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IMPROVING SCHWARZ INEQUALITY IN INNER PRODUCT SPACES

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ABSTRACT. Some improvements of the celebrated Schwarz inequality in complex inner product spaces are given. Applications for n-tuples of complex numbers are provided.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|^2 \|y\|^2 \geq |\langle x, y\rangle|^2 \tag{1.1}$$

for any $x, y \in H$. The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$. This inequality can be written in an equivalent form as

$$\|x\| \|y\| \geq |\langle x, y\rangle| . \tag{1.2}$$

Assume that $P : H \to H$ is an orthogonal projection on H, namely, it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $B(H)$, the Banach algebra of all linear bounded operators on H, that $0 \leq P \leq 1_H$.

In the recent paper [5, Eq. (2.6)] we established among others that

$$\|x\| \|y\| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} + |\langle x, y\rangle - \langle Px, y\rangle| \tag{1.3}$$

for any $x, y \in H$. Since by the triangle inequality we have

$$|\langle x, y\rangle - \langle Px, y\rangle| \geq |\langle x, y\rangle| - |\langle Px, y\rangle|$$

and by the Schwarz inequality for nonnegative selfadjoint operators we have

$$\langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} \geq |\langle Px, y\rangle|$$

for any $x, y \in H$, then we get from (1.3) the following refinement of (1.2)

$$\|x\| \|y\| - |\langle x, y\rangle| \geq \langle Px, x\rangle^{1/2} \langle Py, y\rangle^{1/2} - |\langle Px, y\rangle| \geq 0 \tag{1.4}$$

for any $x, y \in H$.

In 1985 the author [1] (see also [2] or [4, p. 36]) established the following inequality related to Schwarz inequality

$$\left(\|x\|^2 \|z\|^2 - |\langle x, z\rangle|^2\right)\left(\|y\|^2 \|z\|^2 - |\langle y, z\rangle|^2\right) \geq \left|\langle x, y\rangle \|z\|^2 - \langle x, z\rangle \langle z, y\rangle\right|^2 \tag{1.5}$$

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for any x, y, $z \in H$ and obtained, as a consequence, the following refinement of (1.2):

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{1.6}$$

for any x, y, $e \in H$ with $\|e\| = 1$.

If we take the square root in (1.5) and use the triangle inequality, we get for x, y, $z \in H \setminus \{0\}$ that

$$\begin{aligned} &\left(\|x\|^2 \|z\|^2 - |\langle x, z\rangle|^2\right)^{1/2} \left(\|y\|^2 \|z\|^2 - |\langle y, z\rangle|^2\right)^{1/2} \\ &\geq \left|\langle x, y\rangle \|z\|^2 - \langle x, z\rangle \langle z, y\rangle\right| \geq |\langle x, z\rangle \langle z, y\rangle| - |\langle x, y\rangle| \|z\|^2 \end{aligned}$$

which by division with $\|x\| \|y\| \|z\|^2 \neq 0$ produces

$$\frac{|\langle x, y\rangle|}{\|x\| \|y\|} \geq \frac{|\langle x, z\rangle|}{\|x\| \|z\|} \frac{|\langle z, y\rangle|}{\|z\| \|y\|} - \sqrt{1 - \frac{|\langle x, z\rangle|^2}{\|x\|^2 \|z\|^2}} \sqrt{1 - \frac{|\langle y, z\rangle|^2}{\|y\|^2 \|z\|^2}}. \tag{1.7}$$

If the angle between the vectors x, y, $\Psi_{x,y} \in [0, \pi/2]$, is defined by [8]

$$\cos \Psi_{x,y} = \frac{|\langle x, y\rangle|}{\|x\| \|y\|},\ x,\ y \neq 0, \tag{1.8}$$

then the function $\Psi_{x,y}$ is a natural metric on complex projective space, since is satisfies the inequality [8]

$$\Psi_{x,y} \leq \Psi_{x,z} + \Psi_{z,y} \text{ for any } x,\ y,\ z \neq 0. \tag{1.9}$$

