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SOME PROPERTIES AND INEQUALITIES FOR DERIVATIVES OF THE GENERALIZED WALLIS' COSINE FORMULA

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Abstract. In this work, we study some properties and inequalities involving derivatives of a generalized form of the Wallis' cosine (sine) formula. Among other things, log-convexity, monotonicity, subadditivity and subhomogeneity properties of the function are discussed.

1. Introduction

The classical Euler's Gamma function is usually defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

for $x > 0$. Closely associated with the Gamma function is the digamma (or Psi) function $\psi(x)$, which is defined as the logarithmic derivative of the Gamma function. That is,

$$\begin{aligned}\psi(x) &= \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \\ &= -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)},\end{aligned}$$

where $\gamma = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.577215664...$ is the Euler-Mascheroni's constant.

In 1956, Kazarinoff [3] defined the function

$$H(x) = \int_0^{\frac{\pi}{2}} \cos^x t\, dt = \int_0^{\frac{\pi}{2}} \sin^x t\, dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)}, \tag{1}$$

for real numbers $x \in (-1, \infty)$. It is clear that this function is a geralization of the Wallis' cosine (sine) formula [5]:

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n t\, dt = \int_0^{\frac{\pi}{2}} \sin^n t\, dt = \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{n}{2}+\frac{1}{2})}{\Gamma(\frac{n}{2}+1)}, \tag{2}$$

which is defined for $n \in \mathbb{N}$. Lately, the function (2) has been applied in [2], [6] and [7] to study some properties of a sequence originating from geometric probability for pairs of hyperplanes intersecting with a convex body. Also, in the recent work [4], the author studied some interesting properties and inequalities involving the

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generalized function (1). In the present work, our objective is to derive some properties and inequalities involving derivatives of the generalized function. We present our results in the following section.

2. Main Results

By differentiating m times of the generalized function (1), we obtain

$$H^{(m)}(x) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^m \cos^x t \, dt = \frac{d^m}{dx^m} \left\{ \frac{\sqrt{\pi}}{2} \frac{\Gamma(\frac{x}{2}+\frac{1}{2})}{\Gamma(\frac{x}{2}+1)} \right\}, \tag{3}$$

for $x \in (-1, \infty)$ and $m \in \mathbb{N}_0$, where $H^{(0)}(x) = H(x)$. In particular, if $x = 0$, then we obtain

$$H^{(m)}(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^m \, dt = C_m \tag{4}$$

which is known in the literature as the log-cosine function [8]. Also, the right hand side of (3) gives

$$H'(x) = \frac{H(x)}{2} \left\{ \psi\left(\frac{x}{2}+\frac{1}{2}\right) - \psi\left(\frac{x}{2}+1\right) \right\},$$

$$H''(x) = \frac{H(x)}{4} \left\{ \left[\psi\left(\frac{x}{2}+\frac{1}{2}\right) - \psi\left(\frac{x}{2}+1\right) \right]^2 + \left[\psi'\left(\frac{x}{2}+\frac{1}{2}\right) - \psi'\left(\frac{x}{2}+1\right) \right] \right\},$$

from which we derive the following few special cases.

$$H^{(0)}(0) = \frac{\pi}{2}, \tag{5}$$

$$H'(0) = \int_0^{\frac{\pi}{2}} \ln \cos t \, dt = -\frac{\pi}{2} \ln 2, \tag{6}$$

$$H'(1) = \int_0^{\frac{\pi}{2}} (\ln \cos t) \cos t \, dt = -1 + \ln 2, \tag{7}$$

$$H'(2) = \int_0^{\frac{\pi}{2}} (\ln \cos t) \cos^2 t \, dt = \frac{\pi}{8} - \frac{1}{4} \ln 2, \tag{8}$$

$$H''(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \, dt = \frac{\pi^3}{24} + \frac{\pi}{2} (\ln 2)^2, \tag{9}$$

$$H''(1) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \cos t \, dt = 1 - \frac{\pi^2}{12} + (\ln 2 - 1)^2, \tag{10}$$

$$H''(2) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^2 \cos^2 t \, dt = \frac{\pi}{4} (\ln 2 - 1)^2 + \frac{\pi^3}{48} - \frac{3\pi}{16}, \tag{11}$$

$$H'''(0) = \int_0^{\frac{\pi}{2}} (\ln \cos t)^3 \, dt = -\frac{\pi^3}{8} \ln 2 - \frac{\pi}{2} (\ln 2)^3 - \frac{3\pi}{4} \zeta(3), \tag{12}$$

where $\zeta(x)$ is the Riemann zeta function.

