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INEQUALITIES OF JENSEN'S TYPE FOR GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTION f FOR WHICH THE COMPOSITE $f \circ g^{-1}$ IS CONVEX

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of function f for which the composite function $f \circ g^{-1}$ is convex. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a, b]$ are also given. Applications for Hadamard fractional integrals are provided as well.

1. Introduction

The following integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which holds for any convex function $f : [a, b] \to \mathbb{R}$, is well known in the literature as the *Hermite-Hadamard inequality*.

There is an extensive amount of literature devoted to this simple and nice result which has many applications in Theory of Special Means and in Information Theory for divergence measures, from which we would like to refer the reader to the monograph [27], the recent survey paper [19] and the references therein.

In order to extend these type of inequalities for general fractional integrals we need the following preparations.

Assume that the kernel k is defined either on $(0, \infty)$ or on $[0, \infty)$ with complex values and integrable on any finite subinterval. We define the function $K : [0, \infty) \to \mathbb{C}$ by

$$K(t) := \begin{cases} \int_0^t k(s)\,ds \text{ if } 0 < t, \\ \\ 0 \text{ if } t = 0. \end{cases}$$

As a simple example, if $k(t) = t^{\alpha-1}$ then for $\alpha \in (0, 1)$ the function k is defined on $(0, \infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0, \infty)$. If $\alpha \geq 1$, then k is defined on $[0, \infty)$ and $K(t) := \frac{1}{\alpha} t^{\alpha}$ for $t \in [0, \infty)$.

Let g be a strictly increasing function on (a, b), having a continuous derivative g' on (a, b). For the Lebesgue integrable function $f : (a, b) \to \mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of f by

$$S_{k,g,a+} f(x) = \int_a^x k(g(x) - g(t))\, g'(t) f(t)\,dt, \; x \in (a, b] \tag{1.2}$$

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and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b). \tag{1.3}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I^{\alpha}_{a+,g}f(x),\ \ a<x\le b\end{aligned} \tag{1.4}$$

and

$$\begin{aligned}S_{k,g,b-}f(x)&=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I^{\alpha}_{b-,g}f(x),\ \ a\le x<b,\end{aligned} \tag{1.5}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [30, p. 100].

For $g(t)=t$ in (1.5) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [30, p. 111]

$$H^{\alpha}_{a+}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x\le b \tag{1.6}$$

and

$$H^{\alpha}_{b-}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\le a<x<b. \tag{1.7}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R^{\alpha}_{a+}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x\le b \tag{1.8}$$

and

$$R^{\alpha}_{b-}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}\,t^{\alpha+1}},\ \ 0\le a<x<b. \tag{1.9}$$

Also, for $g(t)=\exp(\beta t)$, $\beta>0$, we can consider the *"β-Exponential fractional integrals"*

$$E^{\alpha}_{a+,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_a^x[\exp(\beta x)-\exp(\beta t)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.10}$$

for $a<x\le b$ and

$$E^{\alpha}_{b-,\beta}f(x):=\frac{\beta}{\Gamma(\alpha)}\int_x^b[\exp(\beta t)-\exp(\beta x)]^{\alpha-1}\exp(\beta t)\,f(t)\,dt, \tag{1.11}$$

for $a\le x<b$.

If we take $g(t)=t$ in (1.2) and (1.3), then we can consider the following *k-fractional integrals*

$$S_{k,a+}f(x)=\int_a^x k(x-t)\,f(t)\,dt,\ \ x\in(a,b] \tag{1.12}$$

and

$$S_{k,b-}f(x)=\int_x^b k(t-x)f(t)\,dt,\ x\in[a,b). \tag{1.13}$$

In [33], Raina studied a class of functions defined formally by

$$\mathcal{F}^{\sigma}_{\rho,\lambda}(x):=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k,\ \ |x|<R,\ \text{with } R>0 \tag{1.14}$$

for ρ, $\lambda>0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (1.14), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}f(x):=\int_a^x (x-t)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(w(x-t)^{\rho})f(t)\,dt,\ x>a \tag{1.15}$$

where ρ, $\lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}^{\sigma}_{\rho,\lambda,b-;w}f(x):=\int_x^b (t-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(w(t-x)^{\rho})f(t)\,dt,\ x<b \tag{1.16}$$

where ρ, $\lambda>0$, $w\in\mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t)=t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}(wt^{\rho})$ we re-obtain the definitions of (1.15) and (1.16) from (1.12) and (1.13).

In [31], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}^{\alpha}_{a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\}f(t)\,dt,\ x>a \tag{1.17}$$

and

$$\mathcal{T}^{\alpha}_{b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\}f(t)\,dt,\ x<b \tag{1.18}$$

where $\alpha\in(0,1)$.

We observe that for $k(t)=\frac{1}{\alpha}\exp\left(-\frac{1-\alpha}{\alpha}t\right)$, $t\in\mathbb{R}$ we re-obtain the definitions of (1.17) and (1.18) from (1.12) and (1.13).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+}f(x):=\frac{1}{\alpha}\int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x)-g(t))\right\}g'(t)f(t)\,dt,\ x>a \tag{1.19}$$

and

$$\mathcal{T}^{\alpha}_{g,b-}f(x):=\frac{1}{\alpha}\int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t)-g(x))\right\}g'(t)f(t)\,dt,\ x<b \tag{1.20}$$

where $\alpha\in(0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha>0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+}f(x):=\int_a^x (g(x)-g(t))^{\alpha-1}\ln(g(x)-g(t))\,g'(t)f(t)\,dt, \tag{1.21}$$

for $0<a<x\leq b$ and

$$\mathcal{L}_{g,b-}^{\alpha}f\left(x\right) :=\int_{x}^{b}\left(g\left(t\right) -g\left(x\right) \right) ^{\alpha -1}\ln \left(g\left(t\right) -g\left(x\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt, \tag{1.22}$$

for $0<a\leq x<b$, where $\alpha >0$. These are obtained from (1.12) and (1.13) for the kernel $k\left(t\right) =t^{\alpha -1}\ln t,$ $t>0$.