By using (1.8) we have by (1.7) that

$$\cos \Psi_{x,y} \geq \cos \Psi_{x,z} \cos \Psi_{z,x} - \sin \Psi_{x,z} \sin \Psi_{z,x} = \cos(\Psi_{x,z} + \Psi_{z,x}),$$

which is equivalent to (1.9) since the function cos is decreasing on $[0, \pi]$. This provides a different proof of (1.9) than the one from [8] where it was done by utilising the celebrated *Kreĭn's inequality* [7], see also [6, p. 56],

$$\Phi_{x,y} \leq \Phi_{x,z} + \Phi_{z,y} \text{ for any } x,\ y,\ z \neq 0, \tag{1.10}$$

obtained for angles $\Phi_{x,y}$ between two vectors x, y, where in this case $\Phi_{x,y}$ is defined by

$$\cos \Phi_{x,y} = \frac{\operatorname{Re} \langle x, y\rangle}{\|x\| \|y\|},\ x,\ y \neq 0.$$

The following inequality has been obtained by Wang and Zhang in [10] (see also [11, p. 195])

$$\sqrt{1 - \frac{|\langle x, y\rangle|^2}{\|x\|^2 \|y\|^2}} \leq \sqrt{1 - \frac{|\langle x, z\rangle|^2}{\|x\|^2 \|z\|^2}} + \sqrt{1 - \frac{|\langle y, z\rangle|^2}{\|y\|^2 \|z\|^2}} \tag{1.11}$$

for any x, y, $z \in H \setminus \{0\}$. Using the above notations the inequality (1.11) can be written as [8]

$$\sin \Psi_{x,y} \leq \sin \Psi_{x,z} + \sin \Psi_{z,y} \tag{1.12}$$

for any x, y, $z \in H \setminus \{0\}$. It also provides another triangle type inequality complementing the Kreĭn and Lin inequalities above.

The corresponding result for the angle $\Phi_{x,y}$ was obtained by Lin in [8] as

$$\sin \Phi_{x,y} \leq \sin \Phi_{x,z} + \sin \Phi_{z,y},\ \text{for any } x,\ y,\ z \neq 0, \tag{1.13}$$

or, equivalently, as

$$\sqrt{1-\frac{|\mathrm{Re}\langle x,y\rangle|^2}{\|x\|^2\|y\|^2}}\le\sqrt{1-\frac{|\mathrm{Re}\langle x,z\rangle|^2}{\|x\|^2\|z\|^2}}+\sqrt{1-\frac{|\mathrm{Re}\langle y,z\rangle|^2}{\|y\|^2\|z\|^2}} \tag{1.14}$$

for any $x,\ y,\ z\neq 0$.

In [8] the author has also shown that, in fact, the inequalities (1.11) and (1.14) can be extended for any power $p>2$, namely as

$$\left(1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p}\le\left(1-\frac{|\langle x,z\rangle|^p}{\|x\|^p\|z\|^p}\right)^{1/p}+\left(1-\frac{|\langle y,z\rangle|^p}{\|y\|^p\|z\|^p}\right)^{1/p} \tag{1.15}$$

and

$$\left(1-\frac{|\mathrm{Re}\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p}\le\left(1-\frac{|\mathrm{Re}\langle x,z\rangle|^p}{\|x\|^p\|z\|^p}\right)^{1/p}+\left(1-\frac{|\mathrm{Re}\langle y,z\rangle|^p}{\|y\|^p\|z\|^p}\right)^{1/p} \tag{1.16}$$

for any $x,\ y,\ z\neq 0$.

In this paper we obtain some improvements of Schwarz inequality in complex inner product spaces as follows. For various inequalities related to this famous result see the monographs [3] and [4].

2. Main Results

Employing Lin's inequalities (1.15) and (1.16) we can obtain the following refinements of Schwarz's inequality.

