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SOME WEIGHTED HERMITE-HADAMARD INEQUALITY FOR r-PREINVEX FUNCTIONS ON AN INVEX SET

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Abstract. In this paper, the weighted Hermite-Hadamard inequality for weakly r-preinvex function on an invex set are established. As applications, some inequalities involving two-parameter mean are given.

1. Introduction

It is well-known that the concepts of means are important notions in mathematics, for instance, some definitions of norms are often special means and have explicit geometric meanings [14], and have been applied in heat conduction, chemistry [16], electrostatics [11] and medicine [4].

The classical Hermite-Hadamard inequality for convex functions states that if $f : [a, b] \to R$ is convex, then

$$\frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

In [15], Sun and Yang extend Hermite-Hadamard inequality to the weighted mean of order s of a positive r-convex function on an interval $[a, b]$.

Antczak [1, 2] introduced the concept of r-invex and r-preinvex function and give a new method to solve nonlinear mathematical programing problems. In [10], Noor gave some Hermite-Hadamard inequality for the preinvex and log-preinvex functions. Moreover, in [17], Wasim Ui-Haq and Javed Iqbal introduced Hermite-Hadamard inequality for r-preinvex functions. Recently, Hwang and Dragomir [5] establish the Hermite-Hadamard inequality to a relation of two extended means for weakly r-preinvex functions on an invex set.

The main purpose of this paper is to generalise Hermite-Hadamard inequality that involves weighted mean of two-parameters for weakly r-preinvex functions on an invex set. The obtained results not only establish weighted inequality of the inequality given in [10, 17], but also extend the results given in [12, 15].

2. Preliminary definitions

The power mean $M_r(x, y; \lambda)$ of order r of positive numbers x, y which is defined by

$$M_r(x,y;\lambda) = \begin{cases} (\lambda x^r + (1-\lambda)y^r)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ x^{\lambda}y^{1-\lambda}, & \text{if } r = 0, \end{cases}$$

see [6].

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In [12, 13], Qi gave the following weighted mean values of a positive function f defined on the interval between x and y with two parameters $p, q \in R$ and nonnegative weight w, is not equivalent 0, by

$$M_{w,f}(p, q; x, y) = \begin{cases} \left(\int_x^y w(t) f^p(t) dt \Big/ \int_x^y w(t) f^q(t) dt \right)^{\frac{1}{(p-q)}}, & \text{if } (p-q)(x-y) \neq 0, \\ \exp\left(\int_x^y w(t) f^q(t) \ln f(t) dt \Big/ \int_x^y w(t) f^q(t) dt \right), & \text{if } p = q. \end{cases}$$

and $M_{w,f}(p, q; x, x) = f(x)$. Let $x, y, s \in R$, and w and f be positive and integrable functions on the closed interval [x,y]. The weighted mean of order s of the function f on $[x, y]$ with the weight w is defined in [7] as

$$M^{[s]}(f, w; x, y) = \begin{cases} \left(\int_x^y w(t) f^s(t) dt \Big/ \int_x^y w(t) dt \right)^{\frac{1}{s}}, & \text{if } s \neq 0, \\ \exp\left(\int_x^y w(t) \ln f(t) dt \Big/ \int_x^y w(t) dt \right), & \text{if } s = 0. \end{cases}$$

In addition, $M^{[s]}(f, w; x, x) = f(x)$. By taking $s = p - q, p, q \in R$, and replacing $w(t)$ by $w(t) f^q(t)$ in $M^{[s]}(f, w; x, y)$, we have that $M^{[p-q]}(f, wf^q; x, y) = M_{w,f}(p, q; x, y)$. It is obvious that the weighted mean $M^{[s]}(f, w; x, y)$ is equivalent to the generalized weighted mean values $M_{w,f}(p, q; x, y)$.

In [15], Sun and Yang gave the following theorem for the weighted mean of r-convex functions.

Theorem 1. *Let $f(t)$ be a positive and continuous function on interval $[x, y]$ with continuous derivative $f'(t)$ on $[x, y]$, let $w(t)$ be a positive and continuous function on the range J of the function $f(t)$, and let $h(t) = t$. Then if f is r-convex,*

$$M^{[s]}(f, w \circ f; x, y) \leq M^{[s]}(h, wh^{r-1}; f(x), f(y)) \tag{2.1}$$

For any real number s, while if f is r-concave, the inequality is reversed.