For $\alpha =1$ we get

$$\mathcal{L}_{g,a+}f\left(x\right) :=\int_{a}^{x}\ln \left(g\left(x\right) -g\left(t\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt,\ 0<a<x\leq b \tag{1.23}$$

and

$$\mathcal{L}_{g,b-}f\left(x\right) :=\int_{x}^{b}\ln \left(g\left(t\right) -g\left(x\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt,\ 0<a\leq x<b. \tag{1.24}$$

For $g\left(t\right) =t$, we have the simple forms

$$\mathcal{L}_{a+}^{\alpha}f\left(x\right) :=\int_{a}^{x}\left(x-t\right) ^{\alpha -1}\ln \left(x-t\right) f\left(t\right) dt,\ 0<a<x\leq b, \tag{1.25}$$

$$\mathcal{L}_{b-}^{\alpha}f\left(x\right) :=\int_{x}^{b}\left(t-x\right) ^{\alpha -1}\ln \left(t-x\right) f\left(t\right) dt,\ 0<a\leq x<b, \tag{1.26}$$

$$\mathcal{L}_{a+}f\left(x\right) :=\int_{a}^{x}\ln \left(x-t\right) f\left(t\right) dt,\ 0<a<x\leq b \tag{1.27}$$

and

$$\mathcal{L}_{b-}f\left(x\right) :=\int_{x}^{b}\ln \left(t-x\right) f\left(t\right) dt,\ 0<a\leq x<b. \tag{1.28}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[22], [28]-[41] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned} &S_{k,g,a+,b-}f\left(x\right) \\ &:=\frac{1}{2}\left[S_{k,g,a+}f\left(x\right) +S_{k,g,b-}f\left(x\right) \right] \\ &=\frac{1}{2}\left[\int_{a}^{x}k\left(g\left(x\right) -g\left(t\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt+\int_{x}^{b}k\left(g\left(t\right) -g\left(x\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt\right] \end{aligned} \tag{1.29}$$

for the Lebesgue integrable function $f:\left(a,b\right) \rightarrow \mathbb{C}$ and $x\in \left(a,b\right)$.

Observe that

$$S_{k,g,x+}f\left(b\right) =\int_{x}^{b}k\left(g\left(b\right) -g\left(t\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt,\ x\in \lbrack a,b)$$

and

$$S_{k,g,x-}f\left(a\right) =\int_{a}^{x}k\left(g\left(t\right) -g\left(a\right) \right) g^{\prime }\left(t\right) f\left(t\right) dt,\ x\in (a,b].$$

We can define also the dual mixed operator

$$
\begin{aligned}
&\breve{S}_{k,g,a+,b-}f(x)\\
&:=\frac{1}{2}\left[S_{k,g,x+}f(b)+S_{k,g,x-}f(a)\right]\\
&=\frac{1}{2}\left[\int_x^b k(g(b)-g(t))\,g'(t)\,f(t)\,dt+\int_a^x k(g(t)-g(a))\,g'(t)\,f(t)\,dt\right]
\end{aligned}
$$

for any $x\in(a,b)$.

In the recent paper [26] we obtained the following inequalities for convex functions $f:[a,b]\to\mathbb{R}$:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). If $f:[a,b]\to\mathbb{R}$ is a convex function, then*

$$
\begin{aligned}
&\frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right]\\
&\quad\times f\left(\frac{K(g(x)-g(a))\,a+K(g(b)-g(x))\,b}{K(g(x)-g(a))+K(g(b)-g(x))}\right.\\
&\quad\left.+\frac{\int_a^x K(g(x)-g(t))\,dt-\int_x^b K(g(t)-g(x))\,dt}{K(g(x)-g(a))+K(g(b)-g(x))}\right)\\
&\le\frac{1}{2}\left[f\left(a+\frac{1}{K(g(x)-g(a))}\int_a^x K(g(x)-g(t))\,dt\right)K(g(x)-g(a))\right.\\
&\quad\left.+f\left(b-\frac{1}{K(g(b)-g(x))}\int_x^b K(g(t)-g(x))\,dt\right)K(g(b)-g(x))\right]\\
&\le S_{k,g,a+,b-}f(x)\\
&\le\frac{1}{2}\left[K(g(x)-g(a))\,f(a)+K(g(b)-g(x))\,f(b)\right]\\
&+\frac{1}{2}\left[\frac{f(x)-f(a)}{x-a}\int_a^x K(g(x)-g(t))\,dt-\frac{f(b)-f(x)}{b-x}\int_x^b K(g(t)-g(x))\,dt\right]
\end{aligned}
\tag{1.30}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right]\\
&\quad\times f\left(x+\frac{\int_x^b K(g(b)-g(t))\,dt-\int_a^x K(g(t)-g(a))\,dt}{K(g(x)-g(a))+K(g(b)-g(x))}\right)\\
&\le\frac{1}{2}\left[f\left(x-\frac{1}{K(g(x)-g(a))}\int_a^x K(g(t)-g(a))\,dt\right)K(g(x)-g(a))\right.\\
&\quad\left.+f\left(x+\frac{1}{K(g(b)-g(x))}\int_x^b K(g(b)-g(t))\,dt\right)K(g(b)-g(x))\right]
\end{aligned}
\tag{1.31}
$$

$$\leq \breve{S}_{k,g,a+,b-}f(x) \tag{1.32}$$
$$\leq \frac{1}{2}\left[K(g(x)-g(a))+K(g(b)-g(x))\right]f(x)$$
$$+\frac{1}{2}\left[\frac{f(b)-f(x)}{b-x}\int_x^b K(g(b)-g(t))\,dt-\frac{f(x)-f(a)}{x-a}\int_a^x K(g(t)-g(a))\,dt\right]$$

for $x \in (a,b)$.

Motivated by the above results, in this paper we establish some inequalities of Jensen and Hermite-Hadamard type for the k-g-fractional integrals of continuous functions $f:[a,b]\to\mathbb{R}$ for which the composite function $f\circ g^{-1}$ is convex on $[g(a),g(b)]$. Some examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ are also given. Applications for Hadamard fractional integrals are provided as well.

2. General Results

We have the following simple representation for the k-g-fractional integrals:

Lemma 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f:[a,b]\to\mathbb{C}$ be a complex valued Lebesgue integrable function on the real interval $[a,b]$. Then*

$$S_{k,g,a+}f(x)=\int_0^1 k((g(x)-g(a))s)\,f\circ g^{-1}(sg(a)+(1-s)g(x))\,ds, \tag{2.1}$$

for $x\in(a,b]$ *and*

$$S_{k,g,b-}f(x)=\int_0^1 k((g(b)-g(x))s)\,f\circ g^{-1}((1-s)g(x)+sg(b))\,ds, \tag{2.2}$$

for $x\in[a,b)$.

We also have

$$S_{k,g,x+}f(b)=\int_0^1 k((g(b)-g(x))s)\,f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds \tag{2.3}$$

for $x\in[a,b)$ *and*

$$S_{k,g,x-}f(a)=\int_0^1 k((g(x)-g(a))s)\,f\circ g^{-1}((1-s)g(a)+sg(x))\,ds \tag{2.4}$$

for $x\in(a,b]$.

