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MONOTONICITY AND CONVEXITY PROPERTIES AND SOME INEQUALITIES INVOLVING A GENERALIZED FORM OF THE WALLIS' COSINE FORMULA

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ABSTRACT. This study is focused on monotonicity and convexity properties of a generalized form of the Wallis' cosine formula. The methods and procedures are theoretical in nature. Specifically, by using the integral form of the Nielsen's β-function, it is proved that the generalized Wallis' cosine formula is logarithmically completely monotonic, logarithmically convex and decreasing. Furthermore, by using the classical Wendel's, Hölder's and Young's inequalities, among other analytical techniques, some new inequalities involving the generalized function are established.

1. Introduction and Preliminaries

The Nielsen's β-function, $\beta(x)$ which was introduced in (Nielsen, 1906) may be defined by any of the following equivalent forms.

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t}\,dt, \quad x > 0 \tag{1}$$

$$= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt, \quad x > 0 \tag{2}$$

$$= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\}, \quad x > 0 \tag{3}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function and $\Gamma(x)$ is the Euler's Gamma function. It is known that function $\beta(x)$ satisfies the following properties.

$$\beta(x+1) = \frac{1}{x} - \beta(x), \tag{4}$$

$$\beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}.$$

Some particular values of this function are: $\beta(1) = \ln 2$, $\beta\left(\frac{1}{2}\right) = \frac{\pi}{2}$, $\beta\left(\frac{3}{2}\right) = 2 - \frac{\pi}{2}$ and $\beta(2) = 1 - \ln 2$. Also, some interesting properties and inequalities involving this special function can be found in the recent work (Nantomah, 2017) .

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By differentiating m times of (1), (2) and (3), one respectively obtains

$$\beta^{(m)}(x) = \int_0^1 \frac{(\ln t)^m t^{x-1}}{1+t}\,dt, \quad x>0 \tag{5}$$

$$= (-1)^m \int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\,dt, \quad x>0 \tag{6}$$

$$= \frac{1}{2^{m+1}}\left\{\psi^{(m)}\left(\frac{x+1}{2}\right) - \psi^{(m)}\left(\frac{x}{2}\right)\right\}, \quad x>0 \tag{7}$$

for $m \in \mathbb{N}_0$. It is clear that $\beta^{(0)}(x) = \beta(x)$. In addition, by differentiating m times of (4), one obtains

$$\beta^{(m)}(x+1) = \frac{(-1)^m m!}{x^{m+1}} - \beta^{(m)}(x).$$

Also, it is well known in the literature that

$$\frac{m!}{x^{m+1}} = \int_0^\infty t^m e^{-xt}\,dt \tag{8}$$

for $x>0$ and $m \in \mathbb{N}$.

Definition 1.1. A function $f: I \to \mathbb{R}^+$ is said to be logarithmically convex or in short log-convex if $\ln f$ is convex on I. That is if

$$\ln f(ax+by) \leq a \ln f(x) + b \ln f(y)$$

or equivalently

$$f(ax+by) \leq (f(x))^a (f(y))^b$$

for each $x, y \in I$ and $a, b > 0$ such that $a+b=1$.

Definition 1.2. A function $f: I \to \mathbb{R}$ is said to be completely monotonic on I if f has derivatives of all order on I and

$$(-1)^k f^{(k)}(x) \geq 0$$

for $x \in I$ and $k \in \mathbb{N}$ (Widder, 1946).

Definition 1.3. A function $f: I \to \mathbb{R}^+$ is said to be logarithmically completely monotonic on I if f has derivatives of all order on I and

$$(-1)^k [\ln f(x)]^{(k)} \geq 0$$

for $x \in I$ and $k \in \mathbb{N}$ (Qi & Chen, 2004).

It has been established in (Qi & Chen, 2004) that every logarithmically completely monotonic function is also completely monotonic. However, the converse of this statement is not true.