Remark 2.1. Some families of these type of integrals have been studied in [1] and as pointed out in that work, these types of integrals have a wide range potential applications in mathematical and physical problems.

In what follows, we present some inequalities involving the function $H^{(m)}(x)$. We begin with the following well-known definition.

Definition 2.2. A function $f : I \to \mathbb{R}$ is said to be logarithmically convex or in short log-convex if $\ln f$ is convex on I. That is if

$$\ln f(\alpha x + \beta y) \le \alpha \ln f(x) + \beta \ln f(y)$$

or equivalently

$$f(\alpha x + \beta y) \le (f(x))^{\alpha} (f(y))^{\beta}$$

for each $x, y \in I$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.

Theorem 2.3. *Let $m, n \in \mathbb{N}_0$ be even, $a > 1$, $\frac{1}{a} + \frac{1}{b} = 1$ and $\frac{m}{a} + \frac{n}{b} \in \mathbb{N}_0$. Then the inequality*

$$H^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right) \le \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(n)}(y)\right)^{\frac{1}{b}} \tag{13}$$

is satisfied for $x, y \in (-1, \infty)$.

Proof. The main tool of this proof is the Hölders inequality for integrals. Notice that since $x, y \in (-1, \infty)$, $a > 1$ and $\frac{1}{a} + \frac{1}{b} = 1$, then $\frac{x}{a} + \frac{y}{b} \in (-1, \infty)$. Then by (3), we obtain

$$\begin{aligned} H^{(\frac{m}{a}+\frac{n}{b})}\left(\frac{x}{a} + \frac{y}{b}\right) &= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}+\frac{n}{b}} \cos^{\frac{x}{a}+\frac{y}{b}} t \, dt \\ &= \int_0^{\frac{\pi}{2}} (\ln \cos t)^{\frac{m}{a}} \cos^{\frac{x}{a}} t \cdot (\ln \cos t)^{\frac{n}{b}} \cos^{\frac{y}{b}} t \, dt \\ &\le \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^m \cos^x t \, dt\right)^{\frac{1}{a}} \left(\int_0^{\frac{\pi}{2}} (\ln \cos t)^n \cos^y t \, dt\right)^{\frac{1}{b}} \\ &= \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(n)}(y)\right)^{\frac{1}{b}} \end{aligned}$$

which completes the proof.

Remark 2.4. If $m = n$ in (13), then we obtain

$$H^{(m)}\left(\frac{x}{a} + \frac{y}{b}\right) \le \left(H^{(m)}(x)\right)^{\frac{1}{a}} \left(H^{(m)}(y)\right)^{\frac{1}{b}} \tag{14}$$

which is implies that the function $H^{(m)}(x)$ is log-convex on $(-1, \infty)$.

Remark 2.5. If $m = n$ and $a = b = 2$ in (13), then we obtain

$$H^{(m)}\left(\frac{x+y}{2}\right) \le \sqrt{H^{(m)}(x) H^{(m)}(y)}. \tag{15}$$

Remark 2.6. If $n = m + 2$, $a = b = 2$ and $x = y$ in (13), then we obtain the Turan-type inequality

$$H^{(m)}(x) H^{(m+2)}(x) \ge \left(H^{(m+1)}(x)\right)^2. \tag{16}$$

Corollary 2.7. *The function*

$$h(x)=\frac{H^{(m+1)}(x)}{H^{(m)}(x)} \tag{17}$$

is increasing on $x\in(-1,\infty)$ for $m\in\mathbb{N}_0$ even.