Theorem 1. *Let $x,\ y,\ e\in H$ with $\|e\|=1$ and $p\ge 2$. Then we have the following refinements of Schwarz inequality*

$$\|x\|^p\|y\|^p-|\langle x,y\rangle|^p\ge\left|\det\begin{bmatrix}\|x\| & \left(\|x\|^p-|\langle x,e\rangle|^p\right)^{1/p}\\ \|y\| & \left(\|y\|^p-|\langle y,e\rangle|^p\right)^{1/p}\end{bmatrix}\right|^p \tag{2.1}$$

and

$$\|x\|^p\|y\|^p-|\mathrm{Re}\langle x,y\rangle|^p\ge\left|\det\begin{bmatrix}\|x\| & \left(\|x\|^p-|\mathrm{Re}\langle x,e\rangle|^p\right)^{1/p}\\ \|y\| & \left(\|y\|^p-|\mathrm{Re}\langle y,e\rangle|^p\right)^{1/p}\end{bmatrix}\right|^p. \tag{2.2}$$

Proof. We observe that, by (1.15) and (1.16)

$$d_p(x,y):=\left(1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p}\text{ and }\delta_p(x,y):=\left(1-\frac{|\mathrm{Re}\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p}$$

are distances and by the continuity property of the distance d, namely

$$|d(x,z)-d(y,z)|\le d(x,y)$$

we get

$$\left|\left(1-\frac{|\langle x,e\rangle|^p}{\|x\|^p}\right)^{1/p}-\left(1-\frac{|\langle y,e\rangle|^p}{\|y\|^p}\right)^{1/p}\right|\le\left(1-\frac{|\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p} \tag{2.3}$$

and

$$\left|\left(1-\frac{|\mathrm{Re}\langle x,e\rangle|^p}{\|x\|^p}\right)^{1/p}-\left(1-\frac{|\mathrm{Re}\langle y,e\rangle|^p}{\|y\|^p}\right)^{1/p}\right|\le\left(1-\frac{|\mathrm{Re}\langle x,y\rangle|^p}{\|x\|^p\|y\|^p}\right)^{1/p} \tag{2.4}$$

for any x, $y \neq 0$ and $e \in H$ with $\|e\| = 1$.

If we take the power p in (2.3) and (2.4) and multiply with $\|x\|^p \|y\|^p > 0$, then we get the desired results (2.1) and (2.2). $\square$

The following similar result can be stated as well:

Theorem 2. *Let x, y, $e \in H$ with $\|e\| = 1$. Then we have the following refinement of Schwarz inequality*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \left(\det \begin{bmatrix} |\langle x, e\rangle| & \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \\ & \\ |\langle y, e\rangle| & \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} \end{bmatrix}\right)^2 . \tag{2.5}$$

Proof. We have by Schwarz's inequality that

$$\left|\left\langle x - \alpha e, y - \overline{\beta} e\right\rangle\right|^2 \leq \|x - \alpha e\|^2 \left\|y - \overline{\beta} e\right\|^2 \tag{2.6}$$

for any x, y, $e \in H$ and α, $\beta \in \mathbb{C}$.

Since $\|e\| = 1$, then

$$\left\langle x - \alpha e, y - \overline{\beta} e\right\rangle = \langle x, y\rangle - \alpha \langle e, y\rangle - \beta \langle x, e\rangle + \alpha\beta, \tag{2.7}$$

$$\|x - \alpha e\|^2 = \|x\|^2 - \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right)$$

and

$$\left\|y - \overline{\beta} e\right\|^2 = \|y\|^2 - \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right).$$

This implies that

$$\begin{aligned} & \|x - \alpha e\|^2 \left\|y - \overline{\beta} e\right\|^2 \\ & = \left[\|x\|^2 - \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right)\right] \left[\|y\|^2 - \left(2 \operatorname{Re}\left[\beta \langle x, e\rangle\right] - |\beta|^2\right)\right] \\ & = \|x\|^2 \|y\|^2 - \|y\|^2 \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) - \|x\|^2 \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & + \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & = \|x\|^2 \|y\|^2 - \left(\|y\|^2 - |\langle y, e\rangle|^2\right) \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) \\ & - \left(\|x\|^2 - |\langle x, e\rangle|^2\right) \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & - |\langle y, e\rangle|^2 \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) - |\langle x, e\rangle|^2 \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & + \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & + |\langle y, e\rangle|^2 |\langle x, e\rangle|^2 - |\langle y, e\rangle|^2 |\langle x, e\rangle|^2 \\ & = \|x\|^2 \|y\|^2 - \left(\|y\|^2 - |\langle y, e\rangle|^2\right) \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right) \\ & - \left(\|x\|^2 - |\langle x, e\rangle|^2\right) \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right) \\ & + \left[|\langle x, e\rangle|^2 - \left(2 \operatorname{Re}\left[\overline{\alpha} \langle x, e\rangle\right] - |\alpha|^2\right)\right] \left[|\langle y, e\rangle|^2 - \left(2 \operatorname{Re}\left[\beta \langle y, e\rangle\right] - |\beta|^2\right)\right] \\ & - |\langle y, e\rangle|^2 |\langle x, e\rangle|^2 . \end{aligned} \tag{2.8}$$