The class of logarithmically completely monotonic functions has been a subject of intensive research in recent years. See for instance (Guo & Qi, 2015a) and the related references therein.

Definition 1.4. The Wallis' cosine (sine) formula is given by

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n t\,dt = \int_0^{\frac{\pi}{2}} \sin^n t\,dt = \frac{\sqrt{\pi}}{n}\frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2})} \tag{9}$$

for $n \in \mathbb{N}$ (Qi, 2010). It is also known in the literature as the Wallis' integrals, and it may also be defined as

$$I_n = \frac{1}{2}\frac{\Omega_n}{\Omega_{n-1}} = \frac{\pi}{2}W_{\frac{n}{2}} = \frac{1}{2}B\left(\frac{n+1}{2},\frac{1}{2}\right), \quad n \in \mathbb{N}$$

where $\Omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ is the volume of the unit ball in $\mathbb{R}^n$, $W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}}\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}$ is the Wallis ratio (Qi & Mortici, 2015), and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ is the classical Euler's beta function.

Lately, the Wallis' cosine formula has been applied in (Guo & Qi, 2015b), (Qi & Mansour, 2016) and (Qi, Mortici, & Guo, 2018) to study some properties of a sequence originating from geometric probability for pairs of hyperplanes intersecting with a convex body.

In (Kazarinoff, 1956), a generalization of the Wallis' cosine formula was given as

$$H(x) = \int_0^{\frac{\pi}{2}} \cos^x t\,dt = \frac{\sqrt{\pi}}{x}\frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2})} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}, \quad x \in \mathbb{R}^+ \tag{10}$$

where $H(n) = I_n$ for $n \in \mathbb{N}$.

In this paper, the objective is to prove that the function $H(x)$ is logarithmically completely monotonic, logarithmically convex and decreasing. Additionally, some inequalities involving $H(x)$ are established. The results are presented in the following section.

2. Main Results

Theorem 2.1. *The function $H(x)$ is logarithmically completely monotonic.*

Proof. Note that $\ln H(x) = \ln\sqrt{\pi} + \ln\Gamma(\frac{x}{2}+\frac{1}{2}) - \ln\Gamma(\frac{x}{2}) - \ln x$. Then

$$\begin{aligned}
[\ln H(x)]' &= \frac{1}{2}\frac{\Gamma'(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+\frac{1}{2})} - \frac{1}{2}\frac{\Gamma'(\frac{x}{2})}{\Gamma(\frac{x}{2})} - \frac{1}{x} \\
&= \frac{1}{2}\psi\left(\frac{x}{2}+\frac{1}{2}\right) - \frac{1}{2}\psi\left(\frac{x}{2}\right) - \frac{1}{x} \\
&= \frac{1}{2}\left\{\psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\} - \frac{1}{x} \\
&= \beta(x) - \frac{1}{x}.
\end{aligned}$$

Furthermore, by differentiating n times of $\ln H(x)$, one obtains

$$[\ln H(x)]^{(n)} = \beta^{(n-1)}(x) + \frac{(-1)^n (n-1)!}{x^n} \tag{11}$$

which implies that

$$(-1)^n \left[\ln H(x)\right]^{(n)} = (-1)^n \beta^{(n-1)}(x) + \frac{(n-1)!}{x^n}. \tag{12}$$

Now let $n = m+1$ in the right hand side of (12). Then by (6) and (8), one obtains

$$\begin{aligned}
(-1)^n \left[\ln H(x)\right]^{(n)} &= (-1)^{m+1} \beta^{(m)}(x) + \frac{m!}{x^{m+1}} \\
&= (-1)^{2m+1} \int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\, dt + \int_0^\infty t^m e^{-xt}\, dt \\
&= -\int_0^\infty \frac{t^m e^{-xt}}{1+e^{-t}}\, dt + \int_0^\infty t^m e^{-xt}\, dt \\
&= \int_0^\infty \left(1 - \frac{1}{1+e^{-t}}\right) t^m e^{-xt}\, dt \\
&\geq 0.
\end{aligned}$$

Therefore, $H(x)$ is logarithmically completely monotonic.