Proof. Let $m\in\mathbb{N}_0$ be even. Then direct differentiation yields

$$h'(x)=\frac{H^{(m+2)}(x)H^{(m)}(x)-\left(H^{(m+1)}(x)\right)^2}{[H^{(m)}(x)]^2}\geq 0,$$

which follows easily from the result (16).

Theorem 2.8. *Let $m\in\mathbb{N}_0$ be even and $x,y\in(-1,\infty)$ with $x\leq y$. Then the inequality*

$$\left(\frac{H^{(m)}(y)}{H^{(m)}(x)}\right)^{\lambda}\leq\frac{H^{(m)}(\lambda y)}{H^{(m)}(\lambda x)} \tag{18}$$

holds if $\lambda\geq 1$ and reverses if $0<\lambda<1$.

Proof. Define a function G for $m\in\mathbb{N}_0$ even and $x,y\in(-1,\infty)$ with $x\leq y$ by

$$G(x)=\frac{H^{(m)}(\lambda x)}{[H^{(m)}(x)]^{\lambda}},$$

where $\lambda>0$. Next, let $g(x)=\ln G(x)$. Then,

$$g'(x)=\lambda\left[\frac{H^{(m+1)}(\lambda x)}{H^{(m)}(\lambda x)}-\frac{H^{(m+1)}(x)}{H^{(m)}(x)}\right].$$

Suppose that $\lambda\geq 1$. Then since $h(x)$ is increasing, it follows that $g'(x)\geq 0$, which implies that $g(x)$ is increasing. As a result G is increasing and for $x\leq y$, we obtain $G(x)\leq G(y)$ which gives (18). The case for $0<\lambda<1$ follows the same procedure.

Theorem 2.9. *Let $m\in\mathbb{N}_0$, $u\in\mathbb{N}_0$, such that m and u are even and $m\geq u$. Then the Turan-type inequality*

$$\exp\left\{H^{(m-u)}(x)\right\}\cdot\exp\left\{H^{(m+u)}(x)\right\}\geq\left[\exp\left\{H^{(m)}(x)\right\}\right]^2 \tag{19}$$

holds for $x\in(-1,\infty)$.

Proof. By using (3), we obtain the following estimation.

$$\begin{aligned}
&\frac{H^{(m-u)}(x)}{2}+\frac{H^{(m+u)}(x)}{2}-H^{(m)}(x)\\
&=\frac{1}{2}\left[\int_0^{\frac{\pi}{2}}(\ln\cos t)^{m-u}\cos^x t\,dt+\int_0^{\frac{\pi}{2}}(\ln\cos t)^{m+u}\cos^x t\,dt\right]\\
&\quad-\int_0^{\frac{\pi}{2}}(\ln\cos t)^{m}\cos^x t\,dt\\
&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[\frac{1}{(\ln\cos t)^u}+(\ln\cos t)^u+2\right](\ln\cos t)^{m}\cos^x t\,dt\\
&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\left[1+(\ln\cos t)^u\right]^2(\ln\cos t)^{m-u}\cos^x t\,dt\\
&\geq 0.
\end{aligned}$$

Thus

$$\frac{H^{(m-u)}(x)}{2}+\frac{H^{(m+u)}(x)}{2}\geq H^{(m)}(x),$$

and by taking exponents, we obtain the result (19).

Theorem 2.10. *Let $m\in\mathbb{N}_0$ be even. Then the inequality*

$$H^{(m)}(x+y)\leq H^{(m)}(x)+H^{(m)}(y) \tag{20}$$

holds for $x,y\in[0,\infty)$. In other words, $H^{(m)}(x)$ is subadditive for $m\in\mathbb{N}_0$ even.