We begin with some definitions relating to r-preinvex function. The definitions of invex sets and preinvex functions, see in [8, 9]. In [3], Antczak introduced the following definition of an η-path on invex set.

Definition 1. *Let $K \subset R^n$ be a nonempty invex set with respect to η, $u, v \in K$. For $x \in K$, a set $P_{ux} := \{u + \lambda\eta(v, u) : \lambda \in [0, 1]\}$ is said to be a closed η-path joining the points u and $x = u + \eta(v, u)$ and $P^0_{ux} := \{u + \lambda\eta(v, u) : \lambda \in (0, 1)\}$ is said to be a open η-path joining the points u and $x = u + \eta(v, u)$.*

We note that if $\eta(v, u) = v - u$ then the set $P_{ux} = P_{uv} = \{\lambda v + (1-\lambda)u : \lambda \in [0, 1]\}$ is the line segment with the end points u and v.

In [1], Antczak introduced the class of r-preinvex function with respect to η on the optimization theory.

Definition 2. *Let $K \subset R^n$ be a nonempty invex set with respect to η. A function $f : K \to R^+$ is said to be r-preinvex with respect to η, if there is a vector-value* *function* $\eta : K \times K \to R^n$ *such that*

$$f(u+\lambda\eta(v,u)) \leq \begin{cases} (\lambda f(v)^r + (1-\lambda)f(u)^r)^{\frac{1}{r}}, & \text{if } r \neq 0, \\ f(v)^\lambda f(u)^{1-\lambda}, & \text{if } r = 0. \end{cases}$$

for every $v, u \in K$ *and* $\lambda \in [0,1]$.

Note that 0-preinvex functions are logarithmic preinvex and 1-preinvex functions are preinvex functions. It is obvious that if f is r-preinvex , then f^r is preinvex function for positive r.

The following Condition C and Condition D were given in [8] and [18], respectively.

Condition 1. *(Condition C) Let* $K \subset R^n$ *be a nonempty invex set with respect to* $\eta : K \times K \to R^n$. *We say that the function* η *satisfies the Condition C if for any* $u, v \in K$ *and* $\lambda \in [0,1]$, *the following two identities hold.*

$$< i > \eta(u, u+\lambda\eta(v,u)) = -\lambda\eta(v,u);$$

$$< ii > \eta(v, u+\lambda\eta(v,u)) = (1-\lambda)\eta(v,u).$$

Condition 2. *(Condition D)Let* $K \subset R^n$ *be a nonempty invex set with respect to* $\eta : K \times K \to R^n$, *and let* $f : K \to R$ *be invex with respect to the same* η. *we say that the function* f *satisfies the Condition D if for any* $u, v \in K$, *the following inequality*

$$f(u+\eta(v,u)) \leq f(v)$$

holds.

In [5], Hwang and Dragomir give the following definitions related to power means.

Definition 3. *Let* $K \subset R^n$ *be a nonempty invex set with respect to* η. *A function* $f : K \to R^+$ *is said to be weakly* r*-preinvex with respect to* η, *if there is a vector-value function* $\eta : K \times K \to R^n$ *such that*

$$f(u+\lambda\eta(v,u)) \leq M_r(f(u+\eta(v,u)), f(u); \lambda)$$

for every $v, u \in K$ *and* $\lambda \in [0,1]$.

We note that if f is weakly r-preinvex function, then f^r is weakly preinvex function for positive r, if f is weakly 0-preinvex function, then $log \circ f$ is weakly preinvex function, and if f is weakly 1-preinvex function, then f is weakly preinvex function. We also note that, in Definition 3, if f further satisfies the Condition D, then f is r-preinvex function.

In order to obtain our results, we introduce the following new definitions related to weighted mean of two-parameters for weakly r-preinvex function on an invex set.