Corollary 2.2. *The function $H(x)$ is logarithmically convex and decreasing.*

Proof. By letting $n = 2$ in (11) and using (6) and (8), one obtains

$$\begin{aligned}
\left[\ln H(x)\right]'' &= \beta'(x) + \frac{1}{x^2} \\
&= -\int_0^\infty \frac{te^{-xt}}{1+e^{-t}}\, dt + \int_0^\infty te^{-xt}\, dt \\
&= \int_0^\infty \left(1 - \frac{1}{1+e^{-t}}\right) te^{-xt}\, dt \\
&\geq 0.
\end{aligned}$$

Thus, $H(x)$ is logarithmically convex. Next, let $u(x) = \ln H(x)$. Then

$$\begin{aligned}
u'(x) = \left[\ln H(x)\right]' &= \beta(x) - \frac{1}{x} \\
&= \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\, dt - \int_0^\infty e^{-xt}\, dt \\
&= \int_0^\infty \left(\frac{1}{1+e^{-t}} - 1\right) e^{-xt}\, dt \\
&\leq 0.
\end{aligned}$$

Hence $u(x)$ is decreasing and consequently, $H(x)$ is also decreasing.

Remark 2.3. Since every logarithmically convex function is convex, then $H(x)$ is also convex. This implies that for $x, y > 0$, it is the case that

$$H\left(\frac{\alpha x + \beta y}{\alpha + \beta}\right) \leq \frac{\alpha H(x) + \beta H(y)}{\alpha + \beta}$$

where $\alpha, \beta \geq 0$ and $\alpha + \beta > 0$.

Corollary 2.4. *Let a matrix D be defined for $x > 0$ by*

$$D = \begin{pmatrix} H(x) & H'(x) \\ H'(x) & H''(x) \end{pmatrix}. \tag{13}$$

Then $\det D \geq 0$. *In other words, the function $H(x)$ satisfies the Turan-type inequality*

$$H''(x)H(x) - [H'(x)]^2 \geq 0. \tag{14}$$

Proof. This is a direct consequence of the logarithmic convexity of $H(x)$.

Corollary 2.5. *The inequality*

$$H^2\left(\frac{x+y}{2}\right) \leq H(x)H(y) \tag{15}$$

is valid for $x, y > 0$.

Proof. Since $H(x)$ is logarithmically convex, then for $x, y > 0$, one obtains

$$H\left(\frac{x}{r} + \frac{y}{s}\right) \leq (H(x))^{\frac{1}{r}} (H(y))^{\frac{1}{s}}$$

where $r > 1$, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$. Then by letting $r = s = 2$, the result (15) is obtained.

Lemma 2.6. *For $t > 0$, the inequality*

$$\frac{e^{-t}}{2} + \frac{1}{1 + e^{-t}} < 1 \tag{16}$$

is satisfied.

Proof. Notice that $e^{-t} < 1$ for all $t > 0$. Then it follows easily that

$$\begin{gathered} e^{-t} - 1 < 0, \\ e^{-2t} - e^{-t} < 0, \\ e^{-2t} - e^{-t} + 2e^{-t} < 0 + 2e^{-t}, \\ e^{-2t} + e^{-t} < 2e^{-t}, \\ e^{-2t} + e^{-t} + 2 < 2e^{-t} + 2, \\ e^{-t}(1 + e^{-t}) + 2 < 2(1 + e^{-t}). \end{gathered}$$

Rearranging the last inequality gives the result (16).

Theorem 2.7. *The double-inequality*

$$\frac{\sqrt{\pi}}{2}\left(\frac{x}{2} + \frac{1}{2}\right)^{-\frac{1}{2}} < H(x) < \frac{\pi}{2\sqrt{2}}\left(\frac{x}{2} + \frac{1}{2}\right)^{-\frac{1}{2}} \tag{17}$$

holds for $x > 0$.

