.2). □

Lemma 5. *Let* h *be a continuously differentiable complex-valued function with period* 2π. *Then*

$$\max_{t\in[0,2\pi]} \left| h(t) - \frac{1}{2\pi}\int_0^{2\pi} h(s)\,ds \right|^2 \le \frac{\pi}{6}\int_0^{2\pi} |h'(t)|^2\, dt. \tag{5.5}$$

The inequality is sharp.

Proof. Let $f := h - \frac{1}{2\pi}\int_0^{2\pi} h(s)\,ds$. Then f continuously differentiable, has the period 2π and $\int_0^{2\pi} f(t)\,dt = 0$. By using (5.2) we then get the desired result (5.5). □

Proposition 8. *Let $g : [0, 2\pi] \to \mathbb{C}$ be integrable on $[0, 2\pi]$ and f be a continuously differentiable complex-valued function with period 2π. Then*

$$\left|C_{[0,2\pi]}(f,g)\right| \le \frac{1}{2\pi}\inf_{\gamma\in\mathbb{C}}\|g-\gamma\|_{[0,2\pi],1}\max_{t\in[0,2\pi]}\left|f(t)-\frac{1}{2\pi}\int_0^{2\pi} f(s)\,ds\right| \le \frac{1}{2\sqrt{6}\pi}\inf_{\gamma\in\mathbb{C}}\|g-\gamma\|_{[0,2\pi],1}\|f'\|_{[0,2\pi],2}. \tag{5.6}$$

In particular,

$$\left|C_{[0,2\pi]}(f,g)\right| \le \frac{1}{2\pi}\|g\|_{[0,2\pi],1}\max_{t\in[0,2\pi]}\left|f(t)-\frac{1}{2\pi}\int_0^{2\pi} f(s)\,ds\right| \le \frac{1}{2\sqrt{6}\pi}\|g\|_{[0,2\pi],1}\|f'\|_{[0,2\pi],2} \tag{5.7}$$

and

$$\begin{aligned}\left|C_{[0,2\pi]}(f,g)\right| &\le \frac{1}{2\pi}\left\|g-\frac{1}{2\pi}\int_0^{2\pi} g(s)\,ds\right\|_{[0,2\pi],1}\max_{t\in[0,2\pi]}\left|f(t)-\frac{1}{2\pi}\int_0^{2\pi} f(s)\,ds\right| \\ &\le \frac{1}{2\sqrt{6}\pi}\left\|g-\frac{1}{2\pi}\int_0^{2\pi} g(s)\,ds\right\|_{[0,2\pi],1}\|f'\|_{[0,2\pi],2}.\end{aligned} \tag{5.8}$$

The proof follows by the third inequality in (2.14) and we omit the details.

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