Observe that

$$|\langle x,e\rangle|^2-\left(2\operatorname{Re}\left[\overline{\alpha}\langle x,e\rangle\right]-|\alpha|^2\right)=|\langle x,e\rangle-\alpha|^2$$

and

$$|\langle y,e\rangle|^2-\left(2\operatorname{Re}\left[\beta\langle y,e\rangle\right]-|\beta|^2\right)=\left|\langle y,e\rangle-\overline{\beta}\right|^2.$$

Therefore, by (2.8) we get

$$\begin{aligned}\|x-\alpha e\|^2\left\|y-\overline{\beta}e\right\|^2 &\\ =\|x\|^2\|y\|^2&-\left(\|y\|^2-|\langle y,e\rangle|^2\right)\left(2\operatorname{Re}\left[\overline{\alpha}\langle x,e\rangle\right]-|\alpha|^2\right)\\ &-\left(\|x\|^2-|\langle x,e\rangle|^2\right)\left(2\operatorname{Re}\left[\beta\langle y,e\rangle\right]-|\beta|^2\right)\\ &+|\langle x,e\rangle-\alpha|^2\left|\langle y,e\rangle-\overline{\beta}\right|^2-|\langle y,e\rangle|^2|\langle x,e\rangle|^2.\end{aligned}\tag{2.9}$$

Let $\alpha\in\mathbb{C}$ with $\alpha\neq\langle x,e\rangle$ and put

$$\beta:=\frac{\alpha\overline{\langle y,e\rangle}}{\alpha-\langle x,e\rangle}.\tag{2.10}$$

Then

$$\begin{aligned}|\langle x,e\rangle-\alpha|^2\left|\langle y,e\rangle-\overline{\beta}\right|^2&=|\langle x,e\rangle-\alpha|^2\left|\langle y,e\rangle-\overline{\left(\frac{\alpha\overline{\langle y,e\rangle}}{\alpha-\langle x,e\rangle}\right)}\right|^2\\ &=|\langle x,e\rangle-\alpha|^2\left|\langle y,e\rangle-\frac{\overline{\alpha}\langle y,e\rangle}{\overline{\alpha}-\overline{\langle x,e\rangle}}\right|^2\\ &=|\langle y,e\rangle|^2|\langle x,e\rangle-\alpha|^2\left|\frac{\overline{\langle x,e\rangle}}{\overline{\alpha}-\overline{\langle x,e\rangle}}\right|^2\\ &=|\langle y,e\rangle|^2|\langle x,e\rangle|^2\end{aligned}$$

and

$$\alpha\beta=\alpha\langle e,y\rangle+\beta\langle x,e\rangle.$$

For these choices of α and β we have by (2.6)-(2.9) that

$$\begin{aligned}|\langle x,y\rangle|^2\le\|x\|^2\|y\|^2&-\left(\|y\|^2-|\langle y,e\rangle|^2\right)\left(2\operatorname{Re}\left[\overline{\alpha}\langle x,e\rangle\right]-|\alpha|^2\right)\\ &-\left(\|x\|^2-|\langle x,e\rangle|^2\right)\left(2\operatorname{Re}\left[\beta\langle y,e\rangle\right]-|\beta|^2\right).\end{aligned}\tag{2.11}$$