Definition 4. *Let* $K \subset R^n$ *be a nonempty invex set with respect to a vector-value function* $\eta : K \times K \to R^n$ *and let* $f, w : K \to R^+$ *be an integrable on* η*-path* P_{ux} *for* $x = u + \eta(v,u)$ *where* $v, u \in K$, $\lambda \in [0,1]$. *Set* $x(\lambda) = u + \lambda\eta(v,u)$. *We define the weighted mean of the function* $f(u+\lambda\eta(v,u))$ *on* $[0,1]$ *with respect to* λ *by*

$$M_{p,q}(f,w;u,u+\eta(v,u))$$
$$= \begin{cases} \left(\int_0^1 w(x(\lambda))f^p(x(\lambda))d\lambda \Big/ \int_0^1 w(x(\lambda))f^q(x(\lambda))d\lambda\right)^{\frac{1}{(p-q)}}, & \text{if } p \neq q, \\ \exp\left(\int_0^1 w(x(\lambda))f^q(x(\lambda))\ln f(x(\lambda))d\lambda \Big/ \int_0^1 w(x(\lambda))f^q(x(\lambda))d\lambda\right), & \text{if } p = q. \end{cases}$$

In particular, when $q = 0$, $M_{p,0}(f,w;u,u+\eta(v,u)) = M^{[p]}(f,w;u,u+\eta(v,u))$ is the weighted mean of order p of the function f on $[u, u+\eta(v,u)]$ with the weight w.

In [5], applying Condition C, Hwang and Dragomir have given the following properties for weakly r-preinvex function.

Proposition 1. *Let $K \subset R^n$ be a nonempty invex set with respect to $\eta : K \times K \to R^n$ and suppose that η satisfies Condition C. Let $u \in K$ and let $f : P_{ux} \to R$ for every $v \in K$, $\lambda \in [0,1]$ and $x = u + \eta(v,u) \in K$. Suppose that f is continuous on P_{ux} and is twice-differentiable on P^0_{ux} and $r \geq 0$. Then f is a weakly r-preinvex function with respect to η if and only if*

$$rf^{r-2}(u)\{(r-1)[\eta(v,u)^T\nabla f(u)]^2 + f(u)\eta(v,u)^T\nabla^2 f(u)\eta(v,u)\} \geq 0$$

for $r > 0$,

$$\{\eta(v,u)^T\nabla^2 f(u)\eta(v,u)f(u) - [\eta(v,u)^T\nabla f(u)]^2\}/f^2(u) \geq 0$$

for $r = 0$.

3. Weighted Hermite-Hadamard inequality

For simplicity, in this section, we assume that $K \subset R^n$ be a nonempty invex set with respect to a vector value function $\eta : K \times K \to R^n$. Applying the definitions, conditions and proposition in section 2, we have the following theorems.

Theorem 2. *Let f be a weakly r-preinvex function on invex K with $r \geq 0$. Assume that f be a positive and continuous function on P_{ax} and twice-differentiable on P^0_{ax} for every $a, b \in K$, $\lambda \in [0,1]$ and $a < x = a + \eta(b,a)$, and let η satisfy Condition C. Let m and M be the minimum and maximum of f on P_{ax}, respectively. Further, let w, h be positive and continuous on $[m, M]$ with $h(x) = x$, and let $g_1, g_2 : (0,\infty) \to R$ and suppose that g_2 is a positive integrable on $[m, M]$ and the ratio g_1/g_2 is integrable on $[m, M]$. If g_1/g_2 is increasing on $[m, M]$, then*

$$\frac{\int_0^1 w(f(a+\lambda\eta(b,a)))g_1(f(a+\lambda\eta(b,a)))d\lambda}{\int_0^1 w(f(a+\lambda\eta(b,a)))g_2(f(a+\lambda\eta(b,a)))d\lambda} \leq \frac{\int_{f(a)}^{f(a+\eta(b,a))} w(x)h^{r-1}(x)g_1(h(x))dx}{\int_{f(a)}^{f(a+\eta(b,a))} w(x)h^{r-1}(x)g_2(h(x))dx} \tag{3.1}$$

for $f(a) \neq f(a+\eta(b,a))$; the right-hand side of (3.1) is defined by $g_1(f(a))/g_2(f(a))$ for $f(a) = f(a+\eta(b,a))$. If g_1/g_2 is decreasing, then the inequality (3.1) is reversed.