Proof. Using the change of variable $u=g(t)$, then we have $du=g'(t)\,dt$, $t=g^{-1}(u)$ and

$$S_{k,g,a+}f(x)=\int_{g(a)}^{g(x)} k(g(x)-u)\,f\circ g^{-1}(u)\,du,\ \ x\in(a,b] \tag{2.5}$$

and

$$S_{k,g,b-}f(x)=\int_{g(x)}^{g(b)} k(u-g(x))\,f\circ g^{-1}(u)\,du,\ \ x\in[a,b). \tag{2.6}$$

Further, if we change the variable $u = (1-s)\,g(a) + sg(x)$, with $s \in [0,1]$, then for $a < x \leq b$ we have

$$\begin{aligned}
(2.7)\quad & S_{k,g,a+}f(x) \\
&= \int_0^1 k\left(\left(g(x) - g(a)\right)(1-s)\right) f \circ g^{-1}\left((1-s)\,g(a) + sg(x)\right) ds,\ x \in (a,b] \\
&= \int_0^1 k\left(\left(g(x) - g(a)\right)s\right) f \circ g^{-1}\left(sg(a) + (1-s)\,g(x)\right) ds,\ x \in (a,b].
\end{aligned}$$

If we change the variable $u = (1-s)\,g(x) + sg(b)$, with $s \in [0,1]$, then for $a \leq x < b$ we also have

$$\begin{aligned}
(2.8)\qquad & S_{k,g,b-}f(x) \\
&= \int_0^1 k\left(\left(g(b) - g(x)\right)s\right) f \circ g^{-1}\left((1-s)\,g(x) + sg(b)\right) ds,\ x \in [a,b),
\end{aligned}$$

which proves the first part of the lemma.

Further, if we replace x with b and a with x in (2.7), then we get

$$S_{k,g,x+}f(b) = \int_0^1 k\left(\left(g(b) - g(x)\right)s\right) f \circ g^{-1}\left(sg(x) + (1-s)\,g(b)\right) ds$$

and if we replace x with a and b with x in (2.8), then we obtain

$$S_{k,g,x-}f(a) = \int_0^1 k\left(\left(g(x) - g(a)\right)s\right) f \circ g^{-1}\left((1-s)\,g(a) + sg(x)\right) ds,$$

which proves the last part of the lemma. □

Remark 1. *From the above lemma, we have the representations*

$$\begin{aligned}
(2.9)\quad S_{k,g,a+,b-}f(x) &= \frac{1}{2}\int_0^1 k\left(\left(g(x) - g(a)\right)s\right) f \circ g^{-1}\left(sg(a) + (1-s)\,g(x)\right) ds \\
&\quad + \frac{1}{2}\int_0^1 k\left(\left(g(b) - g(x)\right)s\right) f \circ g^{-1}\left((1-s)\,g(x) + sg(b)\right) ds
\end{aligned}$$

and

$$\begin{aligned}
(2.10)\quad \breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\int_0^1 k\left(\left(g(b) - g(x)\right)s\right) f \circ g^{-1}\left(sg(x) + (1-s)\,g(b)\right) ds \\
&\quad + \frac{1}{2}\int_0^1 k\left(\left(g(x) - g(a)\right)s\right) f \circ g^{-1}\left((1-s)\,g(a) + sg(x)\right) ds
\end{aligned}$$

for $x \in (a,b)$.

We have:

Theorem 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. If $f \circ g^{-1}$ is convex on $(g(a), g(b))$,* *then for any* $x \in (a,b)$ *we have the inequalities*

$$
\begin{aligned}
(2.11)\quad & \frac{1}{2}\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}\right] \\
& \times f\circ g^{-1}\left(\frac{g(a)\frac{K(g(x)-g(a))}{g(x)-g(a)}+g(b)\frac{K(g(b)-g(x))}{g(b)-g(x)}}{\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right. \\
& \left.+\frac{\int_0^1 K((g(x)-g(a))s)\,ds-\int_0^1 (K(g(b)-g(x))s)\,ds}{\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right) \\
& \le \frac{1}{2}\left[f\circ g^{-1}\left(g(a)+\frac{g(x)-g(a)}{K(g(x)-g(a))}\int_0^1 K((g(x)-g(a))s)\,ds\right)\right. \\
& \times\frac{K(g(x)-g(a))}{g(x)-g(a)} \\
& +f\circ g^{-1}\left(g(b)-\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 (K(g(b)-g(x))s)\,ds\right) \\
& \left.\times\frac{K(g(b)-g(x))}{g(b)-g(x)}\right] \\
& \le S_{k,g,a+,b-}f(x) \\
& \le \frac{1}{2}f(x)\left(\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right. \\
& \left.+\frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds\right) \\
& +\frac{1}{2}\left[\frac{f(a)}{g(x)-g(a)}\int_0^1 [K(g(x)-g(a))-K((g(x)-g(a))s)]\,ds\right. \\
& \left.+\frac{f(b)}{g(b)-g(x)}\int_0^1 [K(g(b)-g(x))-(K(g(b)-g(x))s)]\,ds\right]
\end{aligned}
$$

Proof. By the convexity of $f\circ g^{-1}$ on $[g(a),g(b)]$ we have

$$
\begin{aligned}
f\circ g^{-1}(sg(a)+(1-s)g(x)) &\le sf\circ g^{-1}(g(a))+(1-s)f\circ g^{-1}(g(x)) \\
&= sf(a)+(1-s)f(x)
\end{aligned}
$$

and

$$
f\circ g^{-1}((1-s)g(x)+sg(b))\le(1-s)f(x)+sf(b)
$$

for $x\in(a,b)$.

Therefore

$$\begin{aligned} (2.12)\quad S_{k,g,a+,b-}f(x) &\leq \frac{1}{2}\int_0^1 k((g(x)-g(a))s)[sf(a)+(1-s)f(x)]\,ds \\ &+\frac{1}{2}\int_0^1 k((g(b)-g(x))s)[(1-s)f(x)+sf(b)]\,ds \\ =\frac{1}{2}f(a)\int_0^1 k((g(x)-g(a))s)\,sds &+ \frac{1}{2}f(x)\int_0^1 k((g(x)-g(a))s)(1-s)\,ds \\ +\frac{1}{2}f(x)\int_0^1 k((g(b)-g(x))s)(1-s)\,ds &+ \frac{1}{2}f(b)\int_0^1 k((g(b)-g(x))s)\,sds \\ &=: B(x) \end{aligned}$$

for $x \in (a,b)$.

Using the chain rule for the derivative over s we have

$$\begin{aligned} (K((g(x)-g(a))s))' &= K'((g(x)-g(a))s)(g(x)-g(a)) \\ &= k((g(x)-g(a))s)(g(x)-g(a)) \end{aligned}$$

and

$$(K(g(b)-g(x))s)' = k((g(b)-g(x))s)(g(b)-g(x))$$

for $x \in (a,b)$.