By (2.10) we also have

$$2\operatorname{Re}\left[\beta\langle y,e\rangle\right]-|\beta|^2=|\langle y,e\rangle|^2\left[2\operatorname{Re}\left[\frac{\alpha}{\alpha-\langle x,e\rangle}\right]-\left|\frac{\alpha}{\alpha-\langle x,e\rangle}\right|^2\right].\tag{2.12}$$

Take

$$\alpha=\langle x,e\rangle+t\text{ with }t\in\mathbb{R},\ t\neq0.\tag{2.13}$$

Then by (2.12) we have

$$\begin{aligned} B(x,y,e,t) &:= \left(\|y\|^2 - |\langle y,e\rangle|^2\right)\left(2\operatorname{Re}\left[\overline{\alpha}\langle x,e\rangle\right] - |\alpha|^2\right) \\ &\quad + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(2\operatorname{Re}\left[\beta\langle y,e\rangle\right] - |\beta|^2\right) \\ &= \left(\|y\|^2 - |\langle y,e\rangle|^2\right)\left(2\operatorname{Re}\left[\left(\overline{\langle x,e\rangle + t}\right)\langle x,e\rangle\right] - |\langle x,e\rangle + t|^2\right) \\ &\quad + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)|\langle y,e\rangle|^2\left[2\operatorname{Re}\left[\frac{\langle x,e\rangle + t}{t}\right] - \left|\frac{\langle x,e\rangle + t}{t}\right|^2\right]. \end{aligned}$$

Since

$$\begin{aligned} & 2\operatorname{Re}\left[\left(\overline{\langle x,e\rangle + t}\right)\langle x,e\rangle\right] - |\langle x,e\rangle + t|^2 \\ &= 2\operatorname{Re}\left[|\langle x,e\rangle|^2 + t\langle x,e\rangle\right] - |\langle x,e\rangle|^2 - 2t\operatorname{Re}\langle x,e\rangle - t^2 \\ &= |\langle x,e\rangle|^2 - t^2 \end{aligned}$$

and

$$\begin{aligned} & 2\operatorname{Re}\left[\frac{\langle x,e\rangle + t}{t}\right] - \left|\frac{\langle x,e\rangle + t}{t}\right|^2 \\ &= 2\operatorname{Re}\left[\frac{\langle x,e\rangle}{t} + 1\right] - \left|\frac{\langle x,e\rangle}{t} + 1\right|^2 \\ &= \frac{2\operatorname{Re}\langle x,e\rangle}{t} + 2 - \frac{|\langle x,e\rangle|^2}{t^2} - \frac{2\operatorname{Re}\langle x,e\rangle}{t} - 1 = 1 - \frac{|\langle x,e\rangle|^2}{t^2}, \end{aligned}$$

then we get

$$\begin{aligned} B(x,y,e,t) &= \left(\|y\|^2 - |\langle y,e\rangle|^2\right)\left(|\langle x,e\rangle|^2 - t^2\right) \\ &\quad + \left(\|x\|^2 - |\langle x,e\rangle|^2\right)|\langle y,e\rangle|^2\left(1 - \frac{|\langle x,e\rangle|^2}{t^2}\right) \\ &= \left(\|y\|^2 - |\langle y,e\rangle|^2\right)\left(|\langle x,e\rangle|^2 - t^2\right) \\ &\quad - \left(\|x\|^2 - |\langle x,e\rangle|^2\right)|\langle y,e\rangle|^2\left(\frac{|\langle x,e\rangle|^2 - t^2}{t^2}\right) \\ &= \left(|\langle x,e\rangle|^2 - t^2\right)\left[\|y\|^2 - |\langle y,e\rangle|^2 - \frac{\left(\|x\|^2 - |\langle x,e\rangle|^2\right)|\langle y,e\rangle|^2}{t^2}\right] \end{aligned}$$

for $t \in \mathbb{R}$, $t \neq 0$.

Assume that $\langle x,e\rangle, \langle y,e\rangle \neq 0$ and $\|x\| \neq |\langle x,e\rangle|, \|y\| \neq |\langle y,e\rangle|$.