Proof. We give only the proof in the case of $r > 0$ and g_1/g_2 is increasing. The proof in the other case is analogous. Let $\phi(\lambda) = f^r(a + \lambda\eta(b,a))$ for $r \neq 0$ and

$\phi(\lambda) = \ln f(a + \lambda\eta(b, a))$ for $r = 0$. For convenience, let $\psi(\lambda) = f(a + \lambda\eta(b, a))$. Since f is weakly r-preinvex function with respect to η, Proposition 1 gives

$$\phi''(\lambda) \quad = \quad r f^{(r-2)}(a)\{(r-1)[\eta(b,a)^T \nabla f(a)]^2 + f(a)\eta(b,a)^T \nabla^2 f(a)\eta(b,a)\}$$

is positive.

When $f(a) \neq f(a + \eta(b, a))$. The inequality (3.1) is equivalent to

$$\frac{\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda}{\int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda} \leq \frac{\int_0^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda}{\int_0^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda}. \tag{3.2}$$

Consider

$$\begin{aligned} I = & \int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \int_0^1 w(\psi(\mu))\psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)d\mu \\ & - \int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda \int_0^1 w(\psi(\mu))\psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)d\mu \end{aligned}$$

(3.3)

$$= \int_0^1 \int_0^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu.$$

Interchanging λ and μ in (3.3) and adding the resulting equation gives

(3.4)

$$I = \frac{1}{2r}\int_0^1 \int_0^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\big[(\psi^r(\mu))' - (\psi^r(\lambda))'\big]\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu.$$

If the derivative $\phi'(\lambda) = (\psi^r(\lambda))' \geq 0$ for all $\lambda \in (0, 1)$, from $\phi''(\lambda) = (\psi^r(\lambda))'' \geq 0$, we have

$$\big[(\psi^r(\mu))' - (\psi^r(\lambda))'\big)\big]\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big] \leq 0.$$

From (3.4), we get $I \leq 0$. This implies that the inequality (3.2) holds and then (3.1) holds. If $\phi'(\lambda) = (\psi^r(\lambda))' \leq 0$ for all $\lambda \in (0, 1)$, a similar argument gives $I \geq 0$ again the inequality (3.1) holds.

Now suppose that $\phi'(\lambda) = (\psi^r(\lambda))'$ changes sign and $\phi(0) < \phi(1)$. Then $\psi^r(0) \leq \psi^r(1)$ and there exist a point $\alpha \in (0, 1)$ such that $\phi'(\alpha) = (\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \leq 0$ for all $\lambda \in [0, \alpha]$ and $(\psi^r(\lambda))' \geq 0$ for all $\lambda \in [\alpha, 1]$. Therefore, there exist $\beta \in (\alpha, 1)$ such that $\psi(0) = \psi(\beta)$. Thus

$$\begin{aligned} & \int_0^\beta w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda \\ = & \int_{\psi(0)}^{\psi(\alpha)} w(\psi(\lambda))x^{r-1}g_1(x)dx + \int_{\psi(\alpha)}^{\psi(\beta)} w(\psi(\lambda))x^{r-1}g_1(x)dx = 0, \end{aligned}$$

and, similarly,

$$\int_0^\beta w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda = 0.$$

Consequently, the inequality (3.1) is equivalent to

$$\frac{\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda}{\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda} \leq \frac{\int_\beta^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_1(\psi(\lambda))\psi'(\lambda)d\lambda}{\int_\beta^1 w(\psi(\lambda))\psi^{r-1}(\lambda)g_2(\psi(\lambda))\psi'(\lambda)d\lambda}. \tag{3.5}$$

Consider

$$I_2 = \int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \int_\beta^1 w(\psi(\mu))\psi^{r-1}(\mu)g_2(\psi(\mu))\psi'(\mu)d\mu$$
$$- \int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda \int_\beta^1 w(\psi(\mu))\psi^{r-1}(\mu)g_1(\psi(\mu))\psi'(\mu)d\mu$$

(3.6)
$$= \frac{1}{r}\int_0^1 \int_\beta^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu.$$