Proof. Wendel (1948) established the inequality

$$\left(\frac{x}{x+s}\right)^{1-s} \leq \frac{\Gamma(x+s)}{x^s\Gamma(x)} \leq 1, \quad x>0, s\in(0,1) \tag{18}$$

which can be rearranged as

$$1 \leq (x+s)^{1-s}\frac{\Gamma(x+s)}{\Gamma(x+1)} \leq \left(1+\frac{s}{x}\right)^{1-s}.$$

This implies that

$$\lim_{x\to\infty}(x+s)^{1-s}\frac{\Gamma(x+s)}{\Gamma(x+1)} = 1. \tag{19}$$

Also, direct computation gives

$$\lim_{x\to 0^+}(x+s)^{1-s}\frac{\Gamma(x+s)}{\Gamma(x+1)} = s^{1-s}\Gamma(s). \tag{20}$$

Then, by replacing x by $\frac{x}{2}$ and letting $s=\frac{1}{2}$ in (19) and (20), one respectively obtains

$$\lim_{x\to\infty}\left(\frac{x}{2}+\frac{1}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{x}{2}+1\right)} = 1 \tag{21}$$

and

$$\lim_{x\to 0^+}\left(\frac{x}{2}+\frac{1}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{x}{2}+1\right)} = \sqrt{\frac{\pi}{2}}. \tag{22}$$

Now let $G(x)=\left(\frac{x}{2}+\frac{1}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)}{\frac{x}{2}\Gamma\left(\frac{x}{2}\right)}$ and $\phi(x)=\ln G(x)$. That is,

$$\phi(x) = \frac{1}{2}\ln\left(\frac{x}{2}+\frac{1}{2}\right) - \ln\left(\frac{x}{2}\right) + \ln\Gamma\left(\frac{x}{2}+\frac{1}{2}\right) - \ln\Gamma\left(\frac{x}{2}\right). \tag{23}$$

By differentiating (23) and using (2) and (8), one obtains

$$\begin{aligned}
\phi'(x) &= \frac{1}{2(x+1)} - \frac{1}{x} + \frac{1}{2}\left\{\psi\left(\frac{x}{2}+\frac{1}{2}\right) - \psi\left(\frac{x}{2}\right)\right\} \\
&= \frac{1}{2(x+1)} - \frac{1}{x} + \beta(x) \\
&= \frac{1}{2}\int_0^\infty e^{-(x+1)t}\,dt - \int_0^\infty e^{-xt}\,dt + \int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt \\
&= \int_0^\infty \left(\frac{e^{-t}}{2} + \frac{1}{1+e^{-t}} - 1\right)e^{-xt}\,dt \\
&\leq 0
\end{aligned}$$

which follows from (16). Hence $\phi(x)$ is decreasing. Consequently, $G(x)$ is also decreasing. Then for $0<x<\infty$, one gets

$$G(\infty) < G(x) < G(0)$$

which by (21) and (22) results to

$$\left(\frac{x}{2}+\frac{1}{2}\right)^{-\frac{1}{2}} < \frac{\Gamma\left(\frac{x}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{x}{2}+1\right)} < \sqrt{\frac{\pi}{2}}\left(\frac{x}{2}+\frac{1}{2}\right)^{-\frac{1}{2}}. \tag{24}$$

Then, the inequality (17) is obtained from this result.

Remark 2.8. The limits (19) and (20) are already known in the literature. For instance, they were obtained in Theorem 1.2 of (Qi, Niu, Cao, & Chen, 2008) by using different precudures.

Theorem 2.9. *Let $p>1$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the inequality*

$$H(x+y) \leq [H(px)]^{\frac{1}{p}}\,[H(qy)]^{\frac{1}{q}} \tag{25}$$

holds for $x,y>0$.