Therefore

$$\begin{aligned} &\int_0^1 k((g(x)-g(a))s)\,sds \\ &= \frac{1}{g(x)-g(a)}\int_0^1 (K((g(x)-g(a))s))'\,sds \\ &= \frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))s)\,s\big|_0^1 - \int_0^1 K((g(x)-g(a))s)\,ds\right] \\ &= \frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))) - \int_0^1 K((g(x)-g(a))s)\,ds\right] \\ &= \frac{K(g(x)-g(a))}{g(x)-g(a)} - \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds, \end{aligned}$$

$$\begin{aligned} &\int_0^1 k((g(x)-g(a))s)(1-s)\,ds \\ &= \frac{1}{g(x)-g(a)}\int_0^1 (K((g(x)-g(a))s))'((1-s))\,ds \\ &= \frac{1}{g(x)-g(a)}\left[K((g(x)-g(a))s)(1-s)\big|_0^1 + \int_0^1 K((g(x)-g(a))s)\,ds\right] \\ &= \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds, \end{aligned}$$

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)ds \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 \left(K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\right)'\left(\left(1-s\right)\right)ds \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\left[K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)\big|_0^1+\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right] \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)sds \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)'sds \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\left[\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)s\big|_0^1-\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right] \\
&=\frac{1}{g\left(b\right)-g\left(x\right)}\left[K\left(g\left(b\right)-g\left(x\right)\right)-\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right] \\
&=\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds
\end{aligned}$$

for $x\in\left(a,b\right)$.

We have

$$\begin{aligned}
B\left(x\right)&=\frac{1}{2}f\left(a\right)\left[\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}-\frac{1}{g\left(x\right)-g\left(a\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right] \\
&\quad+\frac{1}{2}f\left(x\right)\frac{1}{g\left(x\right)-g\left(a\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds \\
&\quad+\frac{1}{2}f\left(x\right)\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds \\
&\quad+\frac{1}{2}f\left(b\right)\left[\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1 \left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]
\end{aligned}$$

and by (2.12) we get the third inequality in (2.11).

We use Jensen's inequality to get

$$\begin{aligned}
&\frac{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)ds}{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds} \\
&\geq f\circ g^{-1}\left(\frac{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\left[sg\left(a\right)+\left(1-s\right)g\left(x\right)\right]ds}{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds}\right)
\end{aligned} \tag{2.13}$$

and

$$\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds} \tag{2.14}$$
$$\geq f\circ g^{-1}\left(\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left[\left(1-s\right)g\left(x\right)+sg\left(b\right)\right]ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}\right)$$

for $x\in(a,b)$.

Observe that

$$\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds=\frac{1}{g\left(x\right)-g\left(a\right)}\int_0^1\left(K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\right)'ds$$
$$=\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}$$

and

$$\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\left[sg\left(a\right)+\left(1-s\right)g\left(x\right)\right]ds$$
$$=g\left(a\right)\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)sds+g\left(x\right)\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\left(1-s\right)ds$$
$$=g\left(a\right)\left[\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}-\frac{1}{g\left(x\right)-g\left(a\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right]$$
$$+g\left(x\right)\left[\frac{1}{g\left(x\right)-g\left(a\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right]$$
$$=g\left(a\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}+\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds$$

for $x\in(a,b)$.

Then

$$\frac{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)\left[sg\left(a\right)+\left(1-s\right)g\left(x\right)\right]ds}{\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds}$$
$$=\frac{g\left(a\right)\frac{K(g(x)-g(a))}{g(x)-g(a)}+\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds}{\frac{K(g(x)-g(a))}{g(x)-g(a)}}$$
$$=g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds$$

and by (2.13) we get

$$\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)ds \tag{2.15}$$
$$\geq f\circ g^{-1}\left(g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right)\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}$$

for $x\in(a,b)$.

Also

$$\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds=\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}$$

and

$$\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left[\left(1-s\right)g\left(x\right)+sg\left(b\right)\right]ds$$
$$=g\left(x\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left(1-s\right)ds+g\left(b\right)\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)sds$$
$$=\frac{g\left(x\right)}{g\left(b\right)-g\left(x\right)}\int_0^1 K\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds$$
$$+g\left(b\right)\left[\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\frac{1}{g\left(b\right)-g\left(x\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right]$$
$$=g\left(b\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds$$

for $x\in\left(a,b\right)$.

Then

$$\frac{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)\left[\left(1-s\right)g\left(x\right)+sg\left(b\right)\right]ds}{\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}$$
$$=\frac{g\left(b\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}-\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds}{\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}}$$
$$=g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds$$

and by (2.14) we have

$$\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)ds \tag{2.16}$$
$$\geq f\circ g^{-1}\left(g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right)\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}$$

for $x\in\left(a,b\right)$.

Therefore, by (2.9) we have

$$S_{k,g,a+,b-}f\left(x\right) \tag{2.17}$$
$$=\frac{1}{2}\int_0^1 k\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)f\circ g^{-1}\left(sg\left(a\right)+\left(1-s\right)g\left(x\right)\right)ds$$
$$+\frac{1}{2}\int_0^1 k\left(\left(g\left(b\right)-g\left(x\right)\right)s\right)f\circ g^{-1}\left(\left(1-s\right)g\left(x\right)+sg\left(b\right)\right)ds$$
$$\geq\frac{1}{2}\left[f\circ g^{-1}\left(g\left(a\right)+\frac{g\left(x\right)-g\left(a\right)}{K\left(g\left(x\right)-g\left(a\right)\right)}\int_0^1 K\left(\left(g\left(x\right)-g\left(a\right)\right)s\right)ds\right)\right.$$
$$\times\frac{K\left(g\left(x\right)-g\left(a\right)\right)}{g\left(x\right)-g\left(a\right)}$$
$$+f\circ g^{-1}\left(g\left(b\right)-\frac{g\left(b\right)-g\left(x\right)}{K\left(g\left(b\right)-g\left(x\right)\right)}\int_0^1\left(K\left(g\left(b\right)-g\left(x\right)\right)s\right)ds\right)$$
$$\left.\times\frac{K\left(g\left(b\right)-g\left(x\right)\right)}{g\left(b\right)-g\left(x\right)}\right],$$

which proves the second inequality in (2.11).