Split the double integral I_2 into two parts

$$I_{21} = \frac{1}{r}\int_0^\beta \int_\beta^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu,$$

and

$$I_{22} = \frac{1}{r}\int_\beta^1 \int_\beta^1 w(\psi(\lambda))w(\psi(\mu))g_2(\psi(\lambda))g_2(\psi(\mu))\psi^{r-1}(\mu)\psi'(\mu)\Big[\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} - \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}\Big]d\lambda d\mu.$$

When $(\lambda,\mu) \in [0,\beta]\times[\beta,1]$, we have $\lambda \le \mu$ and $(\psi^r(\mu))' = r\psi^{r-1}(\mu)\psi'(\mu) \ge 0$ for all $\mu \in (\beta,1)$. Thus $\psi'(\mu) \ge 0$ for all $\mu \in (\beta,1)$ and

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \le \frac{g_1(\psi(\beta))}{g_2(\psi(\beta))} \le \frac{g_1(\psi(\mu))}{g_2(\psi(\mu))}.$$

This gives $I_{21} \le 0$. By the result proved in case when $\phi'(\lambda) = (\psi^r(\lambda))' \ge 0$, we see that $I_{22} \le 0$. Therefore, $I_2 = I_{21} + I_{22} \le 0$. It follows that (3.5) and also (3.1) holds. Finally, if the sign of the derivative $\phi'(\lambda) = (\psi^r(\lambda))'$ changes and $\psi(0) \ge \psi(1)$ a similar proof again shows that (3.1) holds.

When $f(a) = f(a+\eta(b,a))$, $\psi(0) = \psi(1)$, and so $\phi(0) = \phi(1)$. Since $\phi'' = (\psi^r(\lambda))'' \ge 0$, we see that $\phi' = (\psi^r(\lambda))'$ is continuous and increasing for $\lambda \in (0,1)$. There exist a point $\alpha \in (0,1)$ such that $(\psi^r(\alpha))' = 0$ and $(\psi^r(\lambda))' \le 0$ for all $\lambda \in (0,\alpha)$, and $(\psi^r(\lambda))' \ge 0$ for all $\lambda \in (\alpha,1)$. Hence

$$\frac{g_1(\psi(\lambda))}{g_2(\psi(\lambda))} \le \frac{g_1(\psi(1))}{g_2(\psi(1))},$$

for all $\lambda \in (0,1)$. It follows that

$$\int_0^1 w(\psi(\lambda))g_1(\psi(\lambda))d\lambda \le \frac{g_1(\psi(1))}{g_2(\psi(1))}\int_0^1 w(\psi(\lambda))g_2(\psi(\lambda))d\lambda.$$

Therefore, the inequality (3.1) is valid. This completes the proof of Theorem 2. ▮

Remark 1. *If we take $w \equiv 1$, we get the inequality* (3.1) *reduces to the inequality* (3.1) *in* [5].

Remark 2. *If we take $g_1(x) = x^p, g_2(x) = x^q$ for suitable real number p,q in* (3.1)*, we get the following weighted mean inequality for the twice-differentiable and weakly r-preinvex function f on an invex set with respect to η satisfying condition C,*

$$M_{p,q}(f, w\circ f; a, a+\eta(b,a)) \le M_{p,q}(h, wh^{r-1}; f(a), f(a+\eta(b,a))). \tag{3.7}$$

Moreover, if we take $q = 0$ in (3.7)*, we have the following weighted mean of order p inequality*

$$M^{[p]}(f, w\circ f; a, a+\eta(b,a)) \le M^{[p]}(h, wh^{r-1}; f(a), f(a+\eta(b,a))). \tag{3.8}$$

Taking $p = 1$ *in* (3.8), *gives*

$$\frac{\int_a^{a+\eta(b,a)} w(f(x))f(x)dx}{\int_a^{a+\eta(b,a)} w(f(x))dx} \leq \frac{\int_{f(a)}^{f(a+\eta(b,a))} w(x)x^r dx}{\int_{f(a)}^{f(a+\eta(b,a))} w(x)x^{r-1} dx}. \tag{3.9}$$

The inequality (3.9) *is the weighted type of the inequality given by Ui-Haq and Iqbal in* [17]. *For* $r = 1$ *or* $r = 0$ *in* (3.9), *the inequality* (3.9) *is a weighted type of the inequality given by Noor in* [10].