Proof. Let $a>1$, $b>1$ and $\frac{1}{a}+\frac{1}{b}=1$. Then by the Hölder's inequality, we obtain

$$\begin{aligned}
H^{(m)}(x+y)&=\int_0^{\frac{\pi}{2}}(\ln\cos t)^{\frac{m}{a}+\frac{m}{b}}\cos^{x+y} t\,dt\\
&=\int_0^{\frac{\pi}{2}}(\ln\cos t)^{\frac{m}{a}}\cos^{x} t\cdot(\ln\cos t)^{\frac{m}{b}}\cos^{y} t\,dt\\
&\leq\left(\int_0^{\frac{\pi}{2}}(\ln\cos t)^{m}\cos^{ax} t\,dt\right)^{\frac{1}{a}}\left(\int_0^{\frac{\pi}{2}}(\ln\cos t)^{m}\cos^{by} t\,dt\right)^{\frac{1}{b}}\\
&=\left(H^{(m)}(ax)\right)^{\frac{1}{a}}\left(H^{(m)}(by)\right)^{\frac{1}{b}}.
\end{aligned}$$

That is

$$H^{(m)}(x+y)\leq\left(H^{(m)}(ax)\right)^{\frac{1}{a}}\left(H^{(m)}(by)\right)^{\frac{1}{b}}. \tag{21}$$

Then by the Young's inequality:

$$S^{\frac{1}{a}}T^{\frac{1}{b}}\leq\frac{S}{a}+\frac{T}{b},$$

where $S,T\geq 0$, $a>1$, $\frac{1}{a}+\frac{1}{b}=1$, we obtain

$$\left(H^{(m)}(ax)\right)^{\frac{1}{a}}\left(H^{(m)}(by)\right)^{\frac{1}{b}}\leq\frac{H^{(m)}(ax)}{a}+\frac{H^{(m)}(by)}{b}. \tag{22}$$

Furthermore, since $a > 1$, $b > 1$ and $H^{(m)}(x)$ is decreasing for even m, we have
$$\frac{H^{(m)}(ax)}{a} + \frac{H^{(m)}(by)}{b} \leq H^{(m)}(x) + H^{(m)}(y). \tag{23}$$
Finally, by combining (21), (22) and (23), we obtain the result (20).

Remark 2.11. If $x = y$ in (20), then we obtain
$$H^{(m)}(2x) \leq 2H^{(m)}(x). \tag{24}$$
By repeated applications of (20) and (24), we obtain
$$H^{(m)}(nx) \leq nH^{(m)}(x), \quad n \in \mathbb{N}, \tag{25}$$
which implies that $H^{(m)}(x)$ is $\mathbb{N}$-subhomogeneous for $m \in \mathbb{N}_0$ even.

Remark 2.12. Note that $H^{(0)}(n) = I_n$ for $n \in \mathbb{N}$. Then as a special case, by letting $m = 0$, $x = r \in \mathbb{N}$ and $y = s \in \mathbb{N}$ in (20), we obtain
$$I_{r+s} \leq I_r + I_s$$
which implies that the Wallis' sequence I_n is subadditive.

Theorem 2.13. *Let $m \in \mathbb{N}_0$ be even. Then the inequality*
$$H^{(m)}(x)H^{(m)}(y) \leq C_m H^{(m)}(x+y) \tag{26}$$
holds for $x, y > 0$, where C_m is as defined by (4).

Proof. Let T be defined for $m \in \mathbb{N}_0$ even by
$$T(x,y) = \frac{H^{(m)}(x)H^{(m)}(y)}{H^{(m)}(x+y)}, \quad x > 0, y > 0,$$
and let $\delta(x,y) = \ln T(x,y)$. With no loss of generality, let y be fixed. Then,
$$\delta'(x,y) = \frac{H^{(m+1)}(x)}{H^{(m)}(x)} - \frac{H^{(m+1)}(x+y)}{H^{(m)}(x+y)} \leq 0,$$
since $\frac{H^{(m+1)}(x)}{H^{(m)}(x)}$ is increasing (see Corollary 2.7). Thus, $\delta(x,y)$ is decreasing and consequently, $T(x,y)$ is also decreasing. Then for $x > 0$, we obtain
$$\frac{H^{(m)}(x)H^{(m)}(y)}{H^{(m)}(x+y)} \leq H^{(m)}(0) = C_m,$$
which gives the result (26).

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