The first inequality is obvious by the convexity of $f\circ g^{-1}$. □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* $a,\, b\in I$ as

$$M_g(a,b):=g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I=\mathbb{R}$ and $g(t)=t$ is the *identity function*, then $M_g(a,b)=A(a,b):=\frac{a+b}{2}$, the *arithmetic mean*. If $I=(0,\infty)$ and $g(t)=\ln t$, then $M_g(a,b)=G(a,b):=\sqrt{ab}$, the *geometric mean*. If $I=(0,\infty)$ and $g(t)=\frac{1}{t}$, then $M_g(a,b)=H(a,b):=\frac{2ab}{a+b}$, the *harmonic mean*. If $I=(0,\infty)$ and $g(t)=t^p$, $p\neq 0$, then $M_g(a,b)=M_p(a,b):=\left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent p*. Finally, if $I=\mathbb{R}$ and $g(t)=\exp t$, then

$$M_g(a,b)=LME(a,b):=\ln\left(\frac{\exp a+\exp b}{2}\right),$$

the *LogMeanExp function*.

Corollary 1. *With the assumptions of Theorem 2 we have*

$$\frac{2K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)}f(M_g(a,b)) \tag{2.18}$$
$$\leq\left[f\circ g^{-1}\left(g(a)+\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right.$$
$$\left.+f\circ g^{-1}\left(g(b)-\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right]\frac{K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)}$$
$$\leq S_{k,g,a+,b-}f(M_g(a,b))$$
$$\leq f(M_g(a,b))\frac{2}{g(b)-g(a)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds$$
$$+\left(\frac{f(a)+f(b)}{2}\right)\frac{2}{g(b)-g(a)}\int_0^1\left[K\left(\frac{g(b)-g(a)}{2}\right)-K\left(\left(\frac{g(b)-g(a)}{2}\right)s\right)\right]ds.$$

The following inequalities for the dual operator also hold:

Theorem 3. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with nonnegative values and integrable on any finite subinterval. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b) and $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. If $f\circ g^{-1}$ is convex on $(g(a),g(b))$,* *then for any* $x \in (a,b)$ *we have the inequalities*

$$\begin{aligned}
(2.19)\quad & \frac{1}{2}\left[\frac{K(g(x)-g(a))}{g(x)-g(a)}+\frac{K(g(b)-g(x))}{g(b)-g(x)}\right] \\
& \times f\circ g^{-1}\left(g(x)+\frac{\int_0^1 K((g(b)-g(x))s)\,ds-\int_0^1 K((g(x)-g(a))s)\,ds}{\frac{K(g(b)-g(x))}{g(b)-g(x)}+\frac{K(g(x)-g(a))}{g(x)-g(a)}}\right) \\
& \le \frac{1}{2}\left[f\circ g^{-1}\left(g(x)+\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 K((g(b)-g(x))s)\,ds\right)\right. \\
& \qquad \times\frac{K(g(b)-g(x))}{g(b)-g(x)} \\
& + f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K(g(x)-g(a))}\int_0^1 K((g(x)-g(a))s)\,ds\right) \\
& \qquad \left.\times\frac{K(g(x)-g(a))}{g(x)-g(a)}\right] \\
& \le \breve{S}_{k,g,a+,b-}f(x) \\
& \le \frac{1}{2}f(x)\left(\frac{1}{g(b)-g(x)}\int_0^1 [K(g(b)-g(x))-(K(g(b)-g(x))s)]\,ds\right. \\
& + \left.\frac{1}{g(x)-g(a)}\int_0^1 [K(g(x)-g(a))-K((g(x)-g(a))s)]\,ds\right) \\
& +\frac{1}{2}\left[\frac{f(b)}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds+\frac{f(a)}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right].
\end{aligned}$$

Proof. From the convexity of $f\circ g^{-1}$ we have

$$\begin{aligned}
(2.20)\quad \breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\int_0^1 k((g(b)-g(x))s)\,f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds \\
& + \frac{1}{2}\int_0^1 k((g(x)-g(a))s)\,f\circ g^{-1}((1-s)g(a)+sg(x))\,ds \\
& \le \frac{1}{2}\int_0^1 k((g(b)-g(x))s)\,[sf(x)+(1-s)f(b)]\,ds \\
& + \frac{1}{2}\int_0^1 k((g(x)-g(a))s)\,[(1-s)f(a)+sf(x)]\,ds \\
&= \frac{1}{2}f(x)\int_0^1 k((g(b)-g(x))s)\,sds+\frac{1}{2}f(b)\int_0^1 k((g(b)-g(x))s)(1-s)\,ds \\
& + \frac{1}{2}f(a)\int_0^1 k((g(x)-g(a))s)(1-s)\,ds+f(x)\int_0^1 k((g(x)-g(a))s)\,sds \\
& =: C(x)
\end{aligned}$$

for any $x \in (a,b)$.

We have

$$\begin{aligned}
C(x) &= \frac{1}{2}f(x)\left[\frac{K(g(b)-g(x))}{g(b)-g(x)} - \frac{1}{g(b)-g(x)}\int_0^1 (K(g(b)-g(x))s)\,ds\right] \\
&\quad + \frac{1}{2}f(b)\frac{1}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds \\
&\quad + \frac{1}{2}f(a)\frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds \\
&+ \frac{1}{2}f(x)\left[\frac{K(g(x)-g(a))}{g(x)-g(a)} - \frac{1}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right] \\
&= \frac{1}{2}f(x)\left(\frac{1}{g(b)-g(x)}\int_0^1 [K(g(b)-g(x)) - (K(g(b)-g(x))s)]\,ds\right. \\
&\quad \left. + \frac{1}{g(x)-g(a)}\int_0^1 [K(g(x)-g(a)) - K((g(x)-g(a))s)]\,ds\right) \\
&+ \frac{1}{2}\left[\frac{f(b)}{g(b)-g(x)}\int_0^1 K((g(b)-g(x))s)\,ds + \frac{f(a)}{g(x)-g(a)}\int_0^1 K((g(x)-g(a))s)\,ds\right]
\end{aligned}$$

for any $x \in (a,b)$ and by (2.19) we get the third inequality in (2.19).

By Jensen's inequality we have

$$\begin{aligned}
&\frac{\int_0^1 k((g(b)-g(x))s)\, f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds}{\int_0^1 k((g(b)-g(x))s)\,ds} \qquad (2.21)\\
&\quad \geq f\circ g^{-1}\left(\frac{\int_0^1 k((g(b)-g(x))s)(sg(x)+(1-s)g(b))\,ds}{\int_0^1 k((g(b)-g(x))s)\,ds}\right) \\
&= f\circ g^{-1}\left(\frac{g(x)\int_0^1 k((g(b)-g(x))s)\,s\,ds + g(b)\int_0^1 k((g(b)-g(x))s)(1-s)\,ds}{\int_0^1 k((g(b)-g(x))s)\,ds}\right)
\end{aligned}$$

for any $x \in (a,b)$.