Remark 3. *If we take* $\eta(b,a) = b - a$ *in* (3.7), *we have*

$$M_{p,q}(f, w \circ f; a, b)) \leq M_{p,q}(h, wh^{r-1}; f(a), f(b)). \tag{3.10}$$

We note that the (3.10) *is equivalent to the following inequality*

$$M_{w \circ f, f}(p, q; a, b)) \leq M_{wh^{r-1}, h}(p, q; f(a), f(b)).$$

Take $q = 0$ *in* (3.10), *the inequality* (3.10) *reduce to* (2.1) *in Theorem 1.*

References

[1] T. Antczak, r-preinvexity and r-invexity in mathematical programming. *Computers and Mathematics with Applications*, **50**(3-4) (2005), 551-566. doi:10.1016/j.camwa.2005.01.024

[2] T. Antczak, A new method of solving nonlinear mathematical programming problems involving r-invex functions, *Journal of Mathematical Analysis and Applications*, **311**(1) (2005), 313-323. doi:10.1016/j.jmaa.2005.02.049

[3] T. Antczak, Mean value in invexity analysis, *Nonlinear Analysis*, **60** (2005), 1473-1484. doi:10.1016/j.na.2004.11.005

[4] Y. Ding, Two classes of means and their applications, Shuxue de Shijian yu Renshi (Mathematics in Practice and Theory) 25 (2) (1995), 1620. (Chinese)

[5] Dah-Yan Hwang and S. S. Dragomir, Extensions of the Hermite-Hadamard inequality for r-preinvex functions on an invex set, Bulletin of the Australian Mathematical Society, **95**(3) (2017), 412-423. DOI: https://doi.org/10.1017/S0004972716001374

[6] D. S. Mitrinović, Analysis Inequalities, Springer-Verlag, Berlin, 1970.

[7] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Classical and new inequalities in analysis, *Kluwer Academic, Dordrecht*, 1993.

[8] S. R. Mohan and S. K. Neogy, On Invex Sets and Preinvex Functions, *Journal of Mathematical Analysis and Applications*, **189** (1995), 901-908.

[9] M. A. Noor, Variational-like inequalities, *Optimization*, **30**(4) (1994), 323-330.

[10] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, *Journal of Mathematical Analysis and Approximation Theory*, **2** (2) (2007),126-131.

[11] G. Póly and G. Szegö, Isoperimetric Inequalities in Mathematical Physics, Princeton University Press, Princeton, 1951.

[12] Feng Qi, Generalized weighted mean values with two parameters, Proceedings of the Royal Society of London Series A-Mathematical, Physcial and Engineering Sciences **454**(1978) (1998), 2723-2732.

[13] Feng Qi, On a two-parameter family of nonhomogeneous mean values, Tamkang Journal of Mathematics, **29**(2) (1998), 155-163.

[14] Feng Qi and Qiu-Ming Luo, Refinements and extensions of an inequality, Mathematics and Informatics Quarterly, **9** (1) (1999), 23-25.

[15] Mingbao Sun and Xiaoping Yang, Inequalities for the weighted mean of r-convex functions, Proceeding of the Americal Society, **133** (6) (2005), 1639-1646.

[16] K. Tettamanti, G. Sárkány , D. Krâlik and R. Stomfai, Über die annäherung logarithmischer funktionen durch algebraische funktionen, Period. Polytech. Chem. Engrg. 14 (1970), 99111.

[17] Wasim Ui-Haq and Javed Iqbal, Hermite-Hadamard-type inequalities for r-Preinvex functions, *Journal of Applied Mathematics*, **2013**, 2013, Article ID 126457, 5 pages. http://dx.doi.org/10.1155/2013/126457

[18] X. M. Yang, X. Q. Yang and K. L. Teo, Characterizations and Applications of Prequasi-Invex Functions, *Journal of Optimization Theory and Applications*, **110**(3) (2001), 645-668.

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