If we take $t = t_0 \neq 0$ with

$$t_0^2 = |\langle x,e\rangle\langle y,e\rangle|\sqrt{\frac{\|x\|^2 - |\langle x,e\rangle|^2}{\|y\|^2 - |\langle y,e\rangle|^2}}$$

then we get

$$\begin{aligned}
&B(x,y,e,t_0)\\
&=\left(|\langle x,e\rangle|^2-|\langle x,e\rangle\langle y,e\rangle|\sqrt{\frac{\|x\|^2-|\langle x,e\rangle|^2}{\|y\|^2-|\langle y,e\rangle|^2}}\right)\\
&\times\left[\|y\|^2-|\langle y,e\rangle|^2-\frac{\left(\|x\|^2-|\langle x,e\rangle|^2\right)|\langle y,e\rangle|^2}{|\langle x,e\rangle\langle y,e\rangle|\sqrt{\frac{\|x\|^2-|\langle x,e\rangle|^2}{\|y\|^2-|\langle y,e\rangle|^2}}}\right]\\
&=\frac{|\langle x,e\rangle|}{\sqrt{\|y\|^2-|\langle y,e\rangle|^2}}\left(|\langle x,e\rangle|\sqrt{\|y\|^2-|\langle y,e\rangle|^2}-|\langle y,e\rangle|\sqrt{\|x\|^2-|\langle x,e\rangle|^2}\right)\\
&\times\left[\frac{|\langle x,e\rangle|\sqrt{\|y\|^2-|\langle y,e\rangle|^2}-\sqrt{\|x\|^2-|\langle x,e\rangle|^2}\,|\langle y,e\rangle|}{|\langle x,e\rangle|}\right]\sqrt{\|y\|^2-|\langle y,e\rangle|^2}\\
&=\left(|\langle x,e\rangle|\sqrt{\|y\|^2-|\langle y,e\rangle|^2}-|\langle y,e\rangle|\sqrt{\|x\|^2-|\langle x,e\rangle|^2}\right)^2.
\end{aligned}$$

By using the inequality (2.11) we then have

$$\begin{aligned}
|\langle x,y\rangle|^2&\le\|x\|^2\|y\|^2-B(x,y,e,t_0)\\
&=\|x\|^2\|y\|^2-\left(|\langle x,e\rangle|\sqrt{\|y\|^2-|\langle y,e\rangle|^2}-|\langle y,e\rangle|\sqrt{\|x\|^2-|\langle x,e\rangle|^2}\right)^2,
\end{aligned}$$

which proves the desired result (2.5).

Now, if $\langle x,e\rangle=0$ i.e. $x\perp e$, then (2.5) becomes

$$|\langle y,e\rangle|^2\|x\|^2+|\langle x,y\rangle|^2\le\|x\|^2\|y\|^2$$

which is trivial for $x=0$ and becomes the Bessel's inequality

$$|\langle y,e\rangle|^2+\left|\left\langle y,\frac{x}{\|x\|}\right\rangle\right|^2\le\|y\|^2$$

for the orthonormal family $\left\{e,\frac{x}{\|x\|}\right\}$.

A similar argument applies for $\langle y,e\rangle=0$.

Also, if $\|x\|^2=|\langle x,e\rangle|^2$ then by the equality case in Schwarz inequality for the vectors x and e we get that there exists a constant γ such that $x=\gamma e$. In this situation (2.5) becomes an equality.

A similar argument applies if $\|y\|^2=|\langle y,e\rangle|^2$. □

Remark 1. *If* $(H,\langle\cdot,\cdot\rangle)$ *is a complex inner product space, then* $(H,\langle\cdot,\cdot\rangle_r)$ *with*

$$\langle x,y\rangle_r:=\operatorname{Re}\langle x,y\rangle$$

is a real inner product space and $\langle x,x\rangle^{1/2}=\langle x,x\rangle_r^{1/2}=\|x\|$ *for* $x\in H$. *Therefore by (2.5) for* $\langle\cdot,\cdot\rangle_r$ *we get*

$$\|x\|^2\|y\|^2-|\mathrm{Re}\langle x,y\rangle|^2 \geq \left(\det\begin{bmatrix} |\mathrm{Re}\langle x,e\rangle| & \left(\|x\|^2-|\mathrm{Re}\langle x,e\rangle|^2\right)^{1/2} \\ |\mathrm{Re}\langle y,e\rangle| & \left(\|y\|^2-|\mathrm{Re}\langle y,e\rangle|^2\right)^{1/2}\end{bmatrix}\right)^2 \tag{2.14}$$

for any x, y, $e\in H$ *with* $\|e\|=1$.