Since

$$
\begin{aligned}
& g(x)\int_0^1 k((g(b)-g(x))s)\,sds+g(b)\int_0^1 k((g(b)-g(x))s)(1-s)\,ds \\
&=\frac{g(x)}{g(b)-g(x)}\int_0^1 (K((g(b)-g(x))s))'\,sds \\
&+\frac{g(b)}{g(b)-g(x)}\int_0^1 (K((g(b)-g(x))s))'(1-s)\,ds \\
&=\frac{g(x)}{g(b)-g(x)}\left[(K((g(b)-g(x))s))\,s|_0^1-\int_0^1 K((g(b)-g(x))s)\,ds\right] \\
&+\frac{g(b)}{g(b)-g(x)}\left[(K((g(b)-g(x))s))(1-s)|_0^1+\int_0^1 K((g(b)-g(x))s)\,ds\right] \\
&=\frac{g(x)}{g(b)-g(x)}\left[K(g(b)-g(x))-\int_0^1 K((g(b)-g(x))s)\,ds\right] \\
&+\frac{g(b)}{g(b)-g(x)}\left[\int_0^1 K((g(b)-g(x))s)\,ds\right] \\
&=\frac{g(x)}{g(b)-g(x)}K(g(b)-g(x))+\int_0^1 K((g(b)-g(x))s)\,ds
\end{aligned}
$$

and

$$\int_0^1 k((g(b)-g(x))s)\,ds=\frac{K(g(b)-g(x))}{g(b)-g(x)},$$

then by (2.21) we have

$$
\begin{aligned}
&\int_0^1 k((g(b)-g(x))s)\,f\circ g^{-1}(sg(x)+(1-s)g(b))\,ds \qquad (2.22)\\
&\geq f\circ g^{-1}\left(\frac{\frac{g(x)}{g(b)-g(x)}K(g(b)-g(x))+\int_0^1 K((g(b)-g(x))s)\,ds}{\frac{K(g(b)-g(x))}{g(b)-g(x)}}\right)\frac{K(g(b)-g(x))}{g(b)-g(x)} \\
&=f\circ g^{-1}\left(g(x)+\frac{g(b)-g(x)}{K(g(b)-g(x))}\int_0^1 K((g(b)-g(x))s)\,ds\right)\frac{K(g(b)-g(x))}{g(b)-g(x)}
\end{aligned}
$$

for any $x\in(a,b)$.

By applying Jensen's inequality again, we get

$$
\begin{aligned}
&\frac{\int_0^1 k((g(x)-g(a))s)\,f\circ g^{-1}((1-s)g(a)+sg(x))\,ds}{\int_0^1 k((g(x)-g(a))s)\,ds} \qquad (2.23)\\
&\qquad\geq f\circ g^{-1}\left(\frac{\int_0^1 k((g(x)-g(a))s)((1-s)g(a)+sg(x))\,ds}{\int_0^1 k((g(x)-g(a))s)\,ds}\right) \\
&=f\circ g^{-1}\left(\frac{g(a)\int_0^1 k((g(x)-g(a))s)(1-s)\,ds+g(x)\int_0^1 k((g(x)-g(a))s)\,sds}{\int_0^1 k((g(x)-g(a))s)\,ds}\right)
\end{aligned}
$$

for any $x\in(a,b)$.

Since

$$
\begin{aligned}
&g(a)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)(1-s)\,ds+g(x)\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)sds\\
&=\frac{g(a)}{g(x)-g(a)}\int_0^1\left(K\left(\left(g(x)-g(a)\right)s\right)\right)'(1-s)\,ds\\
&+\frac{g(x)}{g(x)-g(a)}\int_0^1\left(K\left(\left(g(x)-g(a)\right)s\right)\right)'sds\\
&=\frac{g(a)}{g(x)-g(a)}\left[K\left(\left(g(x)-g(a)\right)s\right)(1-s)\Big|_0^1+\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&+\frac{g(x)}{g(x)-g(a)}\left[\left(K\left(\left(g(x)-g(a)\right)s\right)\right)s\Big|_0^1-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&=\frac{g(a)}{g(x)-g(a)}\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\\
&+\frac{g(x)}{g(x)-g(a)}\left[K\left(\left(g(x)-g(a)\right)\right)-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right]\\
&=\frac{g(x)}{g(x)-g(a)}K\left(g(x)-g(a)\right)-\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds
\end{aligned}
$$

and

$$\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)ds=\frac{K\left(g(x)-g(a)\right)}{g(x)-g(a)},$$

then by (2.23) we get

$$
\begin{aligned}
(2.24)\qquad &\int_0^1 k\left(\left(g(x)-g(a)\right)s\right)f\circ g^{-1}\left((1-s)\,g(a)+sg(x)\right)ds\\
&\geq f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K\left(g(x)-g(a)\right)}\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right)\frac{K\left(g(x)-g(a)\right)}{g(x)-g(a)}
\end{aligned}
$$

for $x\in(a,b)$.

Therefore,

$$
\begin{aligned}
\check{S}_{k,g,a+,b-}f(x)\geq{}&\frac{1}{2}f\circ g^{-1}\left(g(x)+\frac{g(b)-g(x)}{K\left(g(b)-g(x)\right)}\int_0^1 K\left(\left(g(b)-g(x)\right)s\right)ds\right)\\
&\times\frac{K\left(g(b)-g(x)\right)}{g(b)-g(x)}\\
&+\frac{1}{2}f\circ g^{-1}\left(g(x)-\frac{g(x)-g(a)}{K\left(g(x)-g(a)\right)}\int_0^1 K\left(\left(g(x)-g(a)\right)s\right)ds\right)\\
&\times\frac{K\left(g(x)-g(a)\right)}{g(x)-g(a)},
\end{aligned}
$$

which proves the second inequality in (2.19).

The first inequality is obvious by the convexity of $f\circ g^{-1}$. □

Corollary 2. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
(2.25)\quad & \frac{2K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)} f\left(M_g(a,b)\right) \\
& \leq \left[f\circ g^{-1}\left(\frac{g(a)+g(b)}{2}+\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right. \\
& \left. + f\circ g^{-1}\left(\frac{g(a)+g(b)}{2}-\frac{g(b)-g(a)}{2K\left(\frac{g(b)-g(a)}{2}\right)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds\right)\right] \\
& \times \frac{K\left(\frac{g(b)-g(a)}{2}\right)}{g(b)-g(a)} \\
& \leq \breve{S}_{k,g,a+,b-} f\left(M_g(a,b)\right) \\
& \leq f\left(M_g(a,b)\right)\frac{2}{g(b)-g(a)}\int_0^1\left[K\left(\frac{g(b)-g(a)}{2}\right)-\left(K\left(\frac{g(b)-g(a)}{2}\right)s\right)\right]ds \\
& + \frac{f(b)+f(a)}{2}\frac{2}{g(b)-g(a)}\int_0^1 K\left(\frac{g(b)-g(a)}{2}s\right)ds.
\end{aligned}
$$

3. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, $\alpha>0$ where Γ is the *Gamma function*, then

$$S_{k,g,a+}f(x)=I_{a+,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[g(x)-g(t)\right]^{\alpha-1}g'(t)f(t)\,dt$$

for $a<x\leq b$ and

$$S_{k,g,b-}f(x)=I_{b-,g}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[g(t)-g(x)\right]^{\alpha-1}g'(t)f(t)\,dt$$

for $a\leq x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [30, p. 100].