Proof. Let $p>1$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then by the Hölder's inequality:

$$\int_a^b f(t)g(t)\,dt \leq \left(\int_a^b f^p(t)\,dt\right)^{\frac{1}{p}}\left(\int_a^b g^q(t)\,dt\right)^{\frac{1}{q}},$$

one obtains

$$\begin{aligned} H(x+y) &= \int_0^{\frac{\pi}{2}} \cos^{x+y} t\,dt \\ &= \int_0^{\frac{\pi}{2}} \cos^{x}\cos^{y} t\,dt \\ &\leq \left(\int_0^{\frac{\pi}{2}} \cos^{px} t\,dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{\pi}{2}} \cos^{qy} t\,dt\right)^{\frac{1}{q}} \\ &= [H(px)]^{\frac{1}{p}}\,[H(qy)]^{\frac{1}{q}} \end{aligned}$$

which completes the proof.

Remark 2.10. Equality holds in (25), if $x=y$ and $p=q=2$.

Remark 2.11. By letting $x=n$, $y=n+1$ where $n\in\mathbb{N}$ and $p=q=2$ in Theorem 2.9, one obtains the Turan-type inequality

$$I_{2n+1}^2 \leq I_{2n}\cdot I_{2n+2}. \tag{26}$$

Corollary 2.12. *Let $p>1$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the inequality*

$$H(x+y) \leq \frac{H(px)}{p}+\frac{H(qy)}{q} \tag{27}$$

holds for $x,y>0$.

Proof. Let $p>1$, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then by (25) and the Young's inequality:

$$x^{\frac{1}{p}}y^{\frac{1}{q}} \leq \frac{x}{p}+\frac{y}{q}, \quad x,y\geq 0,$$

it follows that

$$H(x+y) \leq [H(px)]^{\frac{1}{p}} [H(qy)]^{\frac{1}{q}} \leq \frac{H(px)}{p} + \frac{H(qy)}{q}$$

which gives the desired result.

Corollary 2.13. *The function $H(x)$ is subadditive. That is, the inequality*

$$H(x+y) \leq H(x) + H(y) \tag{28}$$

is holds for $x, y > 0$.

Proof. It follows from (27) that

$$\begin{aligned} H(x+y) &\leq \frac{H(px)}{p} + \frac{H(qy)}{q} \\ &\leq \frac{H(x)}{p} + \frac{H(y)}{q} \leq H(x) + H(y) \end{aligned}$$

which concludes the proof.

Theorem 2.14. *The function $H(x)$ satisfies the inequality*

$$H(x)H(y) \leq \frac{\pi}{2} H(x+y), \tag{29}$$

for $x, y > 0$.

Proof. The log-convexity of $H(x)$ implies that the function $\frac{H'(x)}{H(x)}$ is increasing. Define a function A by

$$A(x,y) = \frac{H(x)H(y)}{H(x+y)}, \quad x, y > 0,$$

and let $u(x,y) = \ln A(x,y)$. Then for a fixed y,

$$u'(x,y) = \frac{H'(x)}{H(x)} - \frac{H'(x+y)}{H(x+y)} \leq 0.$$

Hence, $u(x,y)$ and consequently $A(x,y)$ are decreasing. Then for $x > 0$, one obtains

$$\frac{H(x)H(y)}{H(x+y)} \leq H(0) = \frac{\pi}{2},$$

which gives the result (29).

3. Conclusion

By employing the Nielsen's β-function, it has been proved that the generalized Wallis' cosine formula: $H(x) = \frac{\sqrt{\pi}}{x} \frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2})}$ for $x \in \mathbb{R}^+$ is logarithmically completely monotonic, logarithmically convex and decreasing. Furthermore, by employing the classical Wendel's, Hölder's and Young's inequalities, among other analytical techniques, some new inequalities which involve the generalized function have been established.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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