3. An Application for n-Tuples of Complex Numbers

Let $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$, $e=(e_1,...,e_n)\in\mathbb{C}^n$ with $\sum_{k=1}^n|e_k|^2=1$. Then by writing the above inequalities (2.1) and (2.5) for the inner product $\langle x,y\rangle:=\sum_{k=1}^n x_k\overline{y}_k$ we have for $p\geq 2$, that

$$\left(\sum_{k=1}^n|x_k|^2\right)^{p/2}\left(\sum_{k=1}^n|y_k|^2\right)^{p/2}-\left|\sum_{k=1}^n x_k\overline{y}_k\right|^p \geq \left|\det\begin{bmatrix} \left(\sum_{k=1}^n|x_k|^2\right)^{1/2} & \left(\left(\sum_{k=1}^n|x_k|^2\right)^{p/2}-\left|\sum_{k=1}^n x_k\overline{e}_k\right|^p\right)^{1/p} \\ \left(\sum_{k=1}^n|y_k|^2\right)^{1/2} & \left(\left(\sum_{k=1}^n|y_k|^2\right)^{p/2}-\left|\sum_{k=1}^n y_k\overline{e}_k\right|^p\right)^{1/p}\end{bmatrix}\right|^p, \tag{3.1}$$

and

$$\sum_{k=1}^n|x_k|^2\sum_{k=1}^n|y_k|^2-\left|\sum_{k=1}^n x_k\overline{y}_k\right|^2 \geq \left(\det\begin{bmatrix} \left|\sum_{k=1}^n x_k\overline{e}_k\right| & \left(\sum_{k=1}^n|x_k|^2-\left|\sum_{k=1}^n x_k\overline{e}_k\right|^2\right)^{1/2} \\ \left|\sum_{k=1}^n y_k\overline{e}_k\right| & \left(\sum_{k=1}^n|y_k|^2-\left|\sum_{k=1}^n y_k\overline{e}_k\right|^2\right)^{1/2}\end{bmatrix}\right)^2. \tag{3.2}$$

If we take $e_m=1$ for $m\in\{1,...,n\}$ and $e_k=0$ for any $k\in\{1,...,n\}$, $k\neq m$, then $\sum_{k=1}^n|e_k|^2=1$ and by (3.1) and (3.2) we get

$$\left(\sum_{k=1}^n|x_k|^2\right)^{p/2}\left(\sum_{k=1}^n|y_k|^2\right)^{p/2}-\left|\sum_{k=1}^n x_k\overline{y}_k\right|^p \geq \max_{m\in\{1,...,n\}}\left|\det\begin{bmatrix} \left(\sum_{k=1}^n|x_k|^2\right)^{1/2} & \left(\left(\sum_{k=1}^n|x_k|^2\right)^{p/2}-|x_m|^p\right)^{1/p} \\ \left(\sum_{k=1}^n|y_k|^2\right)^{1/2} & \left(\left(\sum_{k=1}^n|y_k|^2\right)^{p/2}-|y_m|^p\right)^{1/p}\end{bmatrix}\right|^p, \tag{3.3}$$

and

$$\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2-\left|\sum_{k=1}^{n}x_k\overline{y}_k\right|^2 \tag{3.4}$$
$$\geq \max_{m\in\{1,...,n\}}\left(\det\begin{bmatrix}|x_m| & \left(\sum_{1\leq k\neq m\leq n}|x_k|^2\right)^{1/2}\\ |y_m| & \left(\sum_{1\leq k\neq m\leq n}|y_k|^2\right)^{1/2}\end{bmatrix}\right)^2.$$

For $p=2$ we get from (3.3) the simpler inequality

$$\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2-\left|\sum_{k=1}^{n}x_k\overline{y}_k\right|^2 \tag{3.5}$$
$$\geq \max_{m\in\{1,...,n\}}\left(\det\begin{bmatrix}\left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2} & \left(\sum_{1\leq k\neq m\leq n}|x_k|^2\right)^{1/2}\\ \left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2} & \left(\sum_{1\leq k\neq m\leq n}|y_k|^2\right)^{1/2}\end{bmatrix}\right)^2.$$

If we take $e_k=\frac{1}{\sqrt{n}}$ for $k\in\{1,...,n\}$, then $\sum_{k=1}^{n}|e_k|^2=1$ and by (3.1) and (3.2) we get

$$\left(\sum_{k=1}^{n}|x_k|^2\right)^{p/2}\left(\sum_{k=1}^{n}|y_k|^2\right)^{p/2}-\left|\sum_{k=1}^{n}x_k\overline{y}_k\right|^p \tag{3.6}$$
$$\geq n^p\left|\det\begin{bmatrix}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^2\right)^{1/2} & \left(\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^2\right)^{p/2}-\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right|^p\right)^{1/p}\\ \left(\frac{1}{n}\sum_{k=1}^{n}|y_k|^2\right)^{1/2} & \left(\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|^2\right)^{p/2}-\left|\frac{1}{n}\sum_{k=1}^{n}y_k\right|^p\right)^{1/p}\end{bmatrix}\right|^p$$

and

$$\sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2-\left|\sum_{k=1}^{n}x_k\overline{y}_k\right|^2 \tag{3.7}$$
$$\geq n^2\left(\det\begin{bmatrix}\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^2-\left|\frac{1}{n}\sum_{k=1}^{n}x_k\right|^2\right)^{1/2}\\ \left|\frac{1}{n}\sum_{k=1}^{n}y_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n}|y_k|^2-\left|\frac{1}{n}\sum_{k=1}^{n}y_k\right|^2\right)^{1/2}\end{bmatrix}\right)^2.$$

The inequality (3.7) has been obtained recently for real numbers by S. G. Walker in [9], where some interesting applications for the celebrated Cramer-Rao inequality are provided as well.

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References

[1] S. S. Dragomir, Some refinements of Schwarz inequality, Simpozionul de Matematici şi Aplicaţii, Timişoara, Romania, 1-2 Noiembrie 1985, 13–16. ZBL 0594.46018.
[2] S. S. Dragomir and I. Sándor, Some inequalities in pre-Hilbertian spaces. *Studia Univ. Babeş-Bolyai Math.* **32** (1987), no. 1, 71–78.
[3] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Grüss and Bessel Type in Inner Product Spaces*. Nova Science Publishers, Inc., Hauppauge, NY, 2005. viii+249 pp. ISBN: 1-59454-202-3.
[4] S. S. Dragomir, *Advances in Inequalities of the Schwarz, Triangle and Heisenberg Type in Inner Product Spaces*. Nova Science Publishers, Inc., New York, 2007. xii+243 pp. ISBN: 978-1-59454-903-8; 1-59454-903-6.
[5] S. S. Dragomir, Buzano's inequality holds for any projection, *Bull. Aust. Math. Soc.* **93** (2016), 504–510.
[6] K. E. Gustafson and D. K. M. Rao, *Numerical Range,* Springer-Verlag, New York, Inc., 1997.
[7] M. K. Kreĭn, Angular localization of the spectrum of a multiplicative integral in a Hilbert space, *Funct. Anal. Appl.* **3** (1969), 89–90.
[8] M. Lin, Remarks on Kreĭn's inequality, *The Math. Intelligencer*, **34** (2012), No.1, 3-4.
[9] S. G. Walker, A self-improvement to the Cauchy–Schwarz inequality, *Statistics and Probability Letters* **122** (2017), 86–89.
[10] B. Wang, F. Zhang, A trace inequality for unitary matrices, *Amer. Math. Monthly* **101** (1994), 453–455.
[11] F. Zhang, *Matrix Theory: Basic Results and Techniques*, Springer-Verlag, New York, 2011.

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