We consider the mixed operators

$$I_{g,a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[I_{a+,g}^{\alpha}f(x)+I_{b-,g}^{\alpha}f(x)\right] \tag{3.1}$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f(x):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f(b)+I_{x-,g}^{\alpha}f(a)\right] \tag{3.2}$$

for $x\in(a,b)$.

We observe that for $\alpha>0$ we have

$$K(t)=\frac{1}{\Gamma(\alpha)}\int_0^t s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma(\alpha)}=\frac{t^{\alpha}}{\Gamma(\alpha+1)},\ t\geq 0.$$

In what follows we assume that $f:[a,b]\to\mathbb{R}$ is a continuous function on $[a,b]$. If $f\circ g^{-1}$ is convex on $[g(a),g(b)]$, then by using the inequality (2.11) we get

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right] \\
&\quad\times f\circ g^{-1}\left(\frac{\frac{g(x)+\alpha g(a)}{\alpha+1}(g(x)-g(a))^{\alpha-1}}{(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}}\right. \\
&\quad\left.+\frac{\frac{g(x)+\alpha g(b)}{\alpha+1}(g(b)-g(x))^{\alpha-1}}{\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right]}\right) \\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[f\circ g^{-1}\left(\frac{g(x)+\alpha g(a)}{\alpha+1}\right)(g(x)-g(a))^{\alpha-1}\right. \\
&\quad\left.+f\circ g^{-1}\left(\frac{g(x)+\alpha g(b)}{\alpha+1}\right)(g(b)-g(x))^{\alpha-1}\right] \\
&\le I_{g,a+,b-}^{\alpha}f(x) \\
&\le\frac{1}{2\Gamma(\alpha+2)}f(x)\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right] \\
&\quad+\frac{\alpha}{2\Gamma(\alpha+2)}\left[f(a)(g(x)-g(a))^{\alpha-1}+f(b)(g(b)-g(x))^{\alpha-1}\right]
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
&\frac{1}{2\Gamma(\alpha+1)}\left[(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}\right] \\
&\times f\circ g^{-1}\left(\frac{\frac{g(b)+\alpha g(x)}{\alpha+1}(g(b)-g(x))^{\alpha-1}+\frac{g(a)+\alpha g(x)}{\alpha+1}(g(x)-g(a))^{\alpha-1}}{(g(x)-g(a))^{\alpha-1}+(g(b)-g(x))^{\alpha-1}}\right) \\
&\le\frac{1}{2\Gamma(\alpha+1)}\left[f\circ g^{-1}\left(\frac{g(b)+\alpha g(x)}{\alpha+1}\right)(g(b)-g(x))^{\alpha-1}\right. \\
&\quad\left.+f\circ g^{-1}\left(\frac{g(a)+\alpha g(x)}{\alpha+1}\right)(g(x)-g(a))^{\alpha-1}\right] \\
&\le\breve{I}_{g,a+,b-}^{\alpha}f(x) \\
&\le\frac{\alpha}{2\Gamma(\alpha+2)}f(x)\left[(g(b)-g(x))^{\alpha-1}+(g(x)-g(a))^{\alpha-1}\right] \\
&\quad+\frac{1}{2\Gamma(\alpha+2)}\left[f(b)(g(b)-g(x))^{\alpha-1}+f(a)(g(x)-g(a))^{\alpha-1}\right]
\end{aligned}\tag{3.4}$$

for $x\in(a,b)$.

From (2.18) we also have

$$
\begin{aligned}
(3.5)\quad & \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha-1} f\left(M_g(a,b)\right) \\
&\le \left[f\circ g^{-1}\left(\frac{g(b)+(2\alpha+1)g(a)}{2(\alpha+1)}\right)+f\circ g^{-1}\left(\frac{g(a)+(2\alpha+1)g(b)}{2(\alpha+1)}\right)\right] \\
&\qquad\times\frac{\left(g(b)-g(a)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\le I^{\alpha}_{g,a+,b-} f\left(M_g(a,b)\right) \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}\left(g(b)-g(a)\right)^{\alpha-1}\left[f\left(M_g(a,b)\right)+\frac{f(a)+f(b)}{2}\alpha\right] \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha-1}\frac{f(a)+f(b)}{2}
\end{aligned}
$$

while from (2.25) we have

$$
\begin{aligned}
(3.6)\quad & \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha-1} f\left(M_g(a,b)\right) \\
&\le \left[f\circ g^{-1}\left(\frac{\alpha g(a)+(\alpha+2)g(b)}{2(\alpha+1)}\right)+f\circ g^{-1}\left(\frac{\alpha g(b)+(\alpha+2)g(a)}{2(\alpha+1)}\right)\right] \\
&\qquad\times\frac{\left(g(b)-g(a)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\le \breve{I}^{\alpha}_{g,a+,b-} f\left(M_g(a,b)\right) \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}\left(g(b)-g(a)\right)^{\alpha-1}\left[f\left(M_g(a,b)\right)\alpha+\frac{f(a)+f(b)}{2}\right] \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(g(b)-g(a)\right)^{\alpha-1}\frac{f(a)+f(b)}{2}.
\end{aligned}
$$

The last part is obvious by the fact that

$$
f\left(M_g(a,b)\right)\le\frac{f(a)+f(b)}{2}.
$$

4. Applications for GA-Convex Functions

Let $I\subset(0,\infty)$ be an interval; a real-valued function $f:I\to\mathbb{R}$ is said to be *GA-convex* (concave) on I if

$$
(4.1)\qquad f\left(x^{1-\lambda}y^{\lambda}\right)\le(\ge)(1-\lambda)f(x)+\lambda f(y)
$$

for all $x,\ y\in I$ and $\lambda\in[0,1]$.

Since the condition (4.1) can be written as

$$
(4.2)\quad f\circ\exp\left((1-\lambda)\ln x+\lambda\ln y\right)\le(\ge)(1-\lambda)f\circ\exp(\ln x)+\lambda f\circ\exp(\ln y),
$$

then we observe that $f:I\to\mathbb{R}$ is *GA-convex* (concave) on I if and only if $f\circ\exp$ is convex (concave) on $\ln I:=\{\ln z, z\in I\}$. If $I=[a,b]$ then $\ln I=[\ln a,\ln b]$.

It is known that the function $f(x)=\ln(1+x)$ is *GA*-convex on $(0,\infty)$ [7].

For real and positive values of x, the *Euler gamma* function Γ and its *logarithmic derivative* ψ, the so-called *digamma function*, are defined by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt \text{ and } \psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}.$$

It has been shown in [42] that the function $f : (0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \psi(x) + \frac{1}{2x}$$

is GA-concave on $(0, \infty)$ while the function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = \psi(x) + \frac{1}{2x} + \frac{1}{12x^2}$$

is GA-convex on $(0, \infty)$.

For some recent inequalities on GA-convex functions see [16]-[18].

Consider the *Hadamard fractional integrals* [30, p. 111]

$$H_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \le a < x \le b$$

and

$$H_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1} \frac{f(t)\, dt}{t}, \ 0 \le a < x < b,$$

where $\alpha > 0$.

We consider the mixed operators

$$H_{a+,b-}^{\alpha} f(x) := \frac{1}{2}\left[H_{a+}^{\alpha} f(x) + H_{b-}^{\alpha} f(x)\right] \tag{4.3}$$

and

$$\breve{H}_{g,a+,b-}^{\alpha} f(x) := \frac{1}{2}\left[H_{x+}^{\alpha} f(b) + H_{x-}^{\alpha} f(a)\right] \tag{4.4}$$

for $x \in (a, b)$.

If we write the inequalities (3.5) and (3.6) for $g(t) = \ln t$, $t \in [a, b] \subset (0, \infty)$, then for any function $f : [a, b] \to \mathbb{R}$ that is *GA-convex* on $[a, b]$, we have

$$\begin{aligned}
&\frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} \left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} f(G(a,b)) \qquad (4.5)\\
&\le \left[f\left(a^{\frac{2\alpha+1}{2(\alpha+1)}} b^{\frac{1}{2(\alpha+1)}}\right) + f\left(a^{\frac{1}{2(\alpha+1)}} b^{\frac{2\alpha+1}{2(\alpha+1)}}\right)\right] \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
&\le H_{a+,b-}^{\alpha} f(G(a,b)) \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)} \left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} \left[f(G(a,b)) + \frac{f(a)+f(b)}{2}\alpha\right] \\
&\le \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)} \left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} \frac{f(a)+f(b)}{2}
\end{aligned}$$

while from (2.25) we have

$$\begin{aligned}
(4.6)\quad & \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} f(G(a,b)) \\
& \leq \left[f\left(a^{\frac{\alpha}{2(\alpha+1)}} b^{\frac{\alpha+2}{2(\alpha+1)}}\right) + f\left(a^{\frac{\alpha+2}{2(\alpha+1)}} b^{\frac{\alpha}{2(\alpha+1)}}\right)\right] \frac{\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}}{2^{\alpha}\Gamma(\alpha+1)} \\
& \leq \breve{H}^{\alpha}_{a+,b-} f(G(a,b)) \\
& \leq \frac{1}{2^{\alpha-1}\Gamma(\alpha+2)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1}\left[f(G(a,b))\,\alpha + \frac{f(a)+f(b)}{2}\right] \\
& \leq \frac{1}{2^{\alpha-1}\Gamma(\alpha+1)}\left(\ln\left(\frac{b}{a}\right)\right)^{\alpha-1} \frac{f(a)+f(b)}{2},
\end{aligned}$$

where $G(a,b) = \sqrt{ab}$.

References

[1] R. P. Agarwal, M.-J. Luo and R. K. Raina, On Ostrowski type inequalities, *Fasc. Math.* **56** (2016), 5-27.

[2] A. Aglić Aljinović, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[3] T. M. Apostol, *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[4] A. O. Akdemir, Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[5] G. A. Anastassiou, Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[6] G. A. Anastassiou, The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[7] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[8] H. Budak, M. Z. Sarikaya, E. Set, Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[9] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[10] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.

[11] S. S. Dragomir, The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[12] S. S. Dragomir, On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[13] S. S. Dragomir, On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]

[14] S. S. Dragomir, Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[15] S. S. Dragomir, Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[16] S. S. Dragomir, Inequalities of Hermite-Hadamard type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 30. [Online http://rgmia.org/papers/v18/v18a30.pdf].

[17] S. S. Dragomir, Some new inequalities of Hermite-Hadamard type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 33. [Online http://rgmia.org/papers/v18/v18a33.pdf].

[18] S. S. Dragomir, Inequalities of Jensen type for GA-convex functions, RGMIA *Res. Rep. Coll.* **18** (2015), Art. 35. [Online http://rgmia.org/papers/v18/v18a35.pdf].

[19] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[20] S. S. Dragomir, Some inequalities of Hermite-Hadamard type for convex functions and Riemann-Liouville fractional integrals, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 40. [Online http://rgmia.org/papers/v20/v20a40.pdf].

[21] S. S. Dragomir, Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[22] S. S. Dragomir, Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[23] S. S. Dragomir, Further Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 84. [Online http://rgmia.org/papers/v20/v20a84.pdf].

[24] S. S. Dragomir, Ostrowski and trapezoid type inequalities for the generalized k-g-fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 111. [Online http://rgmia.org/papers/v20/v20a111.pdf].

[25] S. S. Dragomir, Some inequalities for the generalized k-g-fractional integrals of functions under complex boundedness conditions, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 119. [Online http://rgmia.org/papers/v20/v20a119.pdf].

[26] S. S. Dragomir, Inequalities of Jensen's type for generalized k-g-fractional integrals, *RGMIA Res. Rep. Coll.* **20** (2017).

[27] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, 2000.[Online http://rgmia.org/monographs/hermite_hadamard.html].

[28] A. Guezane-Lakoud and F. Aissaoui, New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[29] A. Kashuri and R. Liko, Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[30] A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations.* North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0.

[31] M. Kirane, B. T. Torebek, Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type Inequalities for convex functions via fractional integrals, Preprint arXiv:1701.00092.

[32] M. A. Noor, K. I. Noor and S. Iftikhar, Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445

[33] R. K. Raina , On generalized Wright's hypergeometric functions and fractional calculus operators, *East Asian Math. J.*, **21**(2)(2005), 191-203.

[34] M. Z. Sarikaya and H. Filiz, Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[35] M. Z. Sarikaya and H. Budak, Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[36] E. Set, New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[37] M. Tunç, On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[38] M. Tunç, Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[39] H. Yildirim and Z. Kirtay, Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[40] C. Yildiz, E, Özdemir and Z. S. Muhamet, New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[41] H. Yue, Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

[42] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa