# Perfect Square Sum Magic Squares

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**Abstract**

*This paper shows how to create magic squares with a perfect square number for the total sum of their entries. This has been done in two ways: Firstly, by using the sum of consecutive odd numbers, and secondly, by using the sum of consecutive natural numbers, and secondly, by using consecutive natural numbers. In the first case, for all orders of magic squares, one can always obtain a perfect square sum. In the second case, magic squares with perfect square magic sums do exist, but only for odd order magic squares. For the even order magic squares, such as 4, 6, 8, etc. it is not possible to write consecutive natural number magic squares with perfect square sums of their entries. A simplified idea is introduced to check when it is possible to obtain minimum perfect square sums. Also, a uniform method is presented so that, if $k$ is the order of a magic square, then the magic sum of the square is $k^3$, and the sum of all entries of the magic square is $k^4$. Examples are given for the magic squares of orders 3 to 25.*

*An equation means nothing to me unless it expresses a thought of God.*
**– S. Ramanujan** (1887-1920)[22]

## Contents



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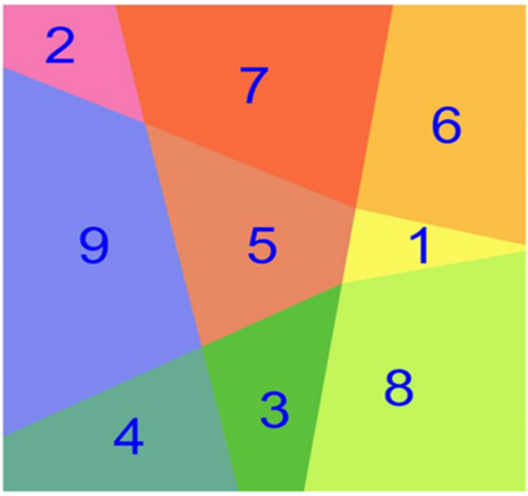
# 1 Introduction

Recently, William Walkington [24] started an interesting discussion as to how to create magic squares with cells that had the same areas as their numbers. Below is a graphic design for a 2017 seasonal greetings card, showing a magic square with approximate areas that was constructed by William Walkington (2016):




William Walkington - 2016

Figure 1

Lee Sallows (2017) [5] also constructed another magic square representing the areas as rectangles. See below:


Lee Sallows - 2017

Figure 2

The sum of all the numbers is given by

$$1+2+3+4+5+6+7+8+9=45. \tag{1}$$

The number 45 is not a perfect square. If we make a slight change, then we can transform the sum into a perfect square:

$$5+6+7+8+9+10+11+12+13=81=9^2 \tag{2}$$

Using this number sequence, Walter Trump (2017) [23] was able to construct the following area magic square:

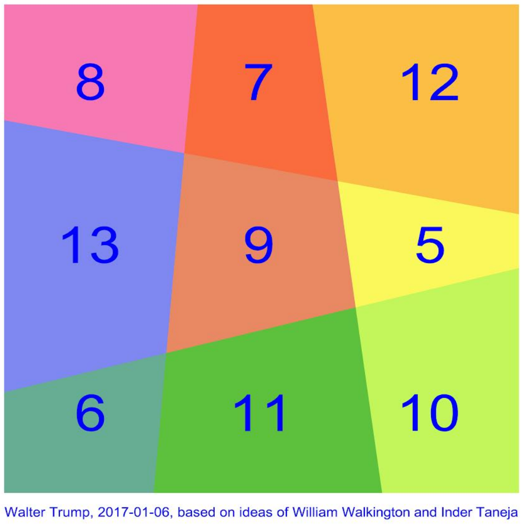



Figure 3

Adding 4 to each number in (1), we obtain the numbers given in (2). Observing area-wise the Figures 1 and 3, there is a considerable difference: For example, from numbers 1 to 2, the cell area is doubled, while from numbers 5 to 6, there is proportionally less increase between the cell areas.

In order to construct a magic square with cell areas that are in proportion to their numbers it is not necessary that the numbers always sum to a perfect square. Below is another example constructed by William Walkington (2017) [24] with sequential numbers from 3 to 11:

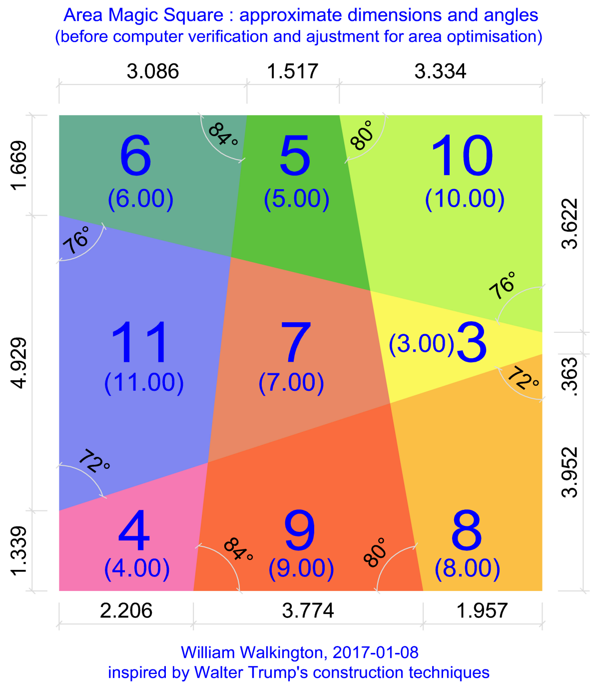



Figure 4

We observe that in Figures 1, 3 and 4, the number cell areas are proportional and aligned in both directions. In Figure 2, the proportionality of the areas is only present in one direction, which is horizontal.

Below are two examples of classical order 4 magic squares with cell areas that are proportional to their numbers:

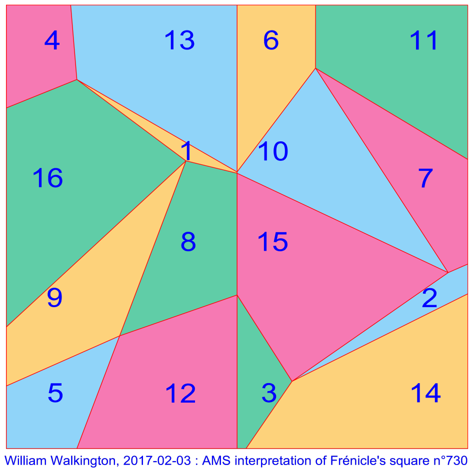



Figure 5

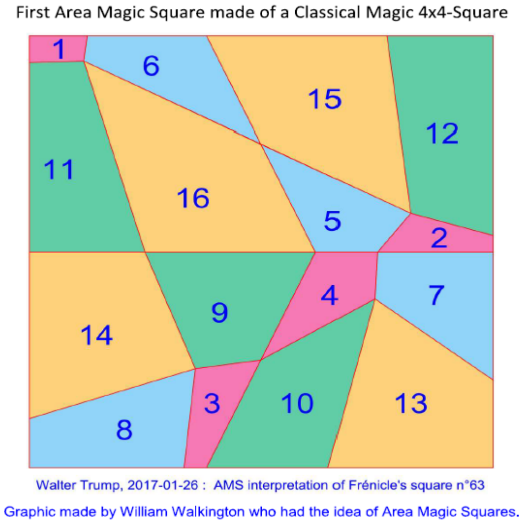



Figure 6

More examples of similar kinds of order 4 area magic squares, together with order 6 area magic squares, can be seen in William Walkington's pages [24]. From equation (2), the question arises, how to create higher order magic squares such that the sum of numbers is always a perfect square. **Based on this idea, in this paper we establish some formulas, so that sum of the numbers in magic squares always becomes a perfect square**.

## 2 Series and Magic Square Sums

This section presents some basic ideas of series and magic square sums.

### 2.1 Natural Number Series Sums

It is well-known that the positive natural number series sum is given by

$$T_n := 1 + 2 + 3 + ... + n = \frac{n(n+1)}{2}, \; n \geq 1 \tag{3}$$

The sequence $T_n$ is also famous for showing **Pascal's triangle values**

## 2.2 Odd Number Series Sums

It is well-known that sum of **odd-number series** is given by

$$F_n := 1 + 3 + 5 + ... + (2n-1) = n^2, \ n \geq 1, \tag{4}$$

Since we are working with magic squares, let us write $n = k^2$, then we have

$$F_{k^2} := 1 + 3 + 5 + ... + (2k^2-1) = k^4, \ k \geq 1, \tag{5}$$

The number $k$ is the order of a magic square. Thus for all $k \geq 3$ we can always have odd numbers for the entries of a magic square with the sum of all numbers being a perfect square.

## 2.3 Magic Square Sums and Sums of the Entries

It is well known that the magic sum of a magic square of order $k$ with elements $1, 2, 3, \ldots k^2$ is given by

$$S_{k\times k} := \frac{k(k^2+1)}{2}, \ k \geq 1, \tag{6}$$

The sum of all of the entries of a magic square is given by

$$F_{k^2} := \frac{k^2\,(k^2+1)}{2}, \ k \geq 1, \tag{7}$$

## 2.4 Perfect Square Sums with Consecutive Odd Numbers

Below is shown a general formula for writing **perfect square sum magic squares** with consecutive natural numbers.

Let us consider

$$G := T(n) - T(n-1) = k\left(n - \frac{k-1}{2}\right), \ n \geq k, \tag{8}$$

a) For even order entries magic squares $k = 2p$:

$$G := k\left(n - \frac{2p-1}{2}\right). \tag{9}$$

In this case, the expression $n - \frac{2p-1}{2}$ is never a natural number.

**Thus, there is no magic square of consecutive even numbers with a perfect square sum for all of its entries.**

b) For odd order magic squares, $k = 2p+1$

$$G := T(n) - T(n-k) = k\left(n - \frac{2p+1-1}{2}\right) = k(n-p), \tag{10}$$

In this case, we can always find a natural number, such that $k(n-p)$ is a perfect square with $n - p \geq k$. By considering $n - p = k$, we obtain $G := k^2$, and the sum of all the numbers is $F_{k^2} := k^4$.

**Result 1.** *Thus, from equations (5) and (10), we conclude that there are at least two ways of writing magic squares with all entries summing to a perfect square:*

*(i) Magic squares with consecutive odd numbers starting from 1;*

*(ii) Magic squares with consecutive natural numbers.*

For the first case (i), the author [21] has recently calculated magic squares with odd numbers not starting from 1 using the idea of Pythagorean triples to obtain a perfect square sum magic squares.

For the second case (ii), the subsection below gives a procedure that can be used to obtain a perfect square sum of all the entries of a magic square.

## 2.5 Procedure

Before considering examples, let us analyse some simple formulas. After simplifying equation (8), one can write

$$\begin{aligned} G(n,k) &= T(n) - T(n-k) \\ &= (n-k+1) + (n-k+3) + \cdots + n \\ &= k\left(n - \frac{k-1}{2}\right), \; n \geq k, \\ &= \frac{k}{2}(2n-k+1), \; n \; \geq k \end{aligned} \tag{11}$$

Since we are working with magic squares and $k$ is the total number of elements in a magic square it should also be the square of the order of the magic square. Therefore if p is the order of magic square then $k = p^2$. In order to obtain a perfect square for $G(n,k)$ we should have

$$\sqrt{G(n,k)} = t, \; t \in \mathbf{N}_+, \; n \; \geq k \tag{12}$$

For each case, we find a natural number $t$.

### 2.5.1 Uniformity

From equation (10), we observe that the first element of the sequence is $n-k+1$ and the last is $n$. In case of uniformity always choose

$$n = k + \frac{k-1}{2} = \frac{3k-1}{2}$$

This gives

$$G = T\left(\frac{3k-1}{2}\right) - T\left(\frac{k-1}{2}\right) = k^2, \; n \geq k$$

Since $k$ is the total number of elements, let us consider $p$ as the order of a magic square, and then the first and last members of the sequence are $F_1 := \frac{p^2+1}{2}$ and $F_{p^2} := \frac{3p^2-1}{2}$ respectively. In this case, the sum of the sequence is given by

$$F_1 + F_2 + \ldots + F_{p^2} := p^4.$$

For simplicity, let's represent the **uniformity** as,

$$\left\langle p, p^2, p^3, p^4 \right\rangle \tag{13}$$

where

$$\begin{aligned} p &\rightarrow \text{order of a magic square;} \\ p^2 &\rightarrow \text{total number of entries;} \\ p^3 &\rightarrow \text{magic square sum;} \\ p^4 &\rightarrow \text{sum of all the entries of a magic square.} \end{aligned}$$

The above process gives the results in a **uniform way, but it does not guarantee a minimum perfect square sum for the entries of a magic square**.

### 2.5.2 General Case

Let $p$ be an order of a magic square, i.e., $k = p^2$. Then from equations (11) or (12), we have

$$p\sqrt{\frac{2n-p^2+1}{2}} = p\sqrt{n-\frac{p^2-1}{2}} = t,\ t \in N_+,\ n \ \geq p^2,$$

The sum of all entries of a magic square is given by

$$F_1 + F_1 + \ldots + F_{p^2}$$

where $F_1 = n - p^2 + 1$ and $F_{p^2} := n$.

Again, let's consider, $n = m^2 + \frac{p^2-1}{2}$, and then we have

$$G(m,p) := p\sqrt{\left(m^2+\frac{p^2-1}{2}\right)-\left(\frac{p^2-1}{2}\right)} = t,\ t \in N_+,\ n \ \geq p^2. \tag{14}$$

In this case, how do we find the minimum value of $m$? For simplicity, let's write,

$$L(n,k) := \left(\sqrt{k}, n-k+1, n, \frac{G(n,k)^2}{\sqrt{k}}, G(n,k)^2, \sqrt{G(n,k)}\right), \tag{15}$$

where $n = m^2 + \frac{p^2-1}{2}$, $k = p^2$ and $p$ is the order of a magic square.

From above we observe that, if $\sqrt{G(n,k)}$ equals the order of a magic square then, we have a **uniformity case**. This only happens when $m = p$. And, first positive entry obtained from $m$ give a minimum perfect square sum (see the results given in 3).

For simplicity, let's write $L(a,b,c,d,e,f)$, for the representations given in expression (15), where

$$\begin{aligned} a &\rightarrow \text{Order of magic square;} \\ b &\rightarrow \text{First member of a sequence;} \\ c &\rightarrow \text{Last member of a sequence;} \\ d &\rightarrow \text{Sum of magic square;} \\ e &\rightarrow \text{Sum of all entries of a magic square;} \\ f &\rightarrow \text{Uniformity, if } a = f. \end{aligned}$$

$$\ldots \tag{16}$$

# 3 Minimum Perfect Square Sums and Uniformity

The examples that follow are of **odd order magic squares** written according to the notations (16) and are based on the equation (15). First example in each case is the **minimum perfect square sum**. The **uniformity cases** are represented as

$$L(a,b,c,d,e,f) \Rightarrow \left\langle p, p^2, p^3, p^4 \right\rangle,$$

where $\left\langle p, p^2, p^3, p^4 \right\rangle$ is expressed as in (13).

**Remark 1.** *In all cases, we use consecutive natural numbers. Amongst the magic squares of orders 3 to 51, magic square of orders 7 and 41 are the only ones starting with the number 1 that yield a minimum perfect square sum. According to Sloane [6], the next magic square with consecutive natural numbers starting with the number 1 that yields a minimum perfect square sum is of order 239. The magic square of order 3 is the only one where the first value of m gives a minimum perfect square sum and has the uniformity property. It should be noted that we are only examining positive entries for the magic squares. We can also obtain perfect square sum magic squares with some negative entries, but these cases are not studied here.*

- **Order 3**

1. $L\left(2^2+\dfrac{3^2-1}{2},\,3^2\right) \rightarrow \left(3,0,8,12,36,2\sqrt{6}\right)$
2. $L\left(3^2+\dfrac{3^2-1}{2},\,3^2\right) \rightarrow (3,5,13,27,81,3) \Rightarrow \left\langle 3,3^2,3^3,3^4\right\rangle$
3. $L\left(4^2+\dfrac{3^2-1}{2},\,3^2\right) \rightarrow \left(3,12,20,48,144,2\sqrt{3}\right)$

- **Order 5**

1. $L\left(4^2+\dfrac{5^2-1}{2},\,5^2\right) \rightarrow \left(5,4,28,80,400,2\sqrt{5}\right)$
2. $L\left(5^2+\dfrac{5^2-1}{2},\,5^2\right) \rightarrow (5,13,37,125,625,5) \Rightarrow \left\langle 5,5^2,5^3,5^4\right\rangle$
3. $L\left(6^2+\dfrac{5^2-1}{2},\,5^2\right) \rightarrow \left(5,24,48,180,900,\sqrt{30}\right)$

- **Order 7**

1. $L\left(5^2+\dfrac{7^2-1}{2},\,7^2\right) \rightarrow \left(7,1,49,175,1225,\sqrt{35}\right)$
2. $L\left(6^2+\dfrac{7^2-1}{2},\,7^2\right) \rightarrow \left(7,12,60,252,1764,\sqrt{42}\right)$
3. $L\left(7^2+\dfrac{7^2-1}{2},\,7^2\right) \rightarrow (7,25,73,343,2401,7) \Rightarrow \left\langle 7,7^2,7^3,7^4\right\rangle$

- **Order 9**

1. $L\left(7^2+\dfrac{9^2-1}{2},\,9^2\right) \rightarrow \left(9,9,89,441,3969,3\sqrt{7}\right)$
2. $L\left(8^2+\dfrac{9^2-1}{2},\,9^2\right) \rightarrow \left(9,24,104,576,5184,6\sqrt{2}\right)$
3. $L\left(9^2+\dfrac{9^2-1}{2},\,9^2\right) \rightarrow (9,41,121,729,6561,9) \Rightarrow \left\langle 9,9^2,9^3,9^4\right\rangle$

- **Order 11**

1. $L\left(8^2+\frac{11^2-1}{2},11^2\right)\rightarrow\left(11,4,124,704,7744,2\sqrt{22}\right)$
2. $L\left(9^2+\frac{11^2-1}{2},11^2\right)\rightarrow\left(11,21,141,891,9801,3\sqrt{11}\right)$
3. $L\left(10^2+\frac{11^2-1}{2},11^2\right)\rightarrow\left(11,40,160,1100,12100,\sqrt{110}\right)$
4. $L\left(11^2+\frac{11^2-1}{2},11^2\right)\rightarrow(11,61,181,1331,14641,11)\Rightarrow\left\langle 11,11^2,11^3,11^4\right\rangle$

## • Order 13

1. $L\left(10^2+\frac{13^2-1}{2},13^2\right)\rightarrow\left(13,16,184,1300,16900,\sqrt{130}\right)$
2. $L\left(11^2+\frac{13^2-1}{2},13^2\right)\rightarrow\left(13,37,205,1573,20449,\sqrt{143}\right)$
3. $L\left(12^2+\frac{13^2-1}{2},13^2\right)\rightarrow\left(13,60,228,1872,24336,2\sqrt{39}\right)$
4. $L\left(13^2+\frac{13^2-1}{2},13^2\right)\rightarrow(13,85,253,2197,28561,13)\Rightarrow\left\langle 13,13^2,13^3,13^4\right\rangle$

## • Order 15

1. $L\left(11^2+\frac{15^2-1}{2},15^2\right)\rightarrow\left(15,9,233,1815,27225,\sqrt{165}\right)$
2. $L\left(12^2+\frac{15^2-1}{2},15^2\right)\rightarrow\left(15,57,281,2535,38025,\sqrt{195}\right)$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

5. $L\left(15^2+\frac{15^2-1}{2},15^2\right)\rightarrow(15,113,337,3375,50625,15)\Rightarrow\left\langle 15,15^2,15^3,15^4\right\rangle$

## • Order 17

1. $L\left(13^2+\frac{17^2-1}{2},17^2\right)\rightarrow\left(17,25,313,2873,48841,\sqrt{221}\right)$
2. $L\left(14^2+\frac{17^2-1}{2},17^2\right)\rightarrow\left(17,52,340,3332,56644,\sqrt{238}\right)$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

5. $L\left(17^2+\frac{17^2-1}{2},17^2\right)\rightarrow\left(145,\ 433,\ 289=17^2,\ 83521=17^4\right)\Rightarrow\left\langle 17,17^2,17^3,17^4\right\rangle$

- **Order 19**

1. $L\left(14^2+\dfrac{19^2-1}{2},\ 19^2\right) \to \left(19, 16, 376, 3724, 70756, \sqrt{266}\right)$
2. $L\left(15^2+\dfrac{19^2-1}{2},\ 19^2\right) \to \left(19, 45, 405, 4275, 81225, \sqrt{285}\right)$

$\dots \quad \dots \qquad \dots \quad \dots$

6. $L\left(19^2+\dfrac{19^2-1}{2},\ 19^2\right) \to (19, 181, 541, 6859, 130321, 19) \Rightarrow \left\langle 19, 19^2, 19^3, 19^4\right\rangle$

- **Order 21**

1. $L\left(15^2+\dfrac{21^2-1}{2},\ 21^2\right) \to \left(21, 5, 445, 4725, 99225, 3\sqrt{35}\right)$
2. $L\left(16^2+\dfrac{21^2-1}{2},\ 21^2\right) \to \left(21, 36, 476, 5376, 112896, 4\sqrt{21}\right)$

$\dots \quad \dots \qquad \dots \quad \dots$

7. $L\left(21^2+\dfrac{21^2-1}{2},\ 21^2\right) \to (21, 221, 661, 9261, 194481, 21) \Rightarrow \left\langle 21, 21^2, 21^3, 21^4\right\rangle$

- **Order 23**

1. $L\left(17^2+\dfrac{23^2-1}{2},\ 23^2\right) \to \left(23, 25, 553, 6647, 152881, \sqrt{391}\right)$
2. $L\left(18^2+\dfrac{23^2-1}{2},\ 23^2\right) \to \left(23, 60, 588, 7452, 171396, 3\sqrt{46}\right)$

$\dots \quad \dots \qquad \dots \quad \dots$

7. $L\left(23^2+\dfrac{23^2-1}{2},\ 23^2\right) \to (23, 265, 793, 12167, 279841, 23) \Rightarrow \left\langle 23, 23^2, 23^3, 23^4\right\rangle$

- **Order 25**

1. $L\left(18^2+\dfrac{25^2-1}{2},\ 25^2\right) \to \left(25, 12, 636, 8100, 202500, 15\sqrt{2}\right)$
2. $L\left(19^2+\dfrac{25^2-1}{2},\ 25^2\right) \to \left(25, 49, 673, 9025, 225625, 5\sqrt{19}\right)$

$\dots \quad \dots \qquad \dots \quad \dots$

8. $L\left(25^2+\dfrac{25^2-1}{2},\ 25^2\right) \to (25, 313, 937, 15625, 390625, 25) \Rightarrow \left\langle 25, 25^2, 25^3, 25^4\right\rangle$

- **Order 27**

1. $L\left(20^2 + \frac{27^2-1}{2}, 27^2\right) \rightarrow \left(27, 36, 764, 10800, 291600, 6\sqrt{15}\right)$
2. $L\left(21^2 + \frac{27^2-1}{2}, 27^2\right) \rightarrow \left(27, 77, 805, 11907, 321489, 9\sqrt{7}\right)$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

8. $L\left(27^2 + \frac{27^2-1}{2}, 27^2\right) \rightarrow (27, 365, 1093, 19683, 531441, 27) \Rightarrow \left\langle 27, 27^2, 27^3, 27^4\right\rangle$

- **Order 29**

1. $L\left(21^2 + \frac{29^2-1}{2}, 29^2\right) \rightarrow \left(29, 21, 861, 12789, 370881, \sqrt{609}\right)$
2. $L\left(22^2 + \frac{29^2-1}{2}, 29^2\right) \rightarrow \left(29, 64, 904, 14036, 407044, \sqrt{638}\right)$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

9. $L\left(29^2 + \frac{29^2-1}{2}, 29^2\right) \rightarrow (29, 421, 1261, 24389, 707281, 29) \Rightarrow \left\langle 29, 29^2, 29^3, 29^4\right\rangle$

- **Order 31**

1. $L\left(22^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow \left(31, 4, 964, 15004, 465124, \sqrt{682}\right)$
2. $L\left(23^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow \left(31, 49, 1009, 16399, 508369, \sqrt{713}\right)$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

10. $L\left(31^2 + \frac{31^2-1}{2}, 31^2\right) \rightarrow (31, 481, 1441, 29791, 923521, 31) \Rightarrow \left\langle 31, 31^2, 31^3, 31^4\right\rangle$

- **Order 33**

1. $L\left(24^2 + \frac{33^2-1}{2}, 33^2\right) \rightarrow \left(33, 32, 1120, 19008, 627264, 6\sqrt{22}\right)$
2. $L\left(25^2 + \frac{33^2-1}{2}, 33^2\right) \rightarrow \left(33, 81, 1169, 20625, 680625, 5\sqrt{33}\right)$

$$\cdots \quad \cdots \quad \cdots \quad \cdots$$

10. $L\left(33^2 + \frac{33^2-1}{2}, 33^2\right) \rightarrow (33, 545, 1633, 35937, 1185921, 33) \Rightarrow \left\langle 33, 33^2, 33^3, 33^4\right\rangle$

## • Order 35

$$
\begin{aligned}
&1. \quad L\left(25^2+\frac{35^2-1}{2},\,35^2\right) \to \left(35, 13, 1237, 21875, 765625, 5\sqrt{35}\right)\\
&2. \quad L\left(26^2+\frac{35^2-1}{2},\,35^2\right) \to \left(35, 64, 1288, 23660, 828100, \sqrt{910}\right)\\
&\qquad \cdots \quad \cdots \qquad \cdots \quad \cdots\\
&11. \quad L\left(35^2+\frac{35^2-1}{2},\,35^2\right) \to (35, 613, 1837, 42875, 1500625, 35) \Rightarrow \left\langle 35, 35^2, 35^3, 35^4\right\rangle
\end{aligned}
$$

## • Order 37

$$
\begin{aligned}
&1. \quad L\left(27^2+\frac{37^2-1}{2},\,37^2\right) \to \left(37, 45, 1413, 26973, 998001, 3\sqrt{111}\right)\\
&2. \quad L\left(28^2+\frac{37^2-1}{2},\,37^2\right) \to \left(37, 100, 1468, 29008, 1073296, 2\sqrt{259}\right)\\
&\qquad \cdots \quad \cdots \qquad \cdots \quad \cdots\\
&11. \quad L\left(37^2+\frac{37^2-1}{2},\,37^2\right) \to (37, 685, 2053, 50653, 1874161, 37) \Rightarrow \left\langle 37, 37^2, 37^3, 37^4\right\rangle
\end{aligned}
$$

## • Order 39

$$
\begin{aligned}
&1. \quad L\left(28^2+\frac{39^2-1}{2},\,39^2\right) \to \left(39, 24, 1544, 30576, 1192464, 2\sqrt{273}\right)\\
&2. \quad L\left(29^2+\frac{39^2-1}{2},\,39^2\right) \to \left(39, 81, 1601, 32799, 1279161, \sqrt{1131}\right)\\
&\qquad \cdots \quad \cdots \qquad \cdots \quad \cdots\\
&12. \quad L\left(39^2+\frac{39^2-1}{2},\,39^2\right) \to (39, 761, 2281, 59319, 2313441, 39) \Rightarrow \left\langle 39, 39^2, 39^3, 39^4\right\rangle
\end{aligned}
$$

## • Order 41

$$
\begin{aligned}
&1. \quad L\left(29^2+\frac{41^2-1}{2},\,41^2\right) \to \left(41, 1, 1681, 34481, 1413721, \sqrt{1189}\right)\\
&2. \quad L\left(30^2+\frac{41^2-1}{2},\,41^2\right) \to \left(41, 60, 1740, 36900, 1512900, \sqrt{1230}\right)\\
&\qquad \cdots \quad \cdots \qquad \cdots \quad \cdots\\
&13. \quad L\left(41^2+\frac{41^2-1}{2},\,41^2\right) \to (41, 841, 2521, 68921, 2825761, 41) \Rightarrow \left\langle 41, 41^2, 41^3, 41^4\right\rangle
\end{aligned}
$$

- **Order 43**

1. $L\left(31^2+\frac{43^2-1}{2},\,43^2\right)\rightarrow\left(43,37,1885,41323,1776889,\sqrt{1333}\right)$
2. $L\left(32^2+\frac{43^2-1}{2},\,43^2\right)\rightarrow\left(43,100,1948,44032,1893376,4\sqrt{86}\right)$

$\ldots\quad\ldots\quad\ldots\quad\ldots$

13. $L\left(43^2+\frac{43^2-1}{2},\,43^2\right)\rightarrow(43,925,2773,79507,3418801,43)\Rightarrow\left\langle 43,43^2,43^3,43^4\right\rangle$

- **Order 45**

1. $L\left(32^2+\frac{45^2-1}{2},\,45^2\right)\rightarrow\left(45,12,2036,46080,2073600,12\sqrt{10}\right)$
2. $L\left(33^2+\frac{45^2-1}{2},\,45^2\right)\rightarrow\left(45,144,2168,52020,2340900,3\sqrt{170}\right)$

$\ldots\quad\ldots\quad\ldots\quad\ldots$

14. $L\left(45^2+\frac{45^2-1}{2},\,45^2\right)\rightarrow(45,1013,3037,91125,4100625,45)\Rightarrow\left\langle 45,45^2,45^3,45^4\right\rangle$

- **Order 47**

1. $L\left(34^2+\frac{47^2-1}{2},\,47^2\right)\rightarrow\left(47,52,2260,54332,2553604,\sqrt{1598}\right)$
2. $L\left(35^2+\frac{47^2-1}{2},\,47^2\right)\rightarrow\left(47,121,2329,57575,2706025,\sqrt{1645}\right)$

$\ldots\quad\ldots\quad\ldots\quad\ldots$

14. $L\left(47^2+\frac{47^2-1}{2},\,47^2\right)\rightarrow(47,1105,3313,103823,4879681,47)\Rightarrow\left\langle 47,47^2,47^3,47^4\right\rangle$

- **Order 49**

1. $L\left(35^2+\frac{49^2-1}{2},\,49^2\right)\rightarrow\left(49,25,2425,60025,2941225,7\sqrt{35}\right)$
2. $L\left(36^2+\frac{49^2-1}{2},\,49^2\right)\rightarrow(49,96,2496,63504,3111696,42)$

$\ldots\quad\ldots\quad\ldots\quad\ldots$

15. $L\left(49^2+\frac{49^2-1}{2},\,49^2\right)\rightarrow(49,1201,3601,117649,5764801,49)\Rightarrow\left\langle 49,49^2,49^3,49^4\right\rangle$

## • Order 51

1. $L\left(37^2+\frac{51^2-1}{2},\ 51^2\right) \to \left(51,69,2669,69819,3560769,\sqrt{1887}\right)$
2. $L\left(38^2+\frac{51^2-1}{2},\ 51^2\right) \to \left(51,144,2744,73644,3755844,\sqrt{1938}\right)$

$\ldots \quad \ldots \quad \ldots \quad \ldots$

15. $L\left(51^2+\frac{51^2-1}{2},\ 51^2\right) \to (51,1301,3901,132651,6765201,51) \Rightarrow \left\langle 51,51^2,51^3,51^4\right\rangle$

# 4 Perfect Square Sum Magic Squares: Examples

Magic squares are constructed following two different methods. The first method uses the formula given in equation (5). In this case, all the magic squares have odd number entries starting from 1, and there are always perfect square sum solutions. The second method, uniquely for odd-orders, uses the expressions given in section 3. In these cases, the construction is presented in two ways: The first of these is for the uniformity cases written section 3, and the second of these is for minimum perfect square sum, i.e., the first listed in each case in section 3. The examples below are for magic squares of orders 3 to 25.

## 4.1 Magic Squares of Order 3

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 3 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. Below, examples of both types are presented.

### 4.1.1 First Approach: Consecutive Odd Numbers

Taking $k=3$ in (7), we get

$$1+3+5+\ldots+(2\times 3^2-1)=3^4$$
$$\Rightarrow 1+3+5+\ldots+17=9^2=3^4 \tag{17}$$

**Example 1.** *According to the numbers given in equation (17), we obtain following magic square of order 3:*

| | | | |
|---|---|---|---|
| | | | 27 |
| 3 | 13 | 11 | 27 |
| 17 | 9 | 1 | 27 |
| 7 | 5 | 15 | 27 |
| 27 | 27 | 27 | 27 |

In this case, we have a magic sum $S_{3\times3}=27=3^3$ and the sum of all entries $F_9=3\times27=81=9^2=3^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\left\langle 3,3^2,3^3,3^4\right\rangle$.

### 4.1.2 Second Approach: Consecutive Natural Numbers

According to equation (10),

$$G:=\frac{n(n+1)}{2}-\frac{(n-9)(n-8)}{2}=9(n-4)$$

Taking $n = 13$, we get a perfect square, i.e.,

$$G := T(13) - T(4) = \frac{13 \times 14}{2} - \frac{4 \times 5}{2} = 9 \times 8 = 81$$

Simplifying, we get

$$5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 81$$

**Example 2.** *According to the values given above, there is a perfect sum magic square of order 3 given by*

| | | | 27 |
|---|---|---|---|
| 6 | 11 | 10 | 27 |
| 13 | 9 | 5 | 27 |
| 8 | 7 | 12 | 27 |
| 27 | 27 | 27 | 27 |

In this case also, we have a magic sum $S_{3\times 3} = 27 = 3^3$ and the sum of all entries $F_9 = 3 \times 27 = 81 = 9^2 = 3^4$. The perfect square sum obtained above satisfy the uniformity property 13 given in subsection 2.5.1, but it is not minimum. See following subsection for the minimum perfect square sum magic square.

### 4.1.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of order 3 magic square given in section 3:

$$L\left(2^2 + \frac{3^2-1}{2}, 3^2\right) \to \left(3, 0, 8, 12, 36, 2\sqrt{6}\right)$$

In this case, we have magic square of order 3 with a perfect square sum $36 = 6^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below a magic square of order 3, where sum of all entries is a minimum perfect square.

**Example 3.** *A magic square of order 3 with perfect square sum is given by*

| | | | 12 |
|---|---|---|---|
| 1 | 6 | 5 | 12 |
| 8 | 4 | 0 | 12 |
| 3 | 2 | 7 | 12 |
| 12 | 12 | 12 | 12 |

In this case the magic sum is $S_{3\times 3} = 12$, and the sum of all entries is $F_9 := 36 = 6^2$.

**Remark 2.** *By choosing $m = 2$ in equation (15), we get a minimum perfect square sum for a magic square given in example 3. The reason is if we choose $m = 1$, we get some entries as negative numbers, see below:*

$$L\left(1^2 + \frac{3^2-1}{2}, 3^2\right) \to \left(3, -3, 5, 3, 9, 2\sqrt{3}\right)$$

*This magic square is formed by entries $-3, -2, -1, 0, 1, 2, 3, 4, 5$ giving a perfect square sum as 9. Since it is formed by negative entries, it is excluded form our study. Our work is limited only to* ***natural numbers****.*

## 4.2 Magic Squares of Order 4

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 4 with odd number entries. Taking $k = 4$ in equation (5), we get

$$1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27 + 29 + 31 = 256 = 16^2$$

**Example 4.** *According to the values given above, the* ***pan diagonal magic square of order 4*** *is given by*

| | | | | | |
|---|---|---|---|---|---|
| | | 64 | 64 | 64 | 64 |
| | 13 | 23 | 1 | 47 | 64 |
| 64 | 3 | 25 | 15 | 21 | 64 |
| 64 | 31 | 5 | 19 | 9 | 64 |
| 64 | 17 | 11 | 29 | 7 | 64 |
| | 64 | 64 | 64 | 64 | 64 |

In this case, the magic sum is $S_{4\times4} := 64 = 4^3$, and the sum of the entries is $F_{16} := 256 = 16^2 = 4^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\langle 4, 4^2, 4^3, 4^4 \rangle$.

## 4.3 Magic Squares of Order 5

According to equations (5) and (10), one can obtain **perfect square sum magic squares** of order 5 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.3.1 First Approach: Consecutive Odd Numbers

Taking $k = 5$ in equation (5), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 5^2 - 1) = 5^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 47 + 49 \qquad = 625 = 25^2 = 5^4 \end{aligned} \tag{18}$$

**Example 5.** *According to the 25 values given in equation (18), the* ***pan diagonal magic square of order 5*** *is given by*

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 125 | 125 | 125 | 125 | 125 |
| | 1 | 17 | 23 | 39 | 45 | 125 |
| 125 | 33 | 49 | 5 | 11 | 27 | 125 |
| 125 | 15 | 21 | 37 | 43 | 9 | 125 |
| 125 | 47 | 3 | 19 | 25 | 31 | 125 |
| 125 | 29 | 35 | 41 | 7 | 13 | 125 |
| | 125 | 125 | 125 | 125 | 125 | 125 |

In this case, the magic sum is $S_{5\times5} := 125 = 5^3$, and the sum of the entries is $F_{25} := 625 = 25^2 = 5^4$.

### 4.3.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-25)(n-24)}{2} = 25(n-12)$$

Taking $n = 37$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(37) - T(12) = \frac{37 \times 38}{2} - \frac{12 \times 13}{2} &= 37 \times 19 - 6 \times 13 \\ &= 703 - 78 = 625 = 25^2 = 4 = 5^4. \end{aligned}$$

Simplifying, we get

$$13 + 14 + 15 + \ldots + 35 + 37 = 625 = 25^2 = 5^4$$

**Example 6.** *Below is a* ***perfect square sum pan diagonal magic square of order 5*** *for the entries 13 to 37:*

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 125 | 125 | 125 | 125 | 125 |
| | 13 | 21 | 24 | 32 | 35 | 125 |
| 125 | 29 | 37 | 15 | 18 | 26 | 125 |
| 125 | 20 | 23 | 31 | 34 | 17 | 125 |
| 125 | 36 | 14 | 22 | 25 | 18 | 125 |
| 125 | 27 | 30 | 33 | 16 | 19 | 125 |
| | 125 | 125 | 125 | 125 | 125 | 125 |

In this case also, the magic sum is $S_{5\times5} = 125 = 5^3$, and the sum of all the entries is $F_{25} := 625 = 5^4$.

Both the examples 5 and 6 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 5, 5^2, 5^3, 5^4 \rangle$.

### 4.3.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of the order 5 magic square given in section 3:

$$L\left(4^2 + \frac{5^2-1}{2}, 5^2\right) \rightarrow \left(5, 4, 28, 80, 400, 2\sqrt{5}\right)$$

In this case, we have magic square of order 5 with a perfect square sum of all entries $400 = 20^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below a **pan diagonal magic square of order 5**, where there is a minimum perfect square sum of all the entries.

**Example 7.** *A* ***pan diagonal magic square*** *of order 5 with perfect square sum is given by*

| | | | | | | |
|---|---|---|---|---|---|---|
| | | 80 | 80 | 80 | 80 | 80 |
| | 4 | 10 | 16 | 22 | 28 | 80 |
| 80 | 21 | 27 | 8 | 9 | 15 | 80 |
| 80 | 13 | 14 | 20 | 26 | 7 | 80 |
| 80 | 25 | 6 | 12 | 18 | 19 | 80 |
| 80 | 17 | 23 | 24 | 5 | 11 | 80 |
| | 80 | 80 | 80 | 80 | 80 | 80 |

In this case the magic sum is $S_{5\times5} = 80$, and the sum of all entries is $F_{25} := 400 = 20^2$.

### 4.4 Magic Squares of Order 6

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 6 with odd number entries. Taking $k = 6$ in equation (5), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 6^2 - 1) = 6^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 69 + 71 \qquad = 1296 = 36^2 = 6^4. \end{aligned} \tag{19}$$

**Example 8.** *According to the values given in equation (19), the magic square of order 6 is given by*

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | 216 |
| 1 | 45 | 55 | 67 | 33 | 15 | 216 |
| 57 | 13 | 69 | 27 | 41 | 9 | 216 |
| 23 | 11 | 25 | 53 | 61 | 43 | 216 |
| 63 | 31 | 7 | 47 | 19 | 49 | 216 |
| 37 | 65 | 21 | 5 | 59 | 29 | 216 |
| 35 | 51 | 39 | 17 | 3 | 71 | 216 |
| 216 | 216 | 216 | 216 | 216 | 216 | 216 |

In this case, the magic sum is $S_{6\times 6} := 216 = 6^3$, and the sum of the entries is $F_{36} := 1296 = 36^2 = 6^4$. This satisfies the property 13 of uniformity 2.5.1, i.e., $\langle 6, 6^2, 6^3, 6^4 \rangle$.

### 4.5 Magic Squares of Order 7

According to equations (5) and (10), one can obtain **perfect square sum magic squares** of order 7 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

#### 4.5.1 First Approach: Consecutive Odd Numbers

Taking $k = 7$ in equation (5), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 7^2 - 1) = 7^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 95 + 97 \qquad = 2401 = 49^2 = 7^4 \end{aligned} \tag{20}$$

**Example 9.** *According to the 49 values given in equation (20), the* ***pan diagonal magic square*** *of order 7 is given by*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 343 | 343 | 343 | 343 | 343 | 343 | 343 |
| | 1 | 17 | 33 | 49 | 65 | 81 | 97 | 343 |
| 343 | 79 | 95 | 13 | 15 | 31 | 47 | 63 | 343 |
| 343 | 45 | 61 | 77 | 93 | 11 | 27 | 29 | 343 |
| 343 | 25 | 41 | 43 | 59 | 75 | 91 | 9 | 343 |
| 343 | 89 | 7 | 23 | 39 | 55 | 57 | 73 | 343 |
| 343 | 69 | 71 | 87 | 5 | 21 | 37 | 53 | 343 |
| 343 | 35 | 51 | 67 | 83 | 85 | 3 | 19 | 343 |
| | 343 | 343 | 343 | 343 | 343 | 343 | 343 | 343 |

#### 4.5.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-49)(n-48)}{2} = 49(n-24)$$

Taking $n = 73$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(73) - T(24) = \frac{73 \times 74}{2} - \frac{24 \times 25}{2} &= 73 \times 37 - 12 \times 25 \\ &= 2701 - 300 = 2401 = 49^2 = 7^4. \end{aligned}$$

Simplifying, we get

$$25 + 26 + 27 + \ldots + 72 + 73 = 2401 = 49^2 = 7^4$$

This gives a perfect square sum for the 49 consecutive natural numbers from 25 to 73.

**Example 10.** *The **pan diagonal magic square of order 7** is given by*

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 343 | 343 | 343 | 343 | 343 | 343 | 343 |
| | 25 | 33 | 41 | 49 | 57 | 65 | 73 | 343 |
| 343 | 64 | 72 | 31 | 32 | 40 | 48 | 56 | 343 |
| 343 | 47 | 55 | 63 | 71 | 30 | 38 | 39 | 343 |
| 343 | 37 | 45 | 46 | 54 | 62 | 70 | 29 | 343 |
| 343 | 69 | 28 | 36 | 44 | 52 | 53 | 61 | 343 |
| 343 | 59 | 60 | 68 | 27 | 35 | 43 | 51 | 343 |
| 343 | 42 | 50 | 58 | 66 | 67 | 26 | 34 | 343 |
| | 343 | 343 | 343 | 343 | 343 | 343 | 343 | 343 |

In both the examples given above the magic sum is $S_{7\times7} = 343 = 7^3$, and the sum of all the entries is $F_{49} := 2401 = 49^2 = 7^4$.

Both the examples 9 and 10 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 7, 7^2, 7^3, 7^4 \rangle$.

#### 4.5.3 Third Approach: Minimum Perfect Square Sum

Let's observe the first example of the order 7 magic square given in section 3:

$$L\left(5^2 + \frac{7^2-1}{2}, 7^2\right) \to \left(7, 1, 49, 175, 1225, \sqrt{35}\right).$$

In this case, we have a magic square of order 7 with a perfect square sum of all entries $1225 = 175^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. See below **pan diagonal magic square** of order 7, where there is a minimum perfect square sum of all the entries:

**Example 11.** *The **pan diagonal magic square of order 7** is given by*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | 175 | 175 | 175 | 175 | 175 | 175 | 175 |
| | 1 | 9 | 17 | 25 | 33 | 41 | 49 | 175 |
| 175 | 40 | 48 | 7 | 8 | 16 | 24 | 32 | 175 |
| 175 | 23 | 31 | 39 | 47 | 6 | 14 | 15 | 175 |
| 175 | 13 | 21 | 22 | 30 | 38 | 46 | 5 | 175 |
| 175 | 45 | 4 | 12 | 20 | 28 | 29 | 37 | 175 |
| 175 | 35 | 36 | 44 | 3 | 11 | 19 | 27 | 175 |
| 175 | 18 | 26 | 34 | 42 | 43 | 2 | 10 | 175 |
| | 175 | 175 | 175 | 175 | 175 | 175 | 175 | 175 |

In this case the magic sum is $S_{7\times7} = 175$, and the sum of all entries is $F_{49} := 7 \times 175 = 1225 = 35^2$. This is the only example (amongst the orders 3 to 51) of a minimum perfect square sum with all entries starting from the number 1.

## 4.6 Bimagic Squares of Order 8

According to equation (9), we cannot obtain a consecutive natural number magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 8 with this property if we use odd number entries. Taking $k = 8$ in equation (5), we get

$$\begin{aligned} 1+3+5+\ldots+(2\times 8^2-1) &= 8^4 \\ \Rightarrow 1+3+5+\ldots+123+125+127 &= 4096 = 64^2 = 8^4. \end{aligned} \tag{21}$$

**Example 12.** *According to the values given in equation (21), the **pan diagonal magic square** of order 8 is given by*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | 512 | 512 | 512 | 512 | 512 | 512 | 512 | 512 |
| | 31 | 81 | 71 | 9 | 53 | 123 | 109 | 35 | 512 |
| 512 | 51 | 125 | 107 | 37 | 25 | 87 | 65 | 15 | 512 |
| 512 | 1 | 79 | 89 | 23 | 43 | 101 | 115 | 61 | 512 |
| 512 | 45 | 99 | 117 | 59 | 7 | 73 | 95 | 17 | 512 |
| 512 | 75 | 5 | 19 | 93 | 97 | 47 | 57 | 119 | 512 |
| 512 | 103 | 41 | 63 | 113 | 77 | 3 | 21 | 91 | 512 |
| 512 | 85 | 27 | 13 | 67 | 127 | 49 | 39 | 105 | 512 |
| 512 | 121 | 55 | 33 | 111 | 83 | 29 | 11 | 69 | 512 |
| | 512 | 512 | 512 | 512 | 512 | 512 | 512 | 512 | 512 |

In this case the magic sum is $S_{8\times8} = 512$, and the sum of the entries is $F_{64} = 4096 = 64^2 = 8^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{8\times8} = 43688$.

**Example 13.** *The bimagic square of order 8 using the entries of example 12 is given by*

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | 43688 |
| $31^2$ | $81^2$ | $71^2$ | $9^2$ | $53^2$ | $123^2$ | $109^2$ | $35^2$ | 43688 |
| $51^2$ | $125^2$ | $107^2$ | $37^2$ | $25^2$ | $87^2$ | $65^2$ | $15^2$ | 43688 |
| $1^2$ | $79^2$ | $89^2$ | $23^2$ | $43^2$ | $101^2$ | $115^2$ | $61^2$ | 43688 |
| $45^2$ | $99^2$ | $117^2$ | $59^2$ | $7^2$ | $73^2$ | $95^2$ | $17^2$ | 43688 |
| $75^2$ | $5^2$ | $19^2$ | $93^2$ | $97^2$ | $47^2$ | $57^2$ | $119^2$ | 43688 |
| $103^2$ | $41^2$ | $63^2$ | $113^2$ | $77^2$ | $3^2$ | $21^2$ | $91^2$ | 43688 |
| $85^2$ | $27^2$ | $13^2$ | $67^2$ | $127^2$ | $49^2$ | $39^2$ | $105^2$ | 43688 |
| $121^2$ | $55^2$ | $33^2$ | $111^2$ | $83^2$ | $29^2$ | $11^2$ | $69^2$ | 43688 |
| 43688 | 43688 | 43688 | 43688 | 43688 | 43688 | 43688 | 43688 | 43688 |

## 4.7 Bimagic Squares of Order 9

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 9 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.7.1 First Approach: Consecutive Odd Numbers

Taking $k = 9$ in equation (5), we get

$$1 + 3 + 5 + \ldots + (2 \times 9^2 - 1) = 9^4$$
$$\Rightarrow 1 + 3 + 5 + \ldots + 159 + 161 \quad = 6561 = 81^2 = 9^4. \tag{22}$$

**Example 14.** *According to the 81 values given in equation (22), the magic square of order 9 is given by*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 729 |
| 1 | 35 | 45 | 69 | 79 | 95 | 119 | 129 | 157 | 729 |
| 65 | 75 | 103 | 109 | 143 | 153 | 15 | 25 | 41 | 729 |
| 123 | 133 | 149 | 11 | 21 | 49 | 55 | 89 | 99 | 729 |
| 53 | 9 | 19 | 97 | 59 | 87 | 147 | 121 | 137 | 729 |
| 93 | 67 | 83 | 161 | 117 | 127 | 43 | 5 | 33 | 729 |
| 151 | 113 | 141 | 39 | 13 | 29 | 107 | 63 | 73 | 729 |
| 27 | 37 | 17 | 77 | 105 | 61 | 139 | 155 | 111 | 729 |
| 85 | 101 | 57 | 135 | 145 | 125 | 23 | 51 | 7 | 729 |
| 131 | 159 | 115 | 31 | 47 | 3 | 81 | 91 | 71 | 729 |
| 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 |

In this case the magic sum is $S_{9\times 9} = 729$, and the sum of the entries is $F_{81} = 6561 = 81^2 = 9^4$. This magic square is also **bimagic** and has a bimagic sum $Sb_{9\times 9} = 78729$. See below:

**Example 15.** *The bimagic square of order 9 using the entries of example 14 is given by*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 78729 |
| $1^2$ | $35^2$ | $45^2$ | $69^2$ | $79^2$ | $95^2$ | $119^2$ | $129^2$ | $157^2$ | 78729 |
| $65^2$ | $75^2$ | $103^2$ | $109^2$ | $143^2$ | $153^2$ | $15^2$ | $25^2$ | $41^2$ | 78729 |
| $123^2$ | $133^2$ | $149^2$ | $11^2$ | $21^2$ | $49^2$ | $55^2$ | $89^2$ | $99^2$ | 78729 |
| $53^2$ | $9^2$ | $19^2$ | $97^2$ | $59^2$ | $87^2$ | $147^2$ | $121^2$ | $137^2$ | 78729 |
| $93^2$ | $67^2$ | $83^2$ | $161^2$ | $117^2$ | $127^2$ | $43^2$ | $5^2$ | $33^2$ | 78729 |
| $151^2$ | $113^2$ | $141^2$ | $39^2$ | $13^2$ | $29^2$ | $107^2$ | $63^2$ | $73^2$ | 78729 |
| $27^2$ | $37^2$ | $17^2$ | $77^2$ | $105^2$ | $61^2$ | $139^2$ | $155^2$ | $111^2$ | 78729 |
| $85^2$ | $101^2$ | $57^2$ | $135^2$ | $145^2$ | $125^2$ | $23^2$ | $51^2$ | $7^2$ | 78729 |
| $131^2$ | $159^2$ | $115^2$ | $31^2$ | $47^2$ | $3^2$ | $81^2$ | $91^2$ | $71^2$ | 78729 |
| 78729 | 78729 | 78729 | 78729 | 78729 | 78729 | 78729 | 78729 | 78729 | 78729 |

### 4.7.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-81)(n-80)}{2} = 81(n-40)$$

Taking $n = 121$, we get a perfect square, i.e.,

$$G := T(121) - T(40) = \frac{121 \times 122}{2} - \frac{40 \times 42}{2} = 121 \times 61 - 20 \times 41$$
$$= 7381 - 820 = 6561 = 81^2 = 9^4.$$

Simplifying, we get

$$41 + 42 + 43 + \ldots + 120 + 121 = 6561 = 81^2 = 9^4$$

This gives a perfect square sum for the 81 entries using consecutive natural numbers from 41 to 121.

**Example 16.** *The magic square of order 9 using the above numbers from 41 to 121 is given by*

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | 729 |
| 41 | 58 | 63 | 75 | 80 | 88 | 100 | 105 | 119 | 729 |
| 73 | 78 | 92 | 95 | 112 | 117 | 48 | 53 | 61 | 729 |
| 102 | 107 | 115 | 46 | 51 | 65 | 68 | 85 | 90 | 729 |
| 67 | 45 | 50 | 89 | 70 | 84 | 114 | 101 | 109 | 729 |
| 87 | 74 | 82 | 121 | 99 | 104 | 62 | 43 | 57 | 729 |
| 116 | 97 | 111 | 60 | 47 | 55 | 94 | 72 | 77 | 729 |
| 54 | 59 | 49 | 79 | 93 | 71 | 110 | 118 | 96 | 729 |
| 83 | 91 | 69 | 108 | 113 | 103 | 52 | 66 | 44 | 729 |
| 106 | 120 | 98 | 56 | 64 | 42 | 81 | 86 | 76 | 729 |
| 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 | 729 |

In both the examples 14 and 16, the magic sum is $S_{9\times 9} = 729 = 9^3$, and the sum of all entries is $F_{81} := 6561 = 81^2 = 9^4$. This magic square shown above is **bimagic** with a bimagic sum $Sb_{9\times 9} = 63969$. We observe that although the examples 14 and 16 show the same magic sums, their bimagic sums are different.

Both the examples 14 and 16 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 9, 9^2, 9^3, 9^4 \rangle$.

### 4.7.3 Third Approach: Minimum Perfect Square Sum

The examples given in 14 and 16 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 9, 9^2, 9^3, 9^4 \rangle$. However, the total entries of the magic square example 16 do not yield a minimum perfect square sum. In order to get, minimum perfect square sum. In order to obtain a minimum perfect square sum, let's consider the first example of the magic square of order 9 given in section 3:

$$L\left(7^2 + \frac{9^2 - 1}{2}, 9^2\right) \to \left(9, 9, 89, 441, 3969, 3\sqrt{7}\right)$$

In this case, we have magic square of order 9 with a perfect square sum of all entries $3969 = 63^2$, but it does not satisfy the property (13) of uniformity given in subsection 2.5.1. See below the magic square of order 9 constructed according to details given above.

**Example 17.** *The magic square of order 9 with 81 consecutive entries from 9 to 89 is given by*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 441 | 441 | 441 | 441 | 441 | 441 | 441 | 441 | 441 |
| | 10 | 26 | 30 | 41 | 45 | 61 | 69 | 76 | 83 | 441 |
| 441 | 42 | 49 | 56 | 64 | 80 | 84 | 14 | 18 | 34 | 441 |
| 441 | 68 | 72 | 88 | 15 | 22 | 29 | 37 | 53 | 57 | 441 |
| 441 | 28 | 17 | 21 | 59 | 36 | 52 | 87 | 67 | 74 | 441 |
| 441 | 60 | 40 | 47 | 82 | 71 | 75 | 32 | 9 | 25 | 441 |
| 441 | 86 | 63 | 79 | 33 | 13 | 20 | 55 | 44 | 48 | 441 |
| 441 | 19 | 35 | 12 | 50 | 54 | 43 | 78 | 85 | 65 | 441 |
| 441 | 51 | 58 | 38 | 73 | 89 | 66 | 23 | 27 | 16 | 441 |
| 441 | 77 | 81 | 70 | 24 | 31 | 11 | 46 | 62 | 39 | 441 |
| | 441 | 441 | 441 | 441 | 441 | 441 | 441 | 441 | 441 | 441 |

The above example 17 has a with magic sum $S_{9\times 9} = 441$, and the sum of all entries is $F_{81} := 9 \times 441 = 3969 = 63^2$. It is **pan diagonal**, but **not bimagic**. It should be noted that the sum of each $3 \times 3$ block is the same for each of the three magic squares of examples 14, 16 and 17.

## 4.8 Magic Squares of Order 10

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 10 with this property if we use odd number entries. Taking $k = 10$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 10^2-1) = 10^4 \\ \Rightarrow &1+3+5+\ldots+197+199 \quad = 10000 = 100^2 = 10^4. \end{aligned} \tag{23}$$

**Example 18.** *According to the values given in equation (23), the magic square of order 10 is given by*

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | *1000* |
| *1* | *159* | *129* | *193* | *77* | *43* | *95* | *171* | *105* | *27* | *1000* |
| *195* | *23* | *17* | *131* | *179* | *147* | *109* | *65* | *81* | *53* | *1000* |
| *93* | *161* | *45* | *157* | *31* | *69* | *187* | *119* | *123* | *15* | *1000* |
| *139* | *113* | *175* | *67* | *3* | *181* | *57* | *29* | *151* | *85* | *1000* |
| *167* | *197* | *103* | *21* | *89* | *135* | *145* | *13* | *59* | *71* | *1000* |
| *25* | *75* | *87* | *19* | *153* | *111* | *163* | *41* | *189* | *137* | *1000* |
| *149* | *91* | *79* | *165* | *55* | *37* | *133* | *183* | *7* | *101* | *1000* |
| *117* | *47* | *191* | *83* | *121* | *5* | *39* | *155* | *73* | *169* | *1000* |
| *51* | *9* | *33* | *115* | *185* | *99* | *61* | *127* | *177* | *143* | *1000* |
| *63* | *125* | *141* | *49* | *107* | *173* | *11* | *97* | *35* | *199* | *1000* |
| *1000* | *1000* | *1000* | *1000* | *1000* | *1000* | *1000* | *1000* | *1000* | *1000* | *1000* |

The above example 18 has a magic sum $S_{10\times 10} = 1000$, and the sum of all entries is $F_{100} := 10 \times 1000 = 10000 = 100^2 = 10^4$. It satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 10, 10^2, 10^3, 10^4 \rangle$.

## 4.9 Magic Squares of Order 11

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.9.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 11^2-1) = 11^4 \\ \Rightarrow &1+3+5+\ldots+239+241 \quad = 14641 = 121^2 = 11^4 \end{aligned} \tag{24}$$

**Example 19.** *According to the 121 values given in equation (24), the* ***pan diagonal magic square*** *of order 11 is given by*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 |
| | 1 | 41 | 59 | 77 | 95 | 113 | 153 | 171 | 189 | 207 | 225 | 1331 |
| 1331 | 201 | 241 | 17 | 35 | 53 | 71 | 89 | 129 | 147 | 165 | 183 | 1331 |
| 1331 | 159 | 177 | 217 | 235 | 11 | 29 | 47 | 87 | 105 | 123 | 141 | 1331 |
| 1331 | 117 | 135 | 175 | 193 | 211 | 229 | 5 | 23 | 63 | 81 | 99 | 1331 |
| 1331 | 75 | 93 | 111 | 151 | 169 | 187 | 205 | 223 | 21 | 39 | 57 | 1331 |
| 1331 | 33 | 51 | 69 | 109 | 127 | 145 | 163 | 181 | 199 | 239 | 15 | 1331 |
| 1331 | 233 | 9 | 27 | 45 | 85 | 103 | 121 | 139 | 157 | 197 | 215 | 1331 |
| 1331 | 191 | 209 | 227 | 3 | 43 | 61 | 79 | 97 | 115 | 133 | 173 | 1331 |
| 1331 | 149 | 167 | 185 | 203 | 221 | 19 | 37 | 55 | 73 | 91 | 131 | 1331 |
| 1331 | 107 | 125 | 143 | 161 | 179 | 219 | 237 | 13 | 31 | 49 | 67 | 1331 |
| 1331 | 65 | 83 | 101 | 119 | 137 | 155 | 195 | 213 | 231 | 7 | 25 | 1331 |
| | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 |

### 4.9.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-121)(n-120)}{2} = 121(n-60)$$

Taking $n = 181$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(181) - T(60) = \frac{181 \times 182}{2} - \frac{60 \times 61}{2} &= 181 \times 91 - 30 \times 61 \\ &= 16471 - 1830 = 14641 = 121^2 = 11^4. \end{aligned}$$

Simplifying, we get

$$61 + 62 + 63 + \ldots + 180 + 181 = 14641 = 121^2 = 11^4.$$

This gives a perfect square sum for the 121 consecutive natural numbers from to 61 to 181.

**Example 20.** *A **pan diagonal magic square** of order 11 for 121 entries from 61 to 181 is given by*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 |
| | 61 | 81 | 90 | 99 | 108 | 117 | 137 | 146 | 155 | 164 | 173 | 1331 |
| 1331 | 161 | 181 | 69 | 78 | 87 | 96 | 105 | 125 | 134 | 143 | 152 | 1331 |
| 1331 | 140 | 149 | 169 | 178 | 66 | 75 | 84 | 104 | 113 | 122 | 131 | 1331 |
| 1331 | 119 | 128 | 148 | 157 | 166 | 175 | 63 | 72 | 92 | 101 | 110 | 1331 |
| 1331 | 98 | 107 | 116 | 136 | 145 | 154 | 163 | 172 | 71 | 80 | 89 | 1331 |
| 1331 | 77 | 86 | 95 | 115 | 124 | 133 | 142 | 151 | 160 | 180 | 68 | 1331 |
| 1331 | 177 | 65 | 74 | 83 | 103 | 112 | 121 | 130 | 139 | 159 | 168 | 1331 |
| 1331 | 156 | 165 | 174 | 62 | 82 | 91 | 100 | 109 | 118 | 127 | 147 | 1331 |
| 1331 | 135 | 144 | 153 | 162 | 171 | 70 | 79 | 88 | 97 | 106 | 126 | 1331 |
| 1331 | 114 | 123 | 132 | 141 | 150 | 170 | 179 | 67 | 76 | 85 | 94 | 1331 |
| 1331 | 93 | 102 | 111 | 120 | 129 | 138 | 158 | 167 | 176 | 64 | 73 | 1331 |
| | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 | 1331 |

In both the examples 19 and 20 given above the magic sum is $S_{11\times 11} = 1331 = 11^3$, and the sum of all numbers is $F_{121} := 14641 = 121^2 = 11^4$.

Both of the examples 19 and 20 satisfy the property (13) of uniformity 2.5.1, i.e., $\left\langle 11, 11^2, 11^3, 11^4 \right\rangle$.

### 4.9.3 Third Approach: Minimum Perfect Square Sum

The examples given in 19 and 20 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 11, 11^2, 11^3, 11^4 \rangle$. However, the sum of the magic square entries of 20 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 11 given in section 3:

$$L\left(8^2 + \frac{11^2-1}{2}, 11^2\right) \rightarrow \left(11, 4, 124, 704, 7744, 2\sqrt{22}\right)$$

In this case, we have a magic square of order 11 with sum of all entries $7744 = 88^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 21.** *A* ***pan diagonal magic square*** *of order 11 with 121 entries starting from 4 to 124 is given by*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 |
| | 4 | 24 | 33 | 42 | 51 | 60 | 80 | 89 | 98 | 107 | 116 | 704 |
| 704 | 104 | 124 | 12 | 21 | 30 | 39 | 48 | 68 | 77 | 86 | 95 | 704 |
| 704 | 83 | 92 | 112 | 121 | 9 | 18 | 27 | 47 | 56 | 65 | 74 | 704 |
| 704 | 62 | 71 | 91 | 100 | 109 | 118 | 6 | 15 | 35 | 44 | 53 | 704 |
| 704 | 41 | 50 | 59 | 79 | 88 | 97 | 106 | 115 | 14 | 23 | 32 | 704 |
| 704 | 20 | 29 | 38 | 58 | 67 | 76 | 85 | 94 | 103 | 123 | 11 | 704 |
| 704 | 120 | 8 | 17 | 26 | 46 | 55 | 64 | 73 | 82 | 102 | 111 | 704 |
| 704 | 99 | 108 | 117 | 5 | 25 | 34 | 43 | 52 | 61 | 70 | 90 | 704 |
| 704 | 78 | 87 | 96 | 105 | 114 | 13 | 22 | 31 | 40 | 49 | 69 | 704 |
| 704 | 57 | 66 | 75 | 84 | 93 | 113 | 122 | 10 | 19 | 28 | 37 | 704 |
| 704 | 36 | 45 | 54 | 63 | 72 | 81 | 101 | 110 | 119 | 7 | 16 | 704 |
| | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 | 704 |

The above example 21 has a magic sum $S_{11\times 11} = 704$, and the sum of all entries is $F_{121} := 11 \times 704 = 7744 = 88^2$. Moreover, it is **pan diagonal**.

## 4.10 Magic Squares of Order 12

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 12 with this property if we use odd number entries. Taking $k = 12$ in equation (5), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 12^2 - 1) = 12^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 285 + 287 \quad = 20736 = 144^2 = 12^4. \end{aligned} \tag{25}$$

**Example 22.** *According to the values given in equation (25), the magic square of order 12 is given by*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | 1728 |
| 135 | 165 | 3 | 273 | 89 | 211 | 53 | 223 | 109 | 191 | 25 | 251 | 1728 |
| 9 | 267 | 141 | 159 | 55 | 221 | 91 | 209 | 35 | 241 | 119 | 181 | 1728 |
| 285 | 15 | 153 | 123 | 235 | 65 | 199 | 77 | 263 | 37 | 179 | 97 | 1728 |
| 147 | 129 | 279 | 21 | 197 | 79 | 233 | 67 | 169 | 107 | 253 | 47 | 1728 |
| 85 | 215 | 49 | 227 | 111 | 189 | 27 | 249 | 137 | 163 | 5 | 271 | 1728 |
| 59 | 217 | 95 | 205 | 33 | 243 | 117 | 183 | 7 | 269 | 139 | 161 | 1728 |
| 239 | 61 | 203 | 73 | 261 | 39 | 177 | 99 | 283 | 17 | 151 | 125 | 1728 |
| 193 | 83 | 229 | 71 | 171 | 105 | 255 | 45 | 149 | 127 | 281 | 19 | 1728 |
| 113 | 187 | 29 | 247 | 133 | 167 | 1 | 275 | 87 | 213 | 51 | 225 | 1728 |
| 31 | 245 | 115 | 185 | 11 | 265 | 143 | 157 | 57 | 219 | 93 | 207 | 1728 |
| 259 | 41 | 175 | 101 | 287 | 13 | 155 | 121 | 237 | 63 | 201 | 75 | 1728 |
| 173 | 103 | 257 | 43 | 145 | 131 | 277 | 23 | 195 | 81 | 231 | 69 | 1728 |
| 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 | 1728 |

In the example 22 given above the magic sum is $S_{12\times12} = 1728 = 12^3$, and the sum of all entries is $F_{144} := 20736 = 144^2 = 12^4$. Thus , the example 22 satisfies the property (13) of uniformity 2.5.1, i.e., $(12, 12^2, 12^3, 12^4)$. Moreover, this is a **block-wise magic square**, i.e., each $4 \times 4$ block is a **pan diagonal magic square** with magic sum 576.

This **block-wise** construction of a magic square of order 12 is a combination of the construction procedures for magic squares of order 4 given in subsection 4.2 and for those of order 3 given in subsection 4.1.

## 4.11 Magic Squares of Order 13

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.11.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 13^2-1) = 13^4 \\ \Rightarrow &1+3+5+\ldots+335+337 \quad = 28561 = 169^2 = 13^4 \end{aligned} \tag{26}$$

**Example 23.** *According to the 169 values given in equation (26), the* ***pan diagonal magic square*** *of order 13 is given by*

| | | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 49 | 71 | 93 | 115 | 137 | 159 | 207 | 229 | 251 | 273 | 295 | 317 | 2197 |
| 2197 | 289 | 337 | 21 | 43 | 65 | 87 | 109 | 131 | 179 | 201 | 223 | 245 | 267 | 2197 |
| 2197 | 239 | 261 | 309 | 331 | 15 | 37 | 59 | 81 | 129 | 151 | 173 | 195 | 217 | 2197 |
| 2197 | 189 | 211 | 259 | 281 | 303 | 325 | 9 | 31 | 53 | 101 | 123 | 145 | 167 | 2197 |
| 2197 | 139 | 161 | 183 | 231 | 253 | 275 | 297 | 319 | 3 | 51 | 73 | 95 | 117 | 2197 |
| 2197 | 89 | 111 | 133 | 181 | 203 | 225 | 247 | 269 | 291 | 313 | 23 | 45 | 67 | 2197 |
| 2197 | 39 | 61 | 83 | 105 | 153 | 175 | 197 | 219 | 241 | 263 | 311 | 333 | 17 | 2197 |
| 2197 | 327 | 11 | 33 | 55 | 103 | 125 | 147 | 169 | 191 | 213 | 235 | 283 | 305 | 2197 |
| 2197 | 277 | 299 | 321 | 5 | 27 | 75 | 97 | 119 | 141 | 163 | 185 | 233 | 255 | 2197 |
| 2197 | 227 | 249 | 271 | 293 | 315 | 25 | 47 | 69 | 91 | 113 | 135 | 157 | 205 | 2197 |
| 2197 | 177 | 199 | 221 | 243 | 265 | 287 | 335 | 19 | 41 | 63 | 85 | 107 | 155 | 2197 |
| 2197 | 127 | 149 | 171 | 193 | 215 | 237 | 285 | 307 | 329 | 13 | 35 | 57 | 79 | 2197 |
| 2197 | 77 | 99 | 121 | 143 | 165 | 187 | 209 | 257 | 279 | 301 | 323 | 7 | 29 | 2197 |
| | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 |

### 4.11.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-169)(n-168)}{2} = 169(n-84)$$

Taking $n = 253$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(253) - T(84) = \frac{253 \times 254}{2} - \frac{84 \times 85}{2} &= 253 \times 127 - 42 \times 85 \\ &= 28561 = 169^2 = 13^4. \end{aligned}$$

Simplifying, we get

$$85 + 86 + 87 + \ldots + 252 + 253 = 28561 = 169^2 = 13^4.$$

This gives a perfect square sum for the 169 consecutive natural numbers from to 85 to 253.

**Example 24.** *The **pan diagonal magic square** of order 13 for the 169 entries from 85 to 253 is given by*

| | | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 85 | 109 | 120 | 131 | 142 | 153 | 164 | 188 | 199 | 210 | 221 | 232 | 243 | 2197 |
| 2197 | 229 | 253 | 95 | 106 | 117 | 128 | 139 | 150 | 174 | 185 | 196 | 207 | 218 | 2197 |
| 2197 | 204 | 215 | 239 | 250 | 92 | 103 | 114 | 125 | 149 | 160 | 171 | 182 | 193 | 2197 |
| 2197 | 179 | 190 | 214 | 225 | 236 | 247 | 89 | 100 | 111 | 135 | 146 | 157 | 168 | 2197 |
| 2197 | 154 | 165 | 176 | 200 | 211 | 222 | 233 | 244 | 86 | 110 | 121 | 132 | 143 | 2197 |
| 2197 | 129 | 140 | 151 | 175 | 186 | 197 | 208 | 219 | 230 | 241 | 96 | 107 | 118 | 2197 |
| 2197 | 104 | 115 | 126 | 137 | 161 | 172 | 183 | 194 | 205 | 216 | 240 | 251 | 93 | 2197 |
| 2197 | 248 | 90 | 101 | 112 | 136 | 147 | 158 | 169 | 180 | 191 | 202 | 226 | 237 | 2197 |
| 2197 | 223 | 234 | 245 | 87 | 98 | 122 | 133 | 144 | 155 | 166 | 177 | 201 | 212 | 2197 |
| 2197 | 198 | 209 | 220 | 231 | 242 | 97 | 108 | 119 | 130 | 141 | 152 | 163 | 187 | 2197 |
| 2197 | 173 | 184 | 195 | 206 | 217 | 228 | 252 | 94 | 105 | 116 | 127 | 138 | 162 | 2197 |
| 2197 | 148 | 159 | 170 | 181 | 192 | 203 | 227 | 238 | 249 | 91 | 102 | 113 | 124 | 2197 |
| 2197 | 123 | 134 | 145 | 156 | 167 | 178 | 189 | 213 | 224 | 235 | 246 | 88 | 99 | 2197 |
| | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 | 2197 |

In both the examples 23 and 24 given above the magic sum is $S_{13\times13} = 2197 = 13^3$, and the sum of all numbers is $F_{169} := 28561 = 169^2 = 13^4$.

Both the examples 23 and 24 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 13, 13^2, 13^3, 13^4 \rangle$.

### 4.11.3 Third Approach: Minimum Perfect Square Sum

The examples given in 23 and 24 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 13, 13^2, 13^3, 13^4 \rangle$. However, the sum of the magic square entries of 24 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 13 given in section 3:

$$L\left(10^2 + \frac{13^2 - 1}{2}, 13^2\right) \to \left(13, 16, 184, 1300, 16900, \sqrt{130}\right)$$

In this case, we have a magic square of order 13 with the sum of all entries $16900 = 130^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 25.** *A **pan diagonal** magic square of order 13 with the 169 entries starting from 16 to 184 is given by*

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 |
| | 16 | 40 | 51 | 62 | 73 | 84 | 95 | 119 | 130 | 141 | 152 | 163 | 174 | 1300 |
| 1300 | 160 | 184 | 26 | 37 | 48 | 59 | 70 | 81 | 105 | 116 | 127 | 138 | 149 | 1300 |
| 1300 | 135 | 146 | 170 | 181 | 23 | 34 | 45 | 56 | 80 | 91 | 102 | 113 | 124 | 1300 |
| 1300 | 110 | 121 | 145 | 156 | 167 | 178 | 20 | 31 | 42 | 66 | 77 | 88 | 99 | 1300 |
| 1300 | 85 | 96 | 107 | 131 | 142 | 153 | 164 | 175 | 17 | 41 | 52 | 63 | 74 | 1300 |
| 1300 | 60 | 71 | 82 | 106 | 117 | 128 | 139 | 150 | 161 | 172 | 27 | 38 | 49 | 1300 |
| 1300 | 35 | 46 | 57 | 68 | 92 | 103 | 114 | 125 | 136 | 147 | 171 | 182 | 24 | 1300 |
| 1300 | 179 | 21 | 32 | 43 | 67 | 78 | 89 | 100 | 111 | 122 | 133 | 157 | 168 | 1300 |
| 1300 | 154 | 165 | 176 | 18 | 29 | 53 | 64 | 75 | 86 | 97 | 108 | 132 | 143 | 1300 |
| 1300 | 129 | 140 | 151 | 162 | 173 | 28 | 39 | 50 | 61 | 72 | 83 | 94 | 118 | 1300 |
| 1300 | 104 | 115 | 126 | 137 | 148 | 159 | 183 | 25 | 36 | 47 | 58 | 69 | 93 | 1300 |
| 1300 | 79 | 90 | 101 | 112 | 123 | 134 | 158 | 169 | 180 | 22 | 33 | 44 | 55 | 1300 |
| 1300 | 54 | 65 | 76 | 87 | 98 | 109 | 120 | 144 | 155 | 166 | 177 | 19 | 30 | 1300 |
| | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 | 1300 |

The above example 25 has a magic sum $S_{11\times11} = 704$, and the sum of all entries is $F_{121} := 11 \times 704 = 7744 = 88^2$. Moreover, it is **pan diagonal**.

## 4.12 Magic Squares of Order 14

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 14 with this property if we use odd number entries. Taking $k = 14$ in equation (5), we get

$$\begin{aligned} & 1 + 3 + 5 + \ldots + (2 \times 14^2 - 1) = 14^4 \\ \Rightarrow & 1 + 3 + 5 + \ldots + 389 + 391 \quad = 38416 = 96^2 = 14^4. \end{aligned} \tag{27}$$

**Example 26.** *According to the values given in equation (27), the magic square of order 14 is given by*

| | | | | | | | | | | | | | | 2744 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 189 | 381 | 223 | 275 | 155 | 115 | 63 | 33 | 317 | 291 | 355 | 249 | 97 | 2744 |
| 293 | 31 | 219 | 351 | 375 | 253 | 173 | 233 | 7 | 105 | 139 | 165 | 75 | 325 | 2744 |
| 251 | 323 | 61 | 273 | 343 | 125 | 159 | 11 | 93 | 51 | 193 | 197 | 297 | 367 | 2744 |
| 379 | 221 | 299 | 91 | 309 | 229 | 9 | 153 | 67 | 339 | 45 | 191 | 279 | 133 | 2744 |
| 327 | 167 | 109 | 23 | 241 | 171 | 57 | 201 | 265 | 123 | 345 | 49 | 371 | 295 | 2744 |
| 207 | 19 | 177 | 73 | 41 | **331** | **357** | **307** | **389** | 147 | 87 | 267 | 117 | 225 | 2744 |
| 37 | 69 | 263 | 113 | 5 | **305** | **391** | **329** | **359** | 183 | 243 | 101 | 143 | 203 | 2744 |
| 89 | 129 | 141 | 179 | 71 | **385** | **303** | **361** | **335** | 237 | 259 | 3 | 205 | 47 | 2744 |
| 59 | 85 | 119 | 145 | 187 | **363** | **333** | **387** | **301** | 213 | 13 | 235 | 43 | 261 | 2744 |
| 135 | 287 | 311 | 53 | 149 | 95 | 209 | 185 | 239 | 271 | 365 | 83 | 21 | 341 | 2744 |
| 161 | 373 | 349 | 227 | 137 | 35 | 269 | 103 | 169 | 27 | 211 | 285 | 319 | 79 | 2744 |
| 277 | 347 | 15 | 321 | 283 | 215 | 231 | 29 | 157 | 369 | 77 | 121 | 107 | 195 | 2744 |
| 353 | 257 | 245 | 383 | 111 | 65 | 39 | 127 | 199 | 281 | 163 | 315 | 181 | 25 | 2744 |
| 175 | 247 | 55 | 289 | 217 | 17 | 99 | 255 | 131 | 81 | 313 | 377 | 337 | 151 | 2744 |
| 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 | 2744 |

In the example 26 given above, the magic sum is $S_{14\times 14} = 2744 = 14^3$, and the sum of all entries is $F_{196} := 38416 = 196^2 = 14^4$. Thus , the example 26 satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 14, 14^2, 14^3, 14^4 \rangle$. Moreover, the inner $4\times$ blocks are magic squares with magic sum $S_{4\times 4} = 1384$.

## 4.13 Magic Squares of Order 15

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 15 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.13.1 First Approach: Consecutive Odd Numbers

Taking $k = 15$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 15^2-1) = 15^4 \\ \Rightarrow &1+3+5+\ldots+447+449 \quad = 50625 = 225^2 = 15^4 \end{aligned} \tag{28}$$

**Example 27.** *According to the 225 values given in equation (28), the magic square of order 15 is given by*

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | 3375 |
| 3 | 171 | 189 | 357 | 375 | 65 | 109 | 247 | 299 | 437 | 31 | 143 | 221 | 325 | 403 | 3375 |
| 339 | 387 | 15 | 153 | 201 | 277 | 449 | 77 | 95 | 259 | 311 | 415 | 43 | 121 | 233 | 3375 |
| 165 | 183 | 351 | 369 | 27 | 107 | 245 | 289 | 427 | 89 | 133 | 211 | 323 | 401 | 55 | 3375 |
| 381 | 9 | 177 | 195 | 333 | 439 | 67 | 119 | 257 | 275 | 413 | 41 | 145 | 223 | 301 | 3375 |
| 207 | 345 | 363 | 21 | 159 | 269 | 287 | 425 | 79 | 97 | 235 | 313 | 391 | 53 | 131 | 3375 |
| 61 | 113 | 251 | 295 | 433 | 33 | 141 | 219 | 327 | 405 | 5 | 169 | 187 | 359 | 377 | 3375 |
| 281 | 445 | 73 | 91 | 263 | 309 | 417 | 45 | 123 | 231 | 337 | 389 | 17 | 155 | 199 | 3375 |
| 103 | 241 | 293 | 431 | 85 | 135 | 213 | 321 | 399 | 57 | 167 | 185 | 349 | 367 | 29 | 3375 |
| 443 | 71 | 115 | 253 | 271 | 411 | 39 | 147 | 225 | 303 | 379 | 7 | 179 | 197 | 335 | 3375 |
| 265 | 283 | 421 | 83 | 101 | 237 | 315 | 393 | 51 | 129 | 209 | 347 | 365 | 19 | 157 | 3375 |
| 35 | 139 | 217 | 329 | 407 | 1 | 173 | 191 | 355 | 373 | 63 | 111 | 249 | 297 | 435 | 3375 |
| 307 | 419 | 47 | 125 | 229 | 341 | 385 | 13 | 151 | 203 | 279 | 447 | 75 | 93 | 261 | 3375 |
| 137 | 215 | 319 | 397 | 59 | 163 | 181 | 353 | 371 | 25 | 105 | 243 | 291 | 429 | 87 | 3375 |
| 409 | 37 | 149 | 227 | 305 | 383 | 11 | 175 | 193 | 331 | 441 | 69 | 117 | 255 | 273 | 3375 |
| 239 | 317 | 395 | 49 | 127 | 205 | 343 | 361 | 23 | 161 | 267 | 285 | 423 | 81 | 99 | 3375 |
| 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 |

In this case the magic sum is $S_{15\times15} = 3375$, and the sum of all entries is $F_{225} := 15 \times 3375 = 50625 = 225^2 = 15^4$. Moreover, each $5 \times 5$ block is a **pan diagonal magic square** which in turn produces another magic square of order 3. See below:

**Example 28.** *The magic square of order 3 arising due to 9 pan diagonal magic squares of order 5 in example 27 is given by*

| | | | |
|---|---|---|---|
| | | | 3375 |
| 1095 | 1157 | 1123 | 3375 |
| 1153 | 1125 | 1097 | 3375 |
| 1127 | 1093 | 1155 | 3375 |
| 3375 | 3375 | 3375 | 3375 |

#### 4.13.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-225)(n-224)}{2} = 225(n-112)$$

Taking $n = 337$, we get a perfect square, i.e.,

$$G := T(337) - T(112) = \frac{337 \times 338}{2} - \frac{112 \times 113}{2} = 50625 = 225^2 = 15^4.$$

Simplifying, we get

$$113 + 114 + 115 + \ldots + 336 + 337 = 50625 = 225^2 = 15^4.$$

This gives a perfect square sum for the 225 entries of consecutive natural numbers from to 113 to 337.

**Example 29.** *The* ***magic square of order 15*** *for 225 entries from 113 to 337 is given by*

| | | | | | | | | | | | | | | | 3375 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 114 | 198 | 207 | 291 | 300 | 145 | 167 | 236 | 262 | 331 | 128 | 184 | 223 | 275 | 314 | 3375 |
| 282 | 306 | 120 | 189 | 213 | 251 | 337 | 151 | 160 | 242 | 268 | 320 | 134 | 173 | 229 | 3375 |
| 195 | 204 | 288 | 297 | 126 | 166 | 235 | 257 | 326 | 157 | 179 | 218 | 274 | 313 | 140 | 3375 |
| 303 | 117 | 201 | 210 | 279 | 332 | 146 | 172 | 241 | 250 | 319 | 133 | 185 | 224 | 263 | 3375 |
| 216 | 285 | 294 | 123 | 192 | 247 | 256 | 325 | 152 | 161 | 230 | 269 | 308 | 139 | 178 | 3375 |
| 143 | 169 | 238 | 260 | 329 | 129 | 183 | 222 | 276 | 315 | 115 | 197 | 206 | 292 | 301 | 3375 |
| 253 | 335 | 149 | 158 | 244 | 267 | 321 | 135 | 174 | 228 | 281 | 307 | 121 | 190 | 212 | 3375 |
| 164 | 233 | 259 | 328 | 155 | 180 | 219 | 273 | 312 | 141 | 196 | 205 | 287 | 296 | 127 | 3375 |
| 334 | 148 | 170 | 239 | 248 | 318 | 132 | 186 | 225 | 264 | 302 | 116 | 202 | 211 | 280 | 3375 |
| 245 | 254 | 323 | 154 | 163 | 231 | 270 | 309 | 138 | 177 | 217 | 286 | 295 | 122 | 191 | 3375 |
| 130 | 182 | 221 | 277 | 316 | 113 | 199 | 208 | 290 | 299 | 144 | 168 | 237 | 261 | 330 | 3375 |
| 266 | 322 | 136 | 175 | 227 | 283 | 305 | 119 | 188 | 214 | 252 | 336 | 150 | 159 | 243 | 3375 |
| 181 | 220 | 272 | 311 | 142 | 194 | 203 | 289 | 298 | 125 | 165 | 234 | 258 | 327 | 156 | 3375 |
| 317 | 131 | 187 | 226 | 265 | 304 | 118 | 200 | 209 | 278 | 333 | 147 | 171 | 240 | 249 | 3375 |
| 232 | 271 | 310 | 137 | 176 | 215 | 284 | 293 | 124 | 193 | 246 | 255 | 324 | 153 | 162 | 3375 |
| 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 | 3375 |

In the examples 27 and 29 given above the magic sum is $S_{15\times 15} = 3375 = 15^3$, and the sum of all entries $F_{225} := 50625 = 225^2 = 15^4$. What is more, each $5\times 5$ block is a **pan diagonal magic square** and when these are combined they produce another magic square of order 3:

**Example 30.** *The magic square of order 3 arising due to 9 pan diagonal magic squares of order 5 in example 29 is given by*

| | | | 3375 |
|---|---|---|---|
| 1110 | 1141 | 1124 | 3375 |
| 1139 | 1125 | 1111 | 3375 |
| 1126 | 1109 | 1140 | 3375 |
| 3375 | 3375 | 3375 | 3375 |

The examples given in 27 and 29 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 15, 15^2, 15^3, 15^4\rangle$.

### 4.13.3 Third Approach: Minimum Perfect Square Sum

The examples given in 27 and 29 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 15, 15^2, 15^3, 15^4\rangle$. However, the magic square example 29 does not have a minimum perfect square sum of its entries. Let's observe the first example of a magic square of order 15 given in section 3:

$$L\left(11^2 + \frac{15^2-1}{2}, 15^2\right) \to \left(15, 9, 233, 1815, 27225, \sqrt{165}\right)$$

In this case, we have a magic square of order 15 with a perfect square sum of all its entries $27225 = 165^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square that follows is constructed according to the details given above.

**Example 31.** *The magic square of order 15 with 225 entries from 9 to 233 is given by*

| | | | | | | | | | | | | | | | 1815 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 114 | 198 | 207 | 291 | 300 | 145 | 167 | 236 | 262 | 331 | 128 | 184 | 223 | 275 | 314 | 1815 |
| 282 | 306 | 120 | 189 | 213 | 251 | 337 | 151 | 160 | 242 | 268 | 320 | 134 | 173 | 229 | 1815 |
| 195 | 204 | 288 | 297 | 126 | 166 | 235 | 257 | 326 | 157 | 179 | 218 | 274 | 313 | 140 | 1815 |
| 303 | 117 | 201 | 210 | 279 | 332 | 146 | 172 | 241 | 250 | 319 | 133 | 185 | 224 | 263 | 1815 |
| 216 | 285 | 294 | 123 | 192 | 247 | 256 | 325 | 152 | 161 | 230 | 269 | 308 | 139 | 178 | 1815 |
| 143 | 169 | 238 | 260 | 329 | 129 | 183 | 222 | 276 | 315 | 115 | 197 | 206 | 292 | 301 | 1815 |
| 253 | 335 | 149 | 158 | 244 | 267 | 321 | 135 | 174 | 228 | 281 | 307 | 121 | 190 | 212 | 1815 |
| 164 | 233 | 259 | 328 | 155 | 180 | 219 | 273 | 312 | 141 | 196 | 205 | 287 | 296 | 127 | 1815 |
| 334 | 148 | 170 | 239 | 248 | 318 | 132 | 186 | 225 | 264 | 302 | 116 | 202 | 211 | 280 | 1815 |
| 245 | 254 | 323 | 154 | 163 | 231 | 270 | 309 | 138 | 177 | 217 | 286 | 295 | 122 | 191 | 1815 |
| 130 | 182 | 221 | 277 | 316 | 113 | 199 | 208 | 290 | 299 | 144 | 168 | 237 | 261 | 330 | 1815 |
| 266 | 322 | 136 | 175 | 227 | 283 | 305 | 119 | 188 | 214 | 252 | 336 | 150 | 159 | 243 | 1815 |
| 181 | 220 | 272 | 311 | 142 | 194 | 203 | 289 | 298 | 125 | 165 | 234 | 258 | 327 | 156 | 1815 |
| 317 | 131 | 187 | 226 | 265 | 304 | 118 | 200 | 209 | 278 | 333 | 147 | 171 | 240 | 249 | 1815 |
| 232 | 271 | 310 | 137 | 176 | 215 | 284 | 293 | 124 | 193 | 246 | 255 | 324 | 153 | 162 | 1815 |
| 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 | 1815 |

The above example 31 has a magic sum $S_{15\times 15} = 1815$, and the sum of all entries is $F_{225} := 15 \times 1815 = 27225 = 165^2$. The magic square of order 15 constructed with the values given in example 31 has a minimum perfect square sum of all its entries. Similar to the other examples 27 and 29, this magic square also has 9 pan diagonal magic squares of order 5, which in turn form a magic square of order 3. See the example below:

**Example 32.** *The 9 blocks of* ***pan diagonal magic squares*** *of order 5 given in example 31in turn form a magic square of order 3 given by*

| | | | 1815 |
|---|---|---|---|
| 590 | 621 | 604 | 1815 |
| 619 | 605 | 591 | 1815 |
| 606 | 589 | 620 | 1815 |
| 1815 | 1815 | 1815 | 1815 |

The block-wise constructions of magic squares of order 15 given in examples 27, 29 and 31 are the compositions of pan diagonal magic squares of order 5 given in subsection 4.3, together with magic squares of order 3 given in subsection 4.1.

## 4.14 Bimagic Squares of Order 16

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 16 with this property if we use odd number entries. Taking $k = 16$ in equation (5), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 16^2-1) = 16^4 \\ \Rightarrow & 1+3+5+\ldots+509+511 \quad = 65536 = 256^2 = 16^4. \end{aligned} \tag{29}$$

**Example 33.** *According to the values given in equation (29), the magic square of order 16 is given by*

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | 4096 |
| 1 | 307 | 477 | 239 | 45 | 287 | 497 | 195 | 87 | 357 | 395 | 185 | 123 | 329 | 423 | 149 | 4096 |
| 463 | 253 | 19 | 289 | 483 | 209 | 63 | 269 | 409 | 171 | 69 | 375 | 437 | 135 | 105 | 347 | 4096 |
| 243 | 449 | 303 | 29 | 223 | 493 | 259 | 49 | 165 | 407 | 377 | 75 | 137 | 443 | 341 | 103 | 4096 |
| 317 | 15 | 225 | 467 | 273 | 35 | 205 | 511 | 363 | 89 | 183 | 389 | 327 | 117 | 155 | 425 | 4096 |
| 91 | 361 | 391 | 181 | 119 | 325 | 427 | 153 | 13 | 319 | 465 | 227 | 33 | 275 | 509 | 207 | 4096 |
| 405 | 167 | 73 | 379 | 441 | 139 | 101 | 343 | 451 | 241 | 31 | 301 | 495 | 221 | 51 | 257 | 4096 |
| 169 | 411 | 373 | 71 | 133 | 439 | 345 | 107 | 255 | 461 | 291 | 17 | 211 | 481 | 271 | 61 | 4096 |
| 359 | 85 | 187 | 393 | 331 | 121 | 151 | 421 | 305 | 3 | 237 | 479 | 285 | 47 | 193 | 499 | 4096 |
| 109 | 351 | 433 | 131 | 65 | 371 | 413 | 175 | 59 | 265 | 487 | 213 | 23 | 293 | 459 | 249 | 4096 |
| 419 | 145 | 127 | 333 | 399 | 189 | 83 | 353 | 501 | 199 | 41 | 283 | 473 | 235 | 5 | 311 | 4096 |
| 159 | 429 | 323 | 113 | 179 | 385 | 367 | 93 | 201 | 507 | 277 | 39 | 229 | 471 | 313 | 11 | 4096 |
| 337 | 99 | 141 | 447 | 381 | 79 | 161 | 403 | 263 | 53 | 219 | 489 | 299 | 25 | 247 | 453 | 4096 |
| 55 | 261 | 491 | 217 | 27 | 297 | 455 | 245 | 97 | 339 | 445 | 143 | 77 | 383 | 401 | 163 | 4096 |
| 505 | 203 | 37 | 279 | 469 | 231 | 9 | 315 | 431 | 157 | 115 | 321 | 387 | 177 | 95 | 365 | 4096 |
| 197 | 503 | 281 | 43 | 233 | 475 | 309 | 7 | 147 | 417 | 335 | 125 | 191 | 397 | 355 | 81 | 4096 |
| 267 | 57 | 215 | 485 | 295 | 21 | 251 | 457 | 349 | 111 | 129 | 435 | 369 | 67 | 173 | 415 | 4096 |
| 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 | 4096 |

In the example 33 given above, the magic sum is $S_{16\times16} = 4096 = 16^3$, and the sum of all entries is $F_{256} := 65536 = 256^2 = 16^4$. Thus, the example 33 satisfies the property (13) of uniformity 2.5.1, i.e., $\langle 16, 16^2, 16^3, 16^4 \rangle$. Moreover, 16 blocks of order 4 are magic squares with magic sum $S_{4\times4} = 1024$. The example shown above is also a **bimagic square** with bimagic sum $Sb_{16\times16} = 1398096$.

**Example 34.** *The bimagic transformation of example 33 is given by*

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | 1398096 |
| $1^2$ | $307^2$ | $477^2$ | $239^2$ | $45^2$ | $287^2$ | $497^2$ | $195^2$ | $87^2$ | $357^2$ | $395^2$ | $185^2$ | $123^2$ | $329^2$ | $423^2$ | $149^2$ | 1398096 |
| $463^2$ | $253^2$ | $19^2$ | $289^2$ | $483^2$ | $209^2$ | $63^2$ | $269^2$ | $409^2$ | $171^2$ | $69^2$ | $375^2$ | $437^2$ | $135^2$ | $105^2$ | $347^2$ | 1398096 |
| $243^2$ | $449^2$ | $303^2$ | $29^2$ | $223^2$ | $493^2$ | $259^2$ | $49^2$ | $165^2$ | $407^2$ | $377^2$ | $75^2$ | $137^2$ | $443^2$ | $341^2$ | $103^2$ | 1398096 |
| $317^2$ | $15^2$ | $225^2$ | $467^2$ | $273^2$ | $35^2$ | $205^2$ | $511^2$ | $363^2$ | $89^2$ | $183^2$ | $389^2$ | $327^2$ | $117^2$ | $155^2$ | $425^2$ | 1398096 |
| $91^2$ | $361^2$ | $391^2$ | $181^2$ | $119^2$ | $325^2$ | $427^2$ | $153^2$ | $13^2$ | $319^2$ | $465^2$ | $227^2$ | $33^2$ | $275^2$ | $509^2$ | $207^2$ | 1398096 |
| $405^2$ | $167^2$ | $73^2$ | $379^2$ | $441^2$ | $139^2$ | $101^2$ | $343^2$ | $451^2$ | $241^2$ | $31^2$ | $301^2$ | $495^2$ | $221^2$ | $51^2$ | $257^2$ | 1398096 |
| $169^2$ | $411^2$ | $373^2$ | $71^2$ | $133^2$ | $439^2$ | $345^2$ | $107^2$ | $255^2$ | $461^2$ | $291^2$ | $17^2$ | $211^2$ | $481^2$ | $271^2$ | $61^2$ | 1398096 |
| $359^2$ | $85^2$ | $187^2$ | $393^2$ | $331^2$ | $121^2$ | $151^2$ | $421^2$ | $305^2$ | $3^2$ | $237^2$ | $479^2$ | $285^2$ | $47^2$ | $193^2$ | $499^2$ | 1398096 |
| $109^2$ | $351^2$ | $433^2$ | $131^2$ | $65^2$ | $371^2$ | $413^2$ | $175^2$ | $59^2$ | $265^2$ | $487^2$ | $213^2$ | $23^2$ | $293^2$ | $459^2$ | $249^2$ | 1398096 |
| $419^2$ | $145^2$ | $127^2$ | $333^2$ | $399^2$ | $189^2$ | $83^2$ | $353^2$ | $501^2$ | $199^2$ | $41^2$ | $283^2$ | $473^2$ | $235^2$ | $5^2$ | $311^2$ | 1398096 |
| $159^2$ | $429^2$ | $323^2$ | $113^2$ | $179^2$ | $385^2$ | $367^2$ | $93^2$ | $201^2$ | $507^2$ | $277^2$ | $39^2$ | $229^2$ | $471^2$ | $313^2$ | $11^2$ | 1398096 |
| $337^2$ | $99^2$ | $141^2$ | $447^2$ | $381^2$ | $79^2$ | $161^2$ | $403^2$ | $263^2$ | $53^2$ | $219^2$ | $489^2$ | $299^2$ | $25^2$ | $247^2$ | $453^2$ | 1398096 |
| $55^2$ | $261^2$ | $491^2$ | $217^2$ | $27^2$ | $297^2$ | $455^2$ | $245^2$ | $97^2$ | $339^2$ | $445^2$ | $143^2$ | $77^2$ | $383^2$ | $401^2$ | $163^2$ | 1398096 |
| $505^2$ | $203^2$ | $37^2$ | $279^2$ | $469^2$ | $231^2$ | $9^2$ | $315^2$ | $431^2$ | $157^2$ | $115^2$ | $321^2$ | $387^2$ | $177^2$ | $95^2$ | $365^2$ | 1398096 |
| $197^2$ | $503^2$ | $281^2$ | $43^2$ | $233^2$ | $475^2$ | $309^2$ | $7^2$ | $147^2$ | $417^2$ | $335^2$ | $125^2$ | $191^2$ | $397^2$ | $355^2$ | $81^2$ | 1398096 |
| $267^2$ | $57^2$ | $215^2$ | $485^2$ | $295^2$ | $21^2$ | $251^2$ | $457^2$ | $349^2$ | $111^2$ | $129^2$ | $435^2$ | $369^2$ | $67^2$ | $173^2$ | $415^2$ | 1398096 |
| 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 | 1398096 |

## 4.15 Magic Squares of Order 17

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 17 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

#### 4.15.1 First Approach: Consecutive Odd Numbers

Taking $k = 11$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 17^2-1) = 17^4 \\ \Rightarrow &1+3+5+\ldots+575+577 \quad = 83521 = 289^2 = 17^4 \end{aligned} \tag{30}$$

**Example 35.** *According to the 289 values given in equation (30), the* ***pan diagonal magic square*** *of order 17 is given by*

| | | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 65 | 95 | 125 | 155 | 185 | 215 | 245 | 275 | 339 | 369 | 399 | 429 | 459 | 489 | 519 | 549 | 4913 |
| 4913 | 513 | 577 | 29 | 59 | 89 | 119 | 149 | 179 | 209 | 239 | 303 | 333 | 363 | 393 | 423 | 453 | 483 | 4913 |
| 4913 | 447 | 477 | 541 | 571 | 23 | 53 | 83 | 113 | 143 | 173 | 237 | 267 | 297 | 327 | 357 | 387 | 417 | 4913 |
| 4913 | 381 | 411 | 475 | 505 | 535 | 565 | 17 | 47 | 77 | 107 | 137 | 201 | 231 | 261 | 291 | 321 | 351 | 4913 |
| 4913 | 315 | 345 | 375 | 439 | 469 | 499 | 529 | 559 | 11 | 41 | 71 | 135 | 165 | 195 | 225 | 255 | 285 | 4913 |
| 4913 | 249 | 279 | 309 | 373 | 403 | 433 | 463 | 493 | 523 | 553 | 5 | 35 | 99 | 129 | 159 | 189 | 219 | 4913 |
| 4913 | 183 | 213 | 243 | 273 | 337 | 367 | 397 | 427 | 457 | 487 | 517 | 547 | 33 | 63 | 93 | 123 | 153 | 4913 |
| 4913 | 117 | 147 | 177 | 207 | 271 | 301 | 331 | 361 | 391 | 421 | 451 | 481 | 511 | 575 | 27 | 57 | 87 | 4913 |
| 4913 | 51 | 81 | 111 | 141 | 171 | 235 | 265 | 295 | 325 | 355 | 385 | 415 | 445 | 509 | 539 | 569 | 21 | 4913 |
| 4913 | 563 | 15 | 45 | 75 | 105 | 169 | 199 | 229 | 259 | 289 | 319 | 349 | 379 | 409 | 473 | 503 | 533 | 4913 |
| 4913 | 497 | 527 | 557 | 9 | 39 | 69 | 133 | 163 | 193 | 223 | 253 | 283 | 313 | 343 | 407 | 437 | 467 | 4913 |
| 4913 | 431 | 461 | 491 | 521 | 551 | 3 | 67 | 97 | 127 | 157 | 187 | 217 | 247 | 277 | 307 | 371 | 401 | 4913 |
| 4913 | 365 | 395 | 425 | 455 | 485 | 515 | 545 | 31 | 61 | 91 | 121 | 151 | 181 | 211 | 241 | 305 | 335 | 4913 |
| 4913 | 299 | 329 | 359 | 389 | 419 | 449 | 479 | 543 | 573 | 25 | 55 | 85 | 115 | 145 | 175 | 205 | 269 | 4913 |
| 4913 | 233 | 263 | 293 | 323 | 353 | 383 | 413 | 443 | 507 | 537 | 567 | 19 | 49 | 79 | 109 | 139 | 203 | 4913 |
| 4913 | 167 | 197 | 227 | 257 | 287 | 317 | 347 | 377 | 441 | 471 | 501 | 531 | 561 | 13 | 43 | 73 | 103 | 4913 |
| 4913 | 101 | 131 | 161 | 191 | 221 | 251 | 281 | 311 | 341 | 405 | 435 | 465 | 495 | 525 | 555 | 7 | 37 | 4913 |
| | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 |

#### 4.15.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-289)(n-288)}{2} = 289(n-144)$$

Taking $n = 433$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(433) - T(144) = \frac{433\times 434}{2} - \frac{144\times 145}{2} &= 433\times 217 - 72\times 145 \\ &= 83521 = 289^2 = 17^4. \end{aligned}$$

Simplifying, we get

$$145+146+147+\ldots+432+433 = 83521 = 289^2 = 17^4.$$

This gives a perfect square sum for the 289 consecutive natural numbers from to 145 to 433.

**Example 36.** *The* ***pan diagonal magic square*** *of order 17 for the 289 entries from 145 to 433 is given by*

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 |
| | 145 | 177 | 192 | 207 | 222 | 237 | 252 | 267 | 282 | 314 | 329 | 344 | 359 | 374 | 389 | 404 | 419 | 4913 |
| 4913 | 401 | 433 | 159 | 174 | 189 | 204 | 219 | 234 | 249 | 264 | 296 | 311 | 326 | 341 | 356 | 371 | 386 | 4913 |
| 4913 | 368 | 383 | 415 | 430 | 156 | 171 | 186 | 201 | 216 | 231 | 263 | 278 | 293 | 308 | 323 | 338 | 353 | 4913 |
| 4913 | 335 | 350 | 382 | 397 | 412 | 427 | 153 | 168 | 183 | 198 | 213 | 245 | 260 | 275 | 290 | 305 | 320 | 4913 |
| 4913 | 302 | 317 | 332 | 364 | 379 | 394 | 409 | 424 | 150 | 165 | 180 | 212 | 227 | 242 | 257 | 272 | 287 | 4913 |
| 4913 | 269 | 284 | 299 | 331 | 346 | 361 | 376 | 391 | 406 | 421 | 147 | 162 | 194 | 209 | 224 | 239 | 254 | 4913 |
| 4913 | 236 | 251 | 266 | 281 | 313 | 328 | 343 | 358 | 373 | 388 | 403 | 418 | 161 | 176 | 191 | 206 | 221 | 4913 |
| 4913 | 203 | 218 | 233 | 248 | 280 | 295 | 310 | 325 | 340 | 355 | 370 | 385 | 400 | 432 | 158 | 173 | 188 | 4913 |
| 4913 | 170 | 185 | 200 | 215 | 230 | 262 | 277 | 292 | 307 | 322 | 337 | 352 | 367 | 399 | 414 | 429 | 155 | 4913 |
| 4913 | 426 | 152 | 167 | 182 | 197 | 229 | 244 | 259 | 274 | 289 | 304 | 319 | 334 | 349 | 381 | 396 | 411 | 4913 |
| 4913 | 393 | 408 | 423 | 149 | 164 | 179 | 211 | 226 | 241 | 256 | 271 | 286 | 301 | 316 | 348 | 363 | 378 | 4913 |
| 4913 | 360 | 375 | 390 | 405 | 420 | 146 | 178 | 193 | 208 | 223 | 238 | 253 | 268 | 283 | 298 | 330 | 345 | 4913 |
| 4913 | 327 | 342 | 357 | 372 | 387 | 402 | 417 | 160 | 175 | 190 | 205 | 220 | 235 | 250 | 265 | 297 | 312 | 4913 |
| 4913 | 294 | 309 | 324 | 339 | 354 | 369 | 384 | 416 | 431 | 157 | 172 | 187 | 202 | 217 | 232 | 247 | 279 | 4913 |
| 4913 | 261 | 276 | 291 | 306 | 321 | 336 | 351 | 366 | 398 | 413 | 428 | 154 | 169 | 184 | 199 | 214 | 246 | 4913 |
| 4913 | 228 | 243 | 258 | 273 | 288 | 303 | 318 | 333 | 365 | 380 | 395 | 410 | 425 | 151 | 166 | 181 | 196 | 4913 |
| 4913 | 195 | 210 | 225 | 240 | 255 | 270 | 285 | 300 | 315 | 347 | 362 | 377 | 392 | 407 | 422 | 148 | 163 | 4913 |
| | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 | 4913 |

In both the examples 35 and 36 given above the magic sum is $S_{17\times 17} = 4913 = 17^3$, and the sum of all numbers is $F_{289} := 83521 = 289^2 = 17^4$.

Both the examples 35 and 36 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 17, 17^2, 17^3, 17^4 \rangle$.

### 4.15.3 Third Approach: Minimum Perfect Square Sum

The examples given in 35 and 36 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 17, 17^2, 17^3, 17^4 \rangle$. However, the sum of the magic square entries of 36 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 17 given in section 3:

$$L\left(13^2 + \frac{17^2-1}{2}, 17^2\right) \to \left(17, 25, 313, 2873, 48841, \sqrt{221}\right)$$

In this case, we have magic square of order 17 with sum of all entries $48841 = 221^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 37.** *The* ***pan diagonal magic square*** *of order 17 with 289 entries starting from 25 to 313 is given by*

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 |
| | 25 | 57 | 72 | 87 | 102 | 117 | 132 | 147 | 162 | 194 | 209 | 224 | 239 | 254 | 269 | 284 | 299 | 2873 |
| 2873 | 281 | 313 | 39 | 54 | 69 | 84 | 99 | 114 | 129 | 144 | 176 | 191 | 206 | 221 | 236 | 251 | 266 | 2873 |
| 2873 | 248 | 263 | 295 | 310 | 36 | 51 | 66 | 81 | 96 | 111 | 143 | 158 | 173 | 188 | 203 | 218 | 233 | 2873 |
| 2873 | 215 | 230 | 262 | 277 | 292 | 307 | 33 | 48 | 63 | 78 | 93 | 125 | 140 | 155 | 170 | 185 | 200 | 2873 |
| 2873 | 182 | 197 | 212 | 244 | 259 | 274 | 289 | 304 | 30 | 45 | 60 | 92 | 107 | 122 | 137 | 152 | 167 | 2873 |
| 2873 | 149 | 164 | 179 | 211 | 226 | 241 | 256 | 271 | 286 | 301 | 27 | 42 | 74 | 89 | 104 | 119 | 134 | 2873 |
| 2873 | 116 | 131 | 146 | 161 | 193 | 208 | 223 | 238 | 253 | 268 | 283 | 298 | 41 | 56 | 71 | 86 | 101 | 2873 |
| 2873 | 83 | 98 | 113 | 128 | 160 | 175 | 190 | 205 | 220 | 235 | 250 | 265 | 280 | 312 | 38 | 53 | 68 | 2873 |
| 2873 | 50 | 65 | 80 | 95 | 110 | 142 | 157 | 172 | 187 | 202 | 217 | 232 | 247 | 279 | 294 | 309 | 35 | 2873 |
| 2873 | 306 | 32 | 47 | 62 | 77 | 109 | 124 | 139 | 154 | 169 | 184 | 199 | 214 | 229 | 261 | 276 | 291 | 2873 |
| 2873 | 273 | 288 | 303 | 29 | 44 | 59 | 91 | 106 | 121 | 136 | 151 | 166 | 181 | 196 | 228 | 243 | 258 | 2873 |
| 2873 | 240 | 255 | 270 | 285 | 300 | 26 | 58 | 73 | 88 | 103 | 118 | 133 | 148 | 163 | 178 | 210 | 225 | 2873 |
| 2873 | 207 | 222 | 237 | 252 | 267 | 282 | 297 | 40 | 55 | 70 | 85 | 100 | 115 | 130 | 145 | 177 | 192 | 2873 |
| 2873 | 174 | 189 | 204 | 219 | 234 | 249 | 264 | 296 | 311 | 37 | 52 | 67 | 82 | 97 | 112 | 127 | 159 | 2873 |
| 2873 | 141 | 156 | 171 | 186 | 201 | 216 | 231 | 246 | 278 | 293 | 308 | 34 | 49 | 64 | 79 | 94 | 126 | 2873 |
| 2873 | 108 | 123 | 138 | 153 | 168 | 183 | 198 | 213 | 245 | 260 | 275 | 290 | 305 | 31 | 46 | 61 | 76 | 2873 |
| 2873 | 75 | 90 | 105 | 120 | 135 | 150 | 165 | 180 | 195 | 227 | 242 | 257 | 272 | 287 | 302 | 28 | 43 | 2873 |
| | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 | 2873 |

The above example 37 has a magic sum $S_{17\times 17} = 2873$, and the sum of all entries is $F_{289} := 17 \times 2873 = 48841 = 221^2$.

## 4.16 Magic Squares of Order 18

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 18 with this property if we use odd number entries. Taking $k = 18$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 18^2-1) = 18^4 \\ \Rightarrow &1+3+5+\ldots+645+647 \quad = 104976 = 324^2 = 18^4. \end{aligned} \tag{31}$$

Below are three different ways of writing a magic square of order 18 with the entries given in (31).

### • First Way

**Example 38.** *According to the values given in equation (31), the magic square of order 18 is given by*

| | | | | | | | | | | | | | | | | | | 5832 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 217 | 261 | 271 | 283 | 249 | 231 | 577 | 621 | 631 | 643 | 609 | 591 | 73 | 117 | 127 | 139 | 105 | 87 | 5832 |
| 273 | 229 | 285 | 243 | 257 | 225 | 633 | 589 | 645 | 603 | 617 | 585 | 129 | 85 | 141 | 99 | 113 | 81 | 5832 |
| 239 | 227 | 241 | 269 | 277 | 259 | 599 | 587 | 601 | 629 | 637 | 619 | 95 | 83 | 97 | 125 | 133 | 115 | 5832 |
| 279 | 247 | 223 | 263 | 235 | 265 | 639 | 607 | 583 | 623 | 595 | 625 | 135 | 103 | 79 | 119 | 91 | 121 | 5832 |
| 253 | 281 | 237 | 221 | 275 | 245 | 613 | 641 | 597 | 581 | 635 | 605 | 109 | 137 | 93 | 77 | 131 | 101 | 5832 |
| 251 | 267 | 255 | 233 | 219 | 287 | 611 | 627 | 615 | 593 | 579 | 647 | 107 | 123 | 111 | 89 | 75 | 143 | 5832 |
| 145 | 189 | 199 | 211 | 177 | 159 | 289 | 333 | 343 | 355 | 321 | 303 | 433 | 477 | 487 | 499 | 465 | 447 | 5832 |
| 201 | 157 | 213 | 171 | 185 | 153 | 345 | 301 | 357 | 315 | 329 | 297 | 489 | 445 | 501 | 459 | 473 | 441 | 5832 |
| 167 | 155 | 169 | 197 | 205 | 187 | 311 | 299 | 313 | 341 | 349 | 331 | 455 | 443 | 457 | 485 | 493 | 475 | 5832 |
| 207 | 175 | 151 | 191 | 163 | 193 | 351 | 319 | 295 | 335 | 307 | 337 | 495 | 463 | 439 | 479 | 451 | 481 | 5832 |
| 181 | 209 | 165 | 149 | 203 | 173 | 325 | 353 | 309 | 293 | 347 | 317 | 469 | 497 | 453 | 437 | 491 | 461 | 5832 |
| 179 | 195 | 183 | 161 | 147 | 215 | 323 | 339 | 327 | 305 | 291 | 359 | 467 | 483 | 471 | 449 | 435 | 503 | 5832 |
| 505 | 549 | 559 | 571 | 537 | 519 | 1 | 45 | 55 | 67 | 33 | 15 | 361 | 405 | 415 | 427 | 393 | 375 | 5832 |
| 561 | 517 | 573 | 531 | 545 | 513 | 57 | 13 | 69 | 27 | 41 | 9 | 417 | 373 | 429 | 387 | 401 | 369 | 5832 |
| 527 | 515 | 529 | 557 | 565 | 547 | 23 | 11 | 25 | 53 | 61 | 43 | 383 | 371 | 385 | 413 | 421 | 403 | 5832 |
| 567 | 535 | 511 | 551 | 523 | 553 | 63 | 31 | 7 | 47 | 19 | 49 | 423 | 391 | 367 | 407 | 379 | 409 | 5832 |
| 541 | 569 | 525 | 509 | 563 | 533 | 37 | 65 | 21 | 5 | 59 | 29 | 397 | 425 | 381 | 365 | 419 | 389 | 5832 |
| 539 | 555 | 543 | 521 | 507 | 575 | 35 | 51 | 39 | 17 | 3 | 71 | 395 | 411 | 399 | 377 | 363 | 431 | 5832 |
| 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 |

In this case the magic sum is $S_{18\times18} = 5832 = 18^3$, and the sum of all entries is $F_{324} := 18 \times 5832 = 104976 = 324^2 = 18^4$.. In this case each $6 \times 6$ block is a **magic square of order 6**, and these blocks in turn form a magic square of order 3:

**Example 39.** *Below is a magic square of order 3 formed by each block of order 6 from example 38:*

| | | | 5832 |
|---|---|---|---|
| 1512 | 3672 | 648 | 5832 |
| 1080 | 1944 | 2808 | 5832 |
| 3240 | 216 | 2376 | 5832 |
| 5832 | 5832 | 5832 | 5832 |

$= 6^3 \times$

| | | | 27 |
|---|---|---|---|
| 7 | 17 | 3 | 27 |
| 5 | 19 | 13 | 27 |
| 15 | 1 | 11 | 27 |
| 27 | 27 | 27 | 27 |

Interestingly, the same happens with the sums of all entries of each block of order 6 which again form a magic square of order 3.

**Example 40.** *Below is a magic square of order 3 formed by the sums of all entries of each block of order 6:*

| | | | 34992 |
|---|---|---|---|
| 9072 | 22032 | 3888 | 34992 |
| 6480 | 11664 | 16848 | 34992 |
| 9072 | 22032 | 3888 | 34992 |
| 34992 | 34992 | 34992 | 34992 |

$= 6^4 \times$

| | | | 27 |
|---|---|---|---|
| 7 | 17 | 3 | 27 |
| 5 | 19 | 13 | 27 |
| 15 | 1 | 11 | 27 |
| 27 | 27 | 27 | 27 |

## • Second Way

By redistributing the total entries in a slightly different way, we get another method for writing a magic square of order 18 with the same values given in (34). See below:

**Example 41.** *According to the values given in equation (31), the magic square of order 18 is given by*

| | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | 5832 |
| 7 | 403 | 493 | 601 | 295 | 133 | 17 | 413 | 503 | 611 | 305 | 143 | 3 | 399 | 489 | 597 | 291 | 129 | 5832 |
| 511 | 115 | 619 | 241 | 367 | 79 | 521 | 125 | 629 | 251 | 377 | 89 | 507 | 111 | 615 | 237 | 363 | 75 | 5832 |
| 205 | 97 | 223 | 475 | 547 | 385 | 215 | 107 | 233 | 485 | 557 | 395 | 201 | 93 | 219 | 471 | 543 | 381 | 5832 |
| 565 | 277 | 61 | 421 | 169 | 439 | 575 | 287 | 71 | 431 | 179 | 449 | 561 | 273 | 57 | 417 | 165 | 435 | 5832 |
| 331 | 583 | 187 | 43 | 529 | 259 | 341 | 593 | 197 | 53 | 539 | 269 | 327 | 579 | 183 | 39 | 525 | 255 | 5832 |
| 313 | 457 | 349 | 151 | 25 | 637 | 323 | 467 | 359 | 161 | 35 | 647 | 309 | 453 | 345 | 147 | 21 | 633 | 5832 |
| 5 | 401 | 491 | 599 | 293 | 131 | 9 | 405 | 495 | 603 | 297 | 135 | 13 | 409 | 499 | 607 | 301 | 139 | 5832 |
| 509 | 113 | 617 | 239 | 365 | 77 | 513 | 117 | 621 | 243 | 369 | 81 | 517 | 121 | 625 | 247 | 373 | 85 | 5832 |
| 203 | 95 | 221 | 473 | 545 | 383 | 207 | 99 | 225 | 477 | 549 | 387 | 211 | 103 | 229 | 481 | 553 | 391 | 5832 |
| 563 | 275 | 59 | 419 | 167 | 437 | 567 | 279 | 63 | 423 | 171 | 441 | 571 | 283 | 67 | 427 | 175 | 445 | 5832 |
| 329 | 581 | 185 | 41 | 527 | 257 | 333 | 585 | 189 | 45 | 531 | 261 | 337 | 589 | 193 | 49 | 535 | 265 | 5832 |
| 311 | 455 | 347 | 149 | 23 | 635 | 315 | 459 | 351 | 153 | 27 | 639 | 319 | 463 | 355 | 157 | 31 | 643 | 5832 |
| 15 | 411 | 501 | 609 | 303 | 141 | 1 | 397 | 487 | 595 | 289 | 127 | 11 | 407 | 497 | 605 | 299 | 137 | 5832 |
| 519 | 123 | 627 | 249 | 375 | 87 | 505 | 109 | 613 | 235 | 361 | 73 | 515 | 119 | 623 | 245 | 371 | 83 | 5832 |
| 213 | 105 | 231 | 483 | 555 | 393 | 199 | 91 | 217 | 469 | 541 | 379 | 209 | 101 | 227 | 479 | 551 | 389 | 5832 |
| 573 | 285 | 69 | 429 | 177 | 447 | 559 | 271 | 55 | 415 | 163 | 433 | 569 | 281 | 65 | 425 | 173 | 443 | 5832 |
| 339 | 591 | 195 | 51 | 537 | 267 | 325 | 577 | 181 | 37 | 523 | 253 | 335 | 587 | 191 | 47 | 533 | 263 | 5832 |
| 321 | 465 | 357 | 159 | 33 | 645 | 307 | 451 | 343 | 145 | 19 | 631 | 317 | 461 | 353 | 155 | 29 | 641 | 5832 |
| 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 |

In this case also the magic sum is $S_{18\times 18} = 5832 = 18^3$, and the sum of all entries is $F_{324} := 18 \times 5832 = 104976 = 324^2 = 18^4$.. Each $6 \times 6$ block is a **magic square of order 6** and these in turn form a magic square of order 3:

**Example 42.** *Below is a magic square of order 3 formed by each block of order 6:*

| | | | |
|---|---|---|---|
| | | | 5832 |
| 1932 | 1992 | 1908 | 5832 |
| 1920 | 1944 | 1968 | 5832 |
| 1980 | 1896 | 1956 | 5832 |
| 5832 | 5832 | 5832 | 5832 |

$= 12 \times$

| | | | |
|---|---|---|---|
| | | | 486 |
| 161 | 166 | 159 | 486 |
| 160 | 162 | 164 | 486 |
| 165 | 158 | 163 | 486 |
| 486 | 486 | 486 | 486 |

Interestingly, the same happens with the sums of all entries of each block of order 6 which again form a magic square of order 3.

**Example 43.** *Below is a magic square of order 3 formed by sums of all entries of each block of order 6:*

| | | | |
|---|---|---|---|
| | | | 34992 |
| 11592 | 11952 | 11448 | 34992 |
| 11520 | 11664 | 11808 | 34992 |
| 11880 | 11376 | 11736 | 34992 |
| 34992 | 34992 | 34992 | 34992 |

$= 72 \times$

| | | | |
|---|---|---|---|
| | | | 486 |
| 161 | 166 | 159 | 486 |
| 160 | 162 | 164 | 486 |
| 165 | 158 | 163 | 486 |
| 486 | 486 | 486 | 486 |

The example 42 has a more uniform distribution of entries than that of the example 34.

## • Third Way

There is yet another way of writing a magic square of order 18 with the same entries given in equation (37). In this case the $6 \times 6$ blocks are not magic squares but the middle $4 \times 4$ block is a magic square of order 4

**Example 44.** *According to the values given in equation (37), the magic square of order 18 is given by*

| | | | | | | | | | | | | | | | | | | 5832 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 411 | 449 | 71 | 479 | 317 | 277 | 85 | 573 | 167 | 111 | 583 | 387 | 355 | 621 | 199 | 237 | 509 | 5832 |
| 275 | 39 | 627 | 413 | 105 | 445 | 315 | 109 | 535 | 215 | 149 | 385 | 353 | 587 | 235 | 21 | 475 | 549 | 5832 |
| 313 | 287 | 77 | 591 | 629 | 139 | 397 | 147 | 497 | 249 | 187 | 351 | 553 | 19 | 57 | 441 | 515 | 383 | 5832 |
| 615 | 311 | 285 | 115 | 555 | 593 | 173 | 185 | 459 | 31 | 225 | 505 | 55 | 93 | 407 | 481 | 395 | 349 | 5832 |
| 207 | 581 | 323 | 283 | 153 | 519 | 557 | 223 | 421 | 65 | 11 | 91 | 129 | 625 | 433 | 393 | 347 | 471 | 5832 |
| 521 | 241 | 547 | 321 | 281 | 191 | 483 | 9 | 635 | 99 | 49 | 165 | 577 | 399 | 391 | 359 | 437 | 127 | 5832 |
| 447 | 485 | 23 | 513 | 319 | 279 | 229 | 47 | 611 | 133 | 73 | 543 | 617 | 389 | 357 | 403 | 163 | 201 | 5832 |
| 221 | 7 | 45 | 83 | 121 | 145 | 183 | 267 | 379 | 309 | 341 | 465 | 503 | 527 | 565 | 603 | 641 | 427 | 5832 |
| 607 | 569 | 531 | 493 | 455 | 431 | 645 | 345 | 305 | 375 | 271 | 41 | 3 | 217 | 193 | 155 | 117 | 79 | 5832 |
| 405 | 623 | 589 | 541 | 507 | 473 | 439 | 377 | 273 | 343 | 303 | 245 | 211 | 177 | 143 | 95 | 61 | 27 | 5832 |
| 43 | 81 | 119 | 157 | 181 | 219 | 5 | 307 | 339 | 269 | 381 | 643 | 429 | 467 | 491 | 529 | 567 | 605 | 5832 |
| 141 | 453 | 487 | 29 | 329 | 369 | 67 | 601 | 75 | 517 | 575 | 419 | 263 | 295 | 171 | 613 | 197 | 231 | 5832 |
| 489 | 523 | 247 | 327 | 367 | 33 | 107 | 639 | 37 | 551 | 599 | 195 | 457 | 261 | 293 | 137 | 409 | 161 | 5832 |
| 559 | 213 | 325 | 365 | 251 | 59 | 525 | 425 | 13 | 585 | 637 | 125 | 159 | 495 | 259 | 291 | 103 | 443 | 5832 |
| 179 | 337 | 363 | 203 | 25 | 561 | 595 | 463 | 227 | 619 | 423 | 477 | 89 | 123 | 533 | 257 | 289 | 69 | 5832 |
| 335 | 361 | 169 | 243 | 597 | 631 | 131 | 501 | 189 | 401 | 461 | 35 | 511 | 53 | 87 | 571 | 255 | 301 | 5832 |
| 373 | 135 | 209 | 633 | 415 | 97 | 333 | 539 | 151 | 435 | 499 | 299 | 239 | 545 | 17 | 51 | 609 | 253 | 5832 |
| 101 | 175 | 417 | 451 | 63 | 331 | 371 | 563 | 113 | 469 | 537 | 265 | 297 | 205 | 579 | 233 | 15 | 647 | 5832 |
| 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 | 5832 |

The middle block is a magic square of order 4 with magic sum $S_{4\times4} = 1296$.

The **block-wise** constructions of magic squares of order 18 given in examples 38 and 41 are the compositions of magic squares of order 6 given in subsection 4.4 together with magic squares of order 3 given in subsection 4.1. The construction of the magic square given in example 44 is based on a procedure given by White [26].

## 4.17 Magic Squares of Order 19

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 11 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.17.1 First Approach: Consecutive Odd Numbers

Taking $k = 19$ in equation (5), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 19^2-1) = 19^4 \\ \Rightarrow & 1+3+5+\ldots+719+721 \quad = 130321 = 361^2 = 19^4 \end{aligned} \tag{32}$$

**Example 45.** *According to the 361 values given in equation (32), the* ***pan diagonal magic square*** *of order 19 is given by*

| | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 |
| | 1 | 73 | 107 | 141 | 175 | 209 | 243 | 277 | 311 | 345 | 417 | 451 | 485 | 519 | 553 | 587 | 621 | 655 | 689 | 6859 |
| 6859 | 649 | 721 | 33 | 67 | 101 | 135 | 169 | 203 | 237 | 271 | 305 | 377 | 411 | 445 | 479 | 513 | 547 | 581 | 615 | 6859 |
| 6859 | 575 | 609 | 681 | 715 | 27 | 61 | 95 | 129 | 163 | 197 | 231 | 303 | 337 | 371 | 405 | 439 | 473 | 507 | 541 | 6859 |
| 6859 | 501 | 535 | 607 | 641 | 675 | 709 | 21 | 55 | 89 | 123 | 157 | 191 | 263 | 297 | 331 | 365 | 399 | 433 | 467 | 6859 |
| 6859 | 427 | 461 | 495 | 567 | 601 | 635 | 669 | 703 | 15 | 49 | 83 | 117 | 189 | 223 | 257 | 291 | 325 | 359 | 393 | 6859 |
| 6859 | 353 | 387 | 421 | 493 | 527 | 561 | 595 | 629 | 663 | 697 | 9 | 43 | 77 | 149 | 183 | 217 | 251 | 285 | 319 | 6859 |
| 6859 | 279 | 313 | 347 | 381 | 453 | 487 | 521 | 555 | 589 | 623 | 657 | 691 | 3 | 75 | 109 | 143 | 177 | 211 | 245 | 6859 |
| 6859 | 205 | 239 | 273 | 307 | 379 | 413 | 447 | 481 | 515 | 549 | 583 | 617 | 651 | 685 | 35 | 69 | 103 | 137 | 171 | 6859 |
| 6859 | 131 | 165 | 199 | 233 | 267 | 339 | 373 | 407 | 441 | 475 | 509 | 543 | 577 | 611 | 683 | 717 | 29 | 63 | 97 | 6859 |
| 6859 | 57 | 91 | 125 | 159 | 193 | 265 | 299 | 333 | 367 | 401 | 435 | 469 | 503 | 537 | 571 | 643 | 677 | 711 | 23 | 6859 |
| 6859 | 705 | 17 | 51 | 85 | 119 | 153 | 225 | 259 | 293 | 327 | 361 | 395 | 429 | 463 | 497 | 569 | 603 | 637 | 671 | 6859 |
| 6859 | 631 | 665 | 699 | 11 | 45 | 79 | 151 | 185 | 219 | 253 | 287 | 321 | 355 | 389 | 423 | 457 | 529 | 563 | 597 | 6859 |
| 6859 | 557 | 591 | 625 | 659 | 693 | 5 | 39 | 111 | 145 | 179 | 213 | 247 | 281 | 315 | 349 | 383 | 455 | 489 | 523 | 6859 |
| 6859 | 483 | 517 | 551 | 585 | 619 | 653 | 687 | 37 | 71 | 105 | 139 | 173 | 207 | 241 | 275 | 309 | 343 | 415 | 449 | 6859 |
| 6859 | 409 | 443 | 477 | 511 | 545 | 579 | 613 | 647 | 719 | 31 | 65 | 99 | 133 | 167 | 201 | 235 | 269 | 341 | 375 | 6859 |
| 6859 | 335 | 369 | 403 | 437 | 471 | 505 | 539 | 573 | 645 | 679 | 713 | 25 | 59 | 93 | 127 | 161 | 195 | 229 | 301 | 6859 |
| 6859 | 261 | 295 | 329 | 363 | 397 | 431 | 465 | 499 | 533 | 605 | 639 | 673 | 707 | 19 | 53 | 87 | 121 | 155 | 227 | 6859 |
| 6859 | 187 | 221 | 255 | 289 | 323 | 357 | 391 | 425 | 459 | 531 | 565 | 599 | 633 | 667 | 701 | 13 | 47 | 81 | 115 | 6859 |
| 6859 | 113 | 147 | 181 | 215 | 249 | 283 | 317 | 351 | 385 | 419 | 491 | 525 | 559 | 593 | 627 | 661 | 695 | 7 | 41 | 6859 |
| | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 |

### 4.17.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-361)(n-360)}{2} = 361(n-180)$$

Taking $n = 541$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(541) - T(180) = \frac{541 \times 542}{2} - \frac{180 \times 181}{2} &= 541 \times 271 - 90 \times 181 \\ &= 130321 = 361^2 = 19^4. \end{aligned}$$

Simplifying, we get

$$181 + 182 + 183 + \ldots + 540 + 541 = 130321 = 361^2 = 19^4.$$

This gives a perfect square sum for the 121 consecutive natural numbers from to 61 to 181.

**Example 46.** *The* ***pan diagonal magic square*** *of order 19 for 361 entries from 181 to 541 is given by*

| | | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 181 | 217 | 234 | 251 | 268 | 285 | 302 | 319 | 336 | 353 | 389 | 406 | 423 | 440 | 457 | 474 | 491 | 508 | 525 | 6859 |
| 6859 | 505 | 541 | 197 | 214 | 231 | 248 | 265 | 282 | 299 | 316 | 333 | 369 | 386 | 403 | 420 | 437 | 454 | 471 | 488 | 6859 |
| 6859 | 468 | 485 | 521 | 538 | 194 | 211 | 228 | 245 | 262 | 279 | 296 | 332 | 349 | 366 | 383 | 400 | 417 | 434 | 451 | 6859 |
| 6859 | 431 | 448 | 484 | 501 | 518 | 535 | 191 | 208 | 225 | 242 | 259 | 276 | 312 | 329 | 346 | 363 | 380 | 397 | 414 | 6859 |
| 6859 | 394 | 411 | 428 | 464 | 481 | 498 | 515 | 532 | 188 | 205 | 222 | 239 | 275 | 292 | 309 | 326 | 343 | 360 | 377 | 6859 |
| 6859 | 357 | 374 | 391 | 427 | 444 | 461 | 478 | 495 | 512 | 529 | 185 | 202 | 219 | 255 | 272 | 289 | 306 | 323 | 340 | 6859 |
| 6859 | 320 | 337 | 354 | 371 | 407 | 424 | 441 | 458 | 475 | 492 | 509 | 526 | 182 | 218 | 235 | 252 | 269 | 286 | 303 | 6859 |
| 6859 | 283 | 300 | 317 | 334 | 370 | 387 | 404 | 421 | 438 | 455 | 472 | 489 | 506 | 523 | 198 | 215 | 232 | 249 | 266 | 6859 |
| 6859 | 246 | 263 | 280 | 297 | 314 | 350 | 367 | 384 | 401 | 418 | 435 | 452 | 469 | 486 | 522 | 539 | 195 | 212 | 229 | 6859 |
| 6859 | 209 | 226 | 243 | 260 | 277 | 313 | 330 | 347 | 364 | 381 | 398 | 415 | 432 | 449 | 466 | 502 | 519 | 536 | 192 | 6859 |
| 6859 | 533 | 189 | 206 | 223 | 240 | 257 | 293 | 310 | 327 | 344 | 361 | 378 | 395 | 412 | 429 | 465 | 482 | 499 | 516 | 6859 |
| 6859 | 496 | 513 | 530 | 186 | 203 | 220 | 256 | 273 | 290 | 307 | 324 | 341 | 358 | 375 | 392 | 409 | 445 | 462 | 479 | 6859 |
| 6859 | 459 | 476 | 493 | 510 | 527 | 183 | 200 | 236 | 253 | 270 | 287 | 304 | 321 | 338 | 355 | 372 | 408 | 425 | 442 | 6859 |
| 6859 | 422 | 439 | 456 | 473 | 490 | 507 | 524 | 199 | 216 | 233 | 250 | 267 | 284 | 301 | 318 | 335 | 352 | 388 | 405 | 6859 |
| 6859 | 385 | 402 | 419 | 436 | 453 | 470 | 487 | 504 | 540 | 196 | 213 | 230 | 247 | 264 | 281 | 298 | 315 | 351 | 368 | 6859 |
| 6859 | 348 | 365 | 382 | 399 | 416 | 433 | 450 | 467 | 503 | 520 | 537 | 193 | 210 | 227 | 244 | 261 | 278 | 295 | 331 | 6859 |
| 6859 | 311 | 328 | 345 | 362 | 379 | 396 | 413 | 430 | 447 | 483 | 500 | 517 | 534 | 190 | 207 | 224 | 241 | 258 | 294 | 6859 |
| 6859 | 274 | 291 | 308 | 325 | 342 | 359 | 376 | 393 | 410 | 446 | 463 | 480 | 497 | 514 | 531 | 187 | 204 | 221 | 238 | 6859 |
| 6859 | 237 | 254 | 271 | 288 | 305 | 322 | 339 | 356 | 373 | 390 | 426 | 443 | 460 | 477 | 494 | 511 | 528 | 184 | 201 | 6859 |
| | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 | 6859 |

In both the examples 45 and 46 given above the magic sum is $S_{19\times 19} = 6859$, and the sum of all entries is $F_{361} := 19 \times 6859 = 130321 = 361^2 = 19^4$.

Both the examples 45 and 46 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 19, 19^2, 19^3, 19^4 \rangle$.

#### 4.17.3 Third Approach: Minimum Perfect Square Sum

The examples given in 45 and 46 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 19, 19^2, 19^3, 19^4 \rangle$. However, the sum of the magic square entries of 46 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 19 given in section 3:

$$L\left(14^2 + \frac{19^2-1}{2}, 19^2\right) \to \left(19, 16, 376, 3724, 70756, \sqrt{266}\right)$$

In this case, we have magic square of order 19 with sum of all entries $70756 = 266^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 47.** *The **pan diagonal magic** square of order 19 with 361 entries starting from 16 to 376 is given by*

| | | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 181 | 217 | 234 | 251 | 268 | 285 | 302 | 319 | 336 | 353 | 389 | 406 | 423 | 440 | 457 | 474 | 491 | 508 | 525 | 3724 |
| 3724 | 505 | 541 | 197 | 214 | 231 | 248 | 265 | 282 | 299 | 316 | 333 | 369 | 386 | 403 | 420 | 437 | 454 | 471 | 488 | 3724 |
| 3724 | 468 | 485 | 521 | 538 | 194 | 211 | 228 | 245 | 262 | 279 | 296 | 332 | 349 | 366 | 383 | 400 | 417 | 434 | 451 | 3724 |
| 3724 | 16 | 52 | 69 | 86 | 103 | 120 | 137 | 154 | 171 | 188 | 224 | 241 | 258 | 275 | 292 | 309 | 326 | 343 | 360 | 3724 |
| 3724 | 340 | 376 | 32 | 49 | 66 | 83 | 100 | 117 | 134 | 151 | 168 | 204 | 221 | 238 | 255 | 272 | 289 | 306 | 323 | 3724 |
| 3724 | 303 | 320 | 356 | 373 | 29 | 46 | 63 | 80 | 97 | 114 | 131 | 167 | 184 | 201 | 218 | 235 | 252 | 269 | 286 | 3724 |
| 3724 | 266 | 283 | 319 | 336 | 353 | 370 | 26 | 43 | 60 | 77 | 94 | 111 | 147 | 164 | 181 | 198 | 215 | 232 | 249 | 3724 |
| 3724 | 229 | 246 | 263 | 299 | 316 | 333 | 350 | 367 | 23 | 40 | 57 | 74 | 110 | 127 | 144 | 161 | 178 | 195 | 212 | 3724 |
| 3724 | 192 | 209 | 226 | 262 | 279 | 296 | 313 | 330 | 347 | 364 | 20 | 37 | 54 | 90 | 107 | 124 | 141 | 158 | 175 | 3724 |
| 3724 | 155 | 172 | 189 | 206 | 242 | 259 | 276 | 293 | 310 | 327 | 344 | 361 | 17 | 53 | 70 | 87 | 104 | 121 | 138 | 3724 |
| 3724 | 118 | 135 | 152 | 169 | 205 | 222 | 239 | 256 | 273 | 290 | 307 | 324 | 341 | 358 | 33 | 50 | 67 | 84 | 101 | 3724 |
| 3724 | 81 | 98 | 115 | 132 | 149 | 185 | 202 | 219 | 236 | 253 | 270 | 287 | 304 | 321 | 357 | 374 | 30 | 47 | 64 | 3724 |
| 3724 | 44 | 61 | 78 | 95 | 112 | 148 | 165 | 182 | 199 | 216 | 233 | 250 | 267 | 284 | 301 | 337 | 354 | 371 | 27 | 3724 |
| 3724 | 368 | 24 | 41 | 58 | 75 | 92 | 128 | 145 | 162 | 179 | 196 | 213 | 230 | 247 | 264 | 300 | 317 | 334 | 351 | 3724 |
| 3724 | 331 | 348 | 365 | 21 | 38 | 55 | 91 | 108 | 125 | 142 | 159 | 176 | 193 | 210 | 227 | 244 | 280 | 297 | 314 | 3724 |
| 3724 | 294 | 311 | 328 | 345 | 362 | 18 | 35 | 71 | 88 | 105 | 122 | 139 | 156 | 173 | 190 | 207 | 243 | 260 | 277 | 3724 |
| 3724 | 257 | 274 | 291 | 308 | 325 | 342 | 359 | 34 | 51 | 68 | 85 | 102 | 119 | 136 | 153 | 170 | 187 | 223 | 240 | 3724 |
| 3724 | 220 | 237 | 254 | 271 | 288 | 305 | 322 | 339 | 375 | 31 | 48 | 65 | 82 | 99 | 116 | 133 | 150 | 186 | 203 | 3724 |
| 3724 | 183 | 200 | 217 | 234 | 251 | 268 | 285 | 302 | 338 | 355 | 372 | 28 | 45 | 62 | 79 | 96 | 113 | 130 | 166 | 3724 |
| 3724 | 146 | 163 | 180 | 197 | 214 | 231 | 248 | 265 | 282 | 318 | 335 | 352 | 369 | 25 | 42 | 59 | 76 | 93 | 129 | 3724 |
| 3724 | 109 | 126 | 143 | 160 | 177 | 194 | 211 | 228 | 245 | 281 | 298 | 315 | 332 | 349 | 366 | 22 | 39 | 56 | 73 | 3724 |
| 3724 | 72 | 89 | 106 | 123 | 140 | 157 | 174 | 191 | 208 | 225 | 261 | 278 | 295 | 312 | 329 | 346 | 363 | 19 | 36 | 3724 |
| | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 | 3724 |

The above example 47 has a magic sum $S_{19\times 19} = 3724$, and the sum of all entries is $F_{361} := 19 \times 3724 = 70756 = 266^2$. Moreover, it is **pan diagonal**.

## 4.18 Magic Squares of Order 20

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 20 with this property if we use odd number entries. Taking $k = 20$ in equation (5), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 20^2-1) = 20^4 \\ \Rightarrow & 1+3+5+\ldots+797+799 \quad = 160000 = 400^2 = 20^4. \end{aligned} \tag{33}$$

We shall construct magic squares of order 20 with odd numbers in two different ways: firstly by using a $5 \times 4$ format and secondly by using a $4 \times 5$ format.

- **First Way:** $5 \times 4$

**Example 48.** *According to the values given in equation (33), the magic square of order 20 with odd numbers is given by*

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 |
| | 43 | 269 | 373 | 595 | 699 | 81 | 231 | 415 | 553 | 737 | 7 | 305 | 329 | 639 | 663 | 125 | 187 | 451 | 517 | 781 | 8000 |
| 8000 | 573 | 715 | 59 | 243 | 389 | 535 | 753 | 97 | 201 | 431 | 609 | 679 | 23 | 287 | 345 | 491 | 797 | 141 | 165 | 467 | 8000 |
| 8000 | 259 | 363 | 589 | 693 | 75 | 217 | 401 | 551 | 735 | 113 | 303 | 327 | 625 | 649 | 39 | 181 | 445 | 507 | 771 | 157 | 8000 |
| 8000 | 709 | 53 | 275 | 379 | 563 | 751 | 95 | 233 | 417 | 521 | 665 | 9 | 319 | 343 | 607 | 787 | 131 | 197 | 461 | 485 | 8000 |
| 8000 | 395 | 579 | 683 | 69 | 253 | 433 | 537 | 721 | 111 | 215 | 359 | 623 | 647 | 25 | 289 | 477 | 501 | 765 | 147 | 171 | 8000 |
| 8000 | 5 | 307 | 331 | 637 | 661 | 127 | 185 | 449 | 519 | 783 | 41 | 271 | 375 | 593 | 697 | 83 | 229 | 413 | 555 | 739 | 8000 |
| 8000 | 611 | 677 | 21 | 285 | 347 | 489 | 799 | 143 | 167 | 465 | 575 | 713 | 57 | 241 | 391 | 533 | 755 | 99 | 203 | 429 | 8000 |
| 8000 | 301 | 325 | 627 | 651 | 37 | 183 | 447 | 505 | 769 | 159 | 257 | 361 | 591 | 695 | 73 | 219 | 403 | 549 | 733 | 115 | 8000 |
| 8000 | 667 | 11 | 317 | 341 | 605 | 785 | 129 | 199 | 463 | 487 | 711 | 55 | 273 | 377 | 561 | 749 | 93 | 235 | 419 | 523 | 8000 |
| 8000 | 357 | 621 | 645 | 27 | 291 | 479 | 503 | 767 | 145 | 169 | 393 | 577 | 681 | 71 | 255 | 435 | 539 | 723 | 109 | 213 | 8000 |
| 8000 | 121 | 191 | 455 | 513 | 777 | 3 | 309 | 333 | 635 | 659 | 85 | 227 | 411 | 557 | 741 | 47 | 265 | 369 | 599 | 703 | 8000 |
| 8000 | 495 | 793 | 137 | 161 | 471 | 613 | 675 | 19 | 283 | 349 | 531 | 757 | 101 | 205 | 427 | 569 | 719 | 63 | 247 | 385 | 8000 |
| 8000 | 177 | 441 | 511 | 775 | 153 | 299 | 323 | 629 | 653 | 35 | 221 | 405 | 547 | 731 | 117 | 263 | 367 | 585 | 689 | 79 | 8000 |
| 8000 | 791 | 135 | 193 | 457 | 481 | 669 | 13 | 315 | 339 | 603 | 747 | 91 | 237 | 421 | 525 | 705 | 49 | 279 | 383 | 567 | 8000 |
| 8000 | 473 | 497 | 761 | 151 | 175 | 355 | 619 | 643 | 29 | 293 | 437 | 541 | 725 | 107 | 211 | 399 | 583 | 687 | 65 | 249 | 8000 |
| 8000 | 87 | 225 | 409 | 559 | 743 | 45 | 267 | 371 | 597 | 701 | 123 | 189 | 453 | 515 | 779 | 1 | 311 | 335 | 633 | 657 | 8000 |
| 8000 | 529 | 759 | 103 | 207 | 425 | 571 | 717 | 61 | 245 | 387 | 493 | 795 | 139 | 163 | 469 | 615 | 673 | 17 | 281 | 351 | 8000 |
| 8000 | 223 | 407 | 545 | 729 | 119 | 261 | 365 | 587 | 691 | 77 | 179 | 443 | 509 | 773 | 155 | 297 | 321 | 631 | 655 | 33 | 8000 |
| 8000 | 745 | 89 | 239 | 423 | 527 | 707 | 51 | 277 | 381 | 565 | 789 | 133 | 195 | 459 | 483 | 671 | 15 | 313 | 337 | 601 | 8000 |
| 8000 | 439 | 543 | 727 | 105 | 209 | 397 | 581 | 685 | 67 | 251 | 475 | 499 | 763 | 149 | 173 | 353 | 617 | 641 | 31 | 295 | 8000 |
| | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 |

In this case the magic sum is $S_{20\times 20} = 8000 = 20^3$, and the sum of all entries is $F_{400} := 20 \times 8000 = 160000 = 400^2 = 20^4$. Moreover, each $5 \times 5$ block is a **pan diagonal magic square** that has a different magic sum, and when these sums are combined, they form another **pan diagonal magic square** of order 4. See the example below:

**Example 49.** *Each $5 \times 5$ block of a magic square of order 20 given in 48 is a* ***pan diagonal magic square*** *with a different magic sum, and when these sums are combined they form another* ***pan diagonal magic square*** *of order 4:*

| | | | | | |
|---|---|---|---|---|---|
| | | 8000 | 8000 | 8000 | 8000 |
| | 1979 | 2017 | 1943 | 2061 | 8000 |
| 8000 | 1941 | 2063 | 1977 | 2019 | 8000 |
| 8000 | 2057 | 1939 | 2021 | 1983 | 8000 |
| 8000 | 2023 | 1981 | 2059 | 1937 | 8000 |
| | 8000 | 8000 | 8000 | 8000 | 8000 |

- **Second Way:** $4 \times 5$

**Example 50.** *According to the values given in equation (33), the magic square of order 20 with odd numbers is given by*

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 |
| | 379 | 401 | 39 | 781 | 333 | 447 | 73 | 747 | 297 | 483 | 117 | 703 | 251 | 529 | 151 | 669 | 215 | 565 | 195 | 625 | 8000 |
| 8000 | 21 | 799 | 361 | 419 | 67 | 753 | 327 | 453 | 103 | 717 | 283 | 497 | 149 | 671 | 249 | 531 | 185 | 635 | 205 | 575 | 8000 |
| 8000 | 761 | 19 | 421 | 399 | 727 | 53 | 467 | 353 | 683 | 97 | 503 | 317 | 649 | 131 | 549 | 271 | 605 | 175 | 585 | 235 | 8000 |
| 8000 | 439 | 381 | 779 | 1 | 473 | 347 | 733 | 47 | 517 | 303 | 697 | 83 | 551 | 269 | 651 | 129 | 595 | 225 | 615 | 165 | 8000 |
| 8000 | 257 | 523 | 157 | 663 | 211 | 569 | 191 | 629 | 375 | 405 | 35 | 785 | 339 | 441 | 79 | 741 | 293 | 487 | 113 | 707 | 8000 |
| 8000 | 143 | 677 | 243 | 537 | 189 | 631 | 209 | 571 | 25 | 795 | 365 | 415 | 61 | 759 | 321 | 459 | 107 | 713 | 287 | 493 | 8000 |
| 8000 | 643 | 137 | 543 | 277 | 609 | 171 | 589 | 231 | 765 | 15 | 425 | 395 | 721 | 59 | 461 | 359 | 687 | 93 | 507 | 313 | 8000 |
| 8000 | 557 | 263 | 657 | 123 | 591 | 229 | 611 | 169 | 435 | 385 | 775 | 5 | 479 | 341 | 739 | 41 | 513 | 307 | 693 | 87 | 8000 |
| 8000 | 335 | 445 | 75 | 745 | 299 | 481 | 119 | 701 | 253 | 527 | 153 | 667 | 217 | 563 | 197 | 623 | 371 | 409 | 31 | 789 | 8000 |
| 8000 | 65 | 755 | 325 | 455 | 101 | 719 | 281 | 499 | 147 | 673 | 247 | 533 | 183 | 637 | 203 | 577 | 29 | 791 | 369 | 411 | 8000 |
| 8000 | 725 | 55 | 465 | 355 | 681 | 99 | 501 | 319 | 647 | 133 | 547 | 273 | 603 | 177 | 583 | 237 | 769 | 11 | 429 | 391 | 8000 |
| 8000 | 475 | 345 | 735 | 45 | 519 | 301 | 699 | 81 | 553 | 267 | 653 | 127 | 597 | 223 | 617 | 163 | 431 | 389 | 771 | 9 | 8000 |
| 8000 | 213 | 567 | 193 | 627 | 377 | 403 | 37 | 783 | 331 | 449 | 71 | 749 | 295 | 485 | 115 | 705 | 259 | 521 | 159 | 661 | 8000 |
| 8000 | 187 | 633 | 207 | 573 | 23 | 797 | 363 | 417 | 69 | 751 | 329 | 451 | 105 | 715 | 285 | 495 | 141 | 679 | 241 | 539 | 8000 |
| 8000 | 607 | 173 | 587 | 233 | 763 | 17 | 423 | 397 | 729 | 51 | 469 | 351 | 685 | 95 | 505 | 315 | 641 | 139 | 541 | 279 | 8000 |
| 8000 | 593 | 227 | 613 | 167 | 437 | 383 | 777 | 3 | 471 | 349 | 731 | 49 | 515 | 305 | 695 | 85 | 559 | 261 | 659 | 121 | 8000 |
| 8000 | 291 | 489 | 111 | 709 | 255 | 525 | 155 | 665 | 219 | 561 | 199 | 621 | 373 | 407 | 33 | 787 | 337 | 443 | 77 | 743 | 8000 |
| 8000 | 109 | 711 | 289 | 491 | 145 | 675 | 245 | 535 | 181 | 639 | 201 | 579 | 27 | 793 | 367 | 413 | 63 | 757 | 323 | 457 | 8000 |
| 8000 | 689 | 91 | 509 | 311 | 645 | 135 | 545 | 275 | 601 | 179 | 581 | 239 | 767 | 13 | 427 | 393 | 723 | 57 | 463 | 357 | 8000 |
| 8000 | 511 | 309 | 691 | 89 | 555 | 265 | 655 | 125 | 599 | 221 | 619 | 161 | 433 | 387 | 773 | 7 | 477 | 343 | 737 | 43 | 8000 |
| | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 | 8000 |

In this case the magic sum is $S_{20\times 20} = 8000 = 20^3$, and the sum of all entries is $F_{400} := 20 \times 8000 = 160000 = 400^2 = 20^4$. Each $4\times 4$ block is a **pan diagonal magic square** of order 4 with the same magic sum $S_{4\times 4} = 1600$. These blocks also have the perfect square sum property, i.e., the sum of all entries in each block is $F_{16} := 4\times 1600 = 6400 = 80^2$.

## 4.19 Magic Squares of Order 21

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 21 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.19.1 First Approach: Consecutive Odd Order

Taking $k = 21$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 21^2-1) = 21^4 \\ \Rightarrow &1+3+5+\ldots+879+881 \quad = 194481 = 441^2 = 21^4. \end{aligned} \tag{34}$$

According to the values given in equation (34), we shall construct the order 21 magic squares in two different ways.

**• First Way**

**Example 51.** *According to the values given in equation (34), the magic square of order 21 is given by*

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | 9261 |
| 295 | 311 | 327 | 343 | 359 | 375 | 391 | 785 | 801 | 817 | 833 | 849 | 865 | 881 | 99 | 115 | 131 | 147 | 163 | 179 | 195 | 9261 |
| 373 | 389 | 307 | 309 | 325 | 341 | 357 | 863 | 879 | 797 | 799 | 815 | 831 | 847 | 177 | 193 | 111 | 113 | 129 | 145 | 161 | 9261 |
| 339 | 355 | 371 | 387 | 305 | 321 | 323 | 829 | 845 | 861 | 877 | 795 | 811 | 813 | 143 | 159 | 175 | 191 | 109 | 125 | 127 | 9261 |
| 319 | 335 | 337 | 353 | 369 | 385 | 303 | 809 | 825 | 827 | 843 | 859 | 875 | 793 | 123 | 139 | 141 | 157 | 173 | 189 | 107 | 9261 |
| 383 | 301 | 317 | 333 | 349 | 351 | 367 | 873 | 791 | 807 | 823 | 839 | 841 | 857 | 187 | 105 | 121 | 137 | 153 | 155 | 171 | 9261 |
| 363 | 365 | 381 | 299 | 315 | 331 | 347 | 853 | 855 | 871 | 789 | 805 | 821 | 837 | 167 | 169 | 185 | 103 | 119 | 135 | 151 | 9261 |
| 329 | 345 | 361 | 377 | 379 | 297 | 313 | 819 | 835 | 851 | 867 | 869 | 787 | 803 | 133 | 149 | 165 | 181 | 183 | 101 | 117 | 9261 |
| 197 | 213 | 229 | 245 | 261 | 277 | 293 | 393 | 409 | 425 | 441 | 457 | 473 | 489 | 589 | 605 | 621 | 637 | 653 | 669 | 685 | 9261 |
| 275 | 291 | 209 | 211 | 227 | 243 | 259 | 471 | 487 | 405 | 407 | 423 | 439 | 455 | 667 | 683 | 601 | 603 | 619 | 635 | 651 | 9261 |
| 241 | 257 | 273 | 289 | 207 | 223 | 225 | 437 | 453 | 469 | 485 | 403 | 419 | 421 | 633 | 649 | 665 | 681 | 599 | 615 | 617 | 9261 |
| 221 | 237 | 239 | 255 | 271 | 287 | 205 | 417 | 433 | 435 | 451 | 467 | 483 | 401 | 613 | 629 | 631 | 647 | 663 | 679 | 597 | 9261 |
| 285 | 203 | 219 | 235 | 251 | 253 | 269 | 481 | 399 | 415 | 431 | 447 | 449 | 465 | 677 | 595 | 611 | 627 | 643 | 645 | 661 | 9261 |
| 265 | 267 | 283 | 201 | 217 | 233 | 249 | 461 | 463 | 479 | 397 | 413 | 429 | 445 | 657 | 659 | 675 | 593 | 609 | 625 | 641 | 9261 |
| 231 | 247 | 263 | 279 | 281 | 199 | 215 | 427 | 443 | 459 | 475 | 477 | 395 | 411 | 623 | 639 | 655 | 671 | 673 | 591 | 607 | 9261 |
| 687 | 703 | 719 | 735 | 751 | 767 | 783 | 1 | 17 | 33 | 49 | 65 | 81 | 97 | 491 | 507 | 523 | 539 | 555 | 571 | 587 | 9261 |
| 765 | 781 | 699 | 701 | 717 | 733 | 749 | 79 | 95 | 13 | 15 | 31 | 47 | 63 | 569 | 585 | 503 | 505 | 521 | 537 | 553 | 9261 |
| 731 | 747 | 763 | 779 | 697 | 713 | 715 | 45 | 61 | 77 | 93 | 11 | 27 | 29 | 535 | 551 | 567 | 583 | 501 | 517 | 519 | 9261 |
| 711 | 727 | 729 | 745 | 761 | 777 | 695 | 25 | 41 | 43 | 59 | 75 | 91 | 9 | 515 | 531 | 533 | 549 | 565 | 581 | 499 | 9261 |
| 775 | 693 | 709 | 725 | 741 | 743 | 759 | 89 | 7 | 23 | 39 | 55 | 57 | 73 | 579 | 497 | 513 | 529 | 545 | 547 | 563 | 9261 |
| 755 | 757 | 773 | 691 | 707 | 723 | 739 | 69 | 71 | 87 | 5 | 21 | 37 | 53 | 559 | 561 | 577 | 495 | 511 | 527 | 543 | 9261 |
| 721 | 737 | 753 | 769 | 771 | 689 | 705 | 35 | 51 | 67 | 83 | 85 | 3 | 19 | 525 | 541 | 557 | 573 | 575 | 493 | 509 | 9261 |
| 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 |

In this case the magic sum is $S_{21\times 21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 21 \times 9261 = 194481 = 441^2 = 21^4$. Moreover, each $7 \times 7$ block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3. See below

**Example 52.** *Each $7 \times 7$ blokc of the magic square of order 21 given in 51 is a* ***pan diagonal magic square*** *with a different magic sum, and the combination of these sums forms another magic square of order 3*

| | | | |
|---|---|---|---|
| | | | 9261 |
| 2401 | 5831 | 1029 | 9261 |
| 1715 | 3087 | 4459 | 9261 |
| 5145 | 343 | 3773 | 9261 |
| 9261 | 9261 | 9261 | 9261 |

$= 7^3 \times$

| | | | |
|---|---|---|---|
| | | | 27 |
| 7 | 17 | 3 | 27 |
| 5 | 9 | 13 | 27 |
| 15 | 1 | 11 | 27 |
| 27 | 27 | 27 | 27 |

## • Second Way

**Example 53.** *According to the values given in equation (34), the magic square of order 21 is given by*

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | 9261 |
| 7 | 151 | 295 | 439 | 583 | 727 | 871 | 17 | 161 | 305 | 449 | 593 | 737 | 881 | 3 | 147 | 291 | 435 | 579 | 723 | 867 | 9261 |
| 709 | 853 | 115 | 133 | 277 | 421 | 565 | 719 | 863 | 125 | 143 | 287 | 431 | 575 | 705 | 849 | 111 | 129 | 273 | 417 | 561 | 9261 |
| 403 | 547 | 691 | 835 | 97 | 241 | 259 | 413 | 557 | 701 | 845 | 107 | 251 | 269 | 399 | 543 | 687 | 831 | 93 | 237 | 255 | 9261 |
| 223 | 367 | 385 | 529 | 673 | 817 | 79 | 233 | 377 | 395 | 539 | 683 | 827 | 89 | 219 | 363 | 381 | 525 | 669 | 813 | 75 | 9261 |
| 799 | 61 | 205 | 349 | 493 | 511 | 655 | 809 | 71 | 215 | 359 | 503 | 521 | 665 | 795 | 57 | 201 | 345 | 489 | 507 | 651 | 9261 |
| 619 | 637 | 781 | 43 | 187 | 331 | 475 | 629 | 647 | 791 | 53 | 197 | 341 | 485 | 615 | 633 | 777 | 39 | 183 | 327 | 471 | 9261 |
| 313 | 457 | 601 | 745 | 763 | 25 | 169 | 323 | 467 | 611 | 755 | 773 | 35 | 179 | 309 | 453 | 597 | 741 | 759 | 21 | 165 | 9261 |
| 5 | 149 | 293 | 437 | 581 | 725 | 869 | 9 | 153 | 297 | 441 | 585 | 729 | 873 | 13 | 157 | 301 | 445 | 589 | 733 | 877 | 9261 |
| 707 | 851 | 113 | 131 | 275 | 419 | 563 | 711 | 855 | 117 | 135 | 279 | 423 | 567 | 715 | 859 | 121 | 139 | 283 | 427 | 571 | 9261 |
| 401 | 545 | 689 | 833 | 95 | 239 | 257 | 405 | 549 | 693 | 837 | 99 | 243 | 261 | 409 | 553 | 697 | 841 | 103 | 247 | 265 | 9261 |
| 221 | 365 | 383 | 527 | 671 | 815 | 77 | 225 | 369 | 387 | 531 | 675 | 819 | 81 | 229 | 373 | 391 | 535 | 679 | 823 | 85 | 9261 |
| 797 | 59 | 203 | 347 | 491 | 509 | 653 | 801 | 63 | 207 | 351 | 495 | 513 | 657 | 805 | 67 | 211 | 355 | 499 | 517 | 661 | 9261 |
| 617 | 635 | 779 | 41 | 185 | 329 | 473 | 621 | 639 | 783 | 45 | 189 | 333 | 477 | 625 | 643 | 787 | 49 | 193 | 337 | 481 | 9261 |
| 311 | 455 | 599 | 743 | 761 | 23 | 167 | 315 | 459 | 603 | 747 | 765 | 27 | 171 | 319 | 463 | 607 | 751 | 769 | 31 | 175 | 9261 |
| 15 | 159 | 303 | 447 | 591 | 735 | 879 | 1 | 145 | 289 | 433 | 577 | 721 | 865 | 11 | 155 | 299 | 443 | 587 | 731 | 875 | 9261 |
| 717 | 861 | 123 | 141 | 285 | 429 | 573 | 703 | 847 | 109 | 127 | 271 | 415 | 559 | 713 | 857 | 119 | 137 | 281 | 425 | 569 | 9261 |
| 411 | 555 | 699 | 843 | 105 | 249 | 267 | 397 | 541 | 685 | 829 | 91 | 235 | 253 | 407 | 551 | 695 | 839 | 101 | 245 | 263 | 9261 |
| 231 | 375 | 393 | 537 | 681 | 825 | 87 | 217 | 361 | 379 | 523 | 667 | 811 | 73 | 227 | 371 | 389 | 533 | 677 | 821 | 83 | 9261 |
| 807 | 69 | 213 | 357 | 501 | 519 | 663 | 793 | 55 | 199 | 343 | 487 | 505 | 649 | 803 | 65 | 209 | 353 | 497 | 515 | 659 | 9261 |
| 627 | 645 | 789 | 51 | 195 | 339 | 483 | 613 | 631 | 775 | 37 | 181 | 325 | 469 | 623 | 641 | 785 | 47 | 191 | 335 | 479 | 9261 |
| 321 | 465 | 609 | 753 | 771 | 33 | 177 | 307 | 451 | 595 | 739 | 757 | 19 | 163 | 317 | 461 | 605 | 749 | 767 | 29 | 173 | 9261 |
| 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 |

In this case the magic sum is $S_{21\times 21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 21 \times 9261 = 194481 = 441^2 = 21^4$. Moreover, each $7 \times 7$ block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3. See below

**Example 54.** *Each $7 \times 7$ block of the magic square of order 21 given in 53 is a* ***pan diagonal magic square*** *with a different magic sum, and the combination of these sums forms another magic square of order 3:*

| | | | |
|---|---|---|---|
| | | | 9261 |
| 3073 | 3143 | 3045 | 9261 |
| 3059 | 3087 | 3115 | 9261 |
| 3129 | 3031 | 3101 | 9261 |
| 9261 | 9261 | 9261 | 9261 |

$= 7 \times$

| | | | |
|---|---|---|---|
| | | | 1323 |
| 439 | 449 | 435 | 1323 |
| 437 | 441 | 445 | 1323 |
| 447 | 433 | 443 | 1323 |
| 1323 | 1323 | 1323 | 1323 |

Interestingly, the same happens with the sums of all entries of each block of order 7, which also form a magic square of order 3:

**Example 55.** *The combination of the sums of all the entries of each $7 \times 7$ block of the magic square of order 21 given in 53 is once again a magic square of order 3:*

| | | | |
|---|---|---|---|
| | | | 64827 |
| 21511 | 22001 | 21315 | 64827 |
| 21413 | 21609 | 21805 | 64827 |
| 21903 | 21217 | 21707 | 64827 |
| 64827 | 64827 | 64827 | 64827 |

$= 7^2 \times$

| | | | |
|---|---|---|---|
| | | | 1323 |
| 439 | 449 | 435 | 1323 |
| 437 | 441 | 445 | 1323 |
| 447 | 433 | 443 | 1323 |
| 1323 | 1323 | 1323 | 1323 |

### 4.19.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-441)(n-440)}{2} = 225(n-220)$$

Taking $n = 661$, we get a perfect square, i.e.,

$$G := T(661) - T(220) = 194481 = 441^2 = 21^4.$$

Simplifying, we get

$$221 + 222 + 223 + \ldots + 660 + 661 = 194481 = 441^2 = 21^4.$$

This gives a perfect square sum for the 441 entries of consecutive natural numbers from to 221 to 661.

**Example 56.** *According to the values from 221 to 661 given above, we can easily construct a magic square of order 21 of consecutive natural numbers with a perfect square sum of all entries:*

| | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | 9261 |
| 368 | 376 | 384 | 392 | 400 | 408 | 416 | 613 | 621 | 629 | 637 | 645 | 653 | 661 | 270 | 278 | 286 | 294 | 302 | 310 | 318 | 9261 |
| 407 | 415 | 374 | 375 | 383 | 391 | 399 | 652 | 660 | 619 | 620 | 628 | 636 | 644 | 309 | 317 | 276 | 277 | 285 | 293 | 301 | 9261 |
| 390 | 398 | 406 | 414 | 373 | 381 | 382 | 635 | 643 | 651 | 659 | 618 | 626 | 627 | 292 | 300 | 308 | 316 | 275 | 283 | 284 | 9261 |
| 380 | 388 | 389 | 397 | 405 | 413 | 372 | 625 | 633 | 634 | 642 | 650 | 658 | 617 | 282 | 290 | 291 | 299 | 307 | 315 | 274 | 9261 |
| 412 | 371 | 379 | 387 | 395 | 396 | 404 | 657 | 616 | 624 | 632 | 640 | 641 | 649 | 314 | 273 | 281 | 289 | 297 | 298 | 306 | 9261 |
| 402 | 403 | 411 | 370 | 378 | 386 | 394 | 647 | 648 | 656 | 615 | 623 | 631 | 639 | 304 | 305 | 313 | 272 | 280 | 288 | 296 | 9261 |
| 385 | 393 | 401 | 409 | 410 | 369 | 377 | 630 | 638 | 646 | 654 | 655 | 614 | 622 | 287 | 295 | 303 | 311 | 312 | 271 | 279 | 9261 |
| 319 | 327 | 335 | 343 | 351 | 359 | 367 | 417 | 425 | 433 | 441 | 449 | 457 | 465 | 515 | 523 | 531 | 539 | 547 | 555 | 563 | 9261 |
| 358 | 366 | 325 | 326 | 334 | 342 | 350 | 456 | 464 | 423 | 424 | 432 | 440 | 448 | 554 | 562 | 521 | 522 | 530 | 538 | 546 | 9261 |
| 341 | 349 | 357 | 365 | 324 | 332 | 333 | 439 | 447 | 455 | 463 | 422 | 430 | 431 | 537 | 545 | 553 | 561 | 520 | 528 | 529 | 9261 |
| 331 | 339 | 340 | 348 | 356 | 364 | 323 | 429 | 437 | 438 | 446 | 454 | 462 | 421 | 527 | 535 | 536 | 544 | 552 | 560 | 519 | 9261 |
| 363 | 322 | 330 | 338 | 346 | 347 | 355 | 461 | 420 | 428 | 436 | 444 | 445 | 453 | 559 | 518 | 526 | 534 | 542 | 543 | 551 | 9261 |
| 353 | 354 | 362 | 321 | 329 | 337 | 345 | 451 | 452 | 460 | 419 | 427 | 435 | 443 | 549 | 550 | 558 | 517 | 525 | 533 | 541 | 9261 |
| 336 | 344 | 352 | 360 | 361 | 320 | 328 | 434 | 442 | 450 | 458 | 459 | 418 | 426 | 532 | 540 | 548 | 556 | 557 | 516 | 524 | 9261 |
| 564 | 572 | 580 | 588 | 596 | 604 | 612 | 221 | 229 | 237 | 245 | 253 | 261 | 269 | 466 | 474 | 482 | 490 | 498 | 506 | 514 | 9261 |
| 603 | 611 | 570 | 571 | 579 | 587 | 595 | 260 | 268 | 227 | 228 | 236 | 244 | 252 | 505 | 513 | 472 | 473 | 481 | 489 | 497 | 9261 |
| 586 | 594 | 602 | 610 | 569 | 577 | 578 | 243 | 251 | 259 | 267 | 226 | 234 | 235 | 488 | 496 | 504 | 512 | 471 | 479 | 480 | 9261 |
| 576 | 584 | 585 | 593 | 601 | 609 | 568 | 233 | 241 | 242 | 250 | 258 | 266 | 225 | 478 | 486 | 487 | 495 | 503 | 511 | 470 | 9261 |
| 608 | 567 | 575 | 583 | 591 | 592 | 600 | 265 | 224 | 232 | 240 | 248 | 249 | 257 | 510 | 469 | 477 | 485 | 493 | 494 | 502 | 9261 |
| 598 | 599 | 607 | 566 | 574 | 582 | 590 | 255 | 256 | 264 | 223 | 231 | 239 | 247 | 500 | 501 | 509 | 468 | 476 | 484 | 492 | 9261 |
| 581 | 589 | 597 | 605 | 606 | 565 | 573 | 238 | 246 | 254 | 262 | 263 | 222 | 230 | 483 | 491 | 499 | 507 | 508 | 467 | 475 | 9261 |
| 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 | 9261 |

In the examples given above the magic sum is $S_{21\times21} = 9261 = 21^3$, and the sum of all entries is $F_{441} := 194481 = 441^2 = 21^4$. In this case, each $7\times7$ block is a **pan diagonal magic square** that has a different magic sum, and the combination of these sums forms another magic square of order 3:

**Example 57.** *The following magic square of order 3 results from the combination of the different magic sums of each $7\times7$ **pan diagonal magic square** block of the order 21 magic square in example 56:*

| | | | |
|---|---|---|---|
| | | | 9261 |
| 2744 | 4459 | 2058 | 9261 |
| 2401 | 3087 | 3773 | 9261 |
| 4116 | 1715 | 3430 | 9261 |
| 9261 | 9261 | 9261 | 9261 |

$= 7^3 \times$

| | | | |
|---|---|---|---|
| | | | 27 |
| 8 | 13 | 6 | 27 |
| 7 | 9 | 11 | 27 |
| 12 | 5 | 10 | 27 |
| 27 | 27 | 27 | 27 |

Interestingly, the same happens with the sums of all entries of each $7 \times 7$ block which again form a magic square of order 3.

**Example 58.** *The following magic square of order 3 results from the combination of the different sums of the entries of each $7 \times 7$ pan diagonal magic square block of the order 21 magic square in example 56:*

| | | | |
|---|---|---|---|
| | | | 64827 |
| 19208 | 31213 | 14406 | 64827 |
| 16807 | 21609 | 26411 | 64827 |
| 28812 | 12005 | 24010 | 64827 |
| 64827 | 64827 | 64827 | 64827 |

$= 7^4 \times$

| | | | |
|---|---|---|---|
| | | | 27 |
| 8 | 13 | 6 | 27 |
| 7 | 9 | 11 | 27 |
| 12 | 5 | 10 | 27 |
| 27 | 27 | 27 | 27 |

### 4.19.3 Third Approach: Minimum Perfect Square Sum

The examples given in 51, 53 and 56 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 21, 21^2, 21^3, 21^4 \rangle$. However, the example 56 with consecutive natural numbers does not have a mininum perfect square sum. Let's see the first example of the magic square of order 21 given in section 3:

$$L\left(15^2 + \frac{21^2 - 1}{2}, 21^2\right) \to \left(21, 5, 445, 4725, 99225, 3\sqrt{35}\right).$$

In this case, we have magic square of order 21 with a perfect square sum of all entries $99225 = 315^2$, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 59.** *The minimum perfect square sum magic square of order 21 is given by*

| | | | | | | | | | | | | | | | | | | | | | 4725 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 8 | 80 | 152 | 224 | 296 | 368 | 440 | 13 | 85 | 157 | 229 | 301 | 373 | 445 | 6 | 78 | 150 | 222 | 294 | 366 | 438 | 4725 |
| 359 | 431 | 62 | 71 | 143 | 215 | 287 | 364 | 436 | 67 | 76 | 148 | 220 | 292 | 357 | 429 | 60 | 69 | 141 | 213 | 285 | 4725 |
| 206 | 278 | 350 | 422 | 53 | 125 | 134 | 211 | 283 | 355 | 427 | 58 | 130 | 139 | 204 | 276 | 348 | 420 | 51 | 123 | 132 | 4725 |
| 116 | 188 | 197 | 269 | 341 | 413 | 44 | 121 | 193 | 202 | 274 | 346 | 418 | 49 | 114 | 186 | 195 | 267 | 339 | 411 | 42 | 4725 |
| 404 | 35 | 107 | 179 | 251 | 260 | 332 | 409 | 40 | 112 | 184 | 256 | 265 | 337 | 402 | 33 | 105 | 177 | 249 | 258 | 330 | 4725 |
| 314 | 323 | 395 | 26 | 98 | 170 | 242 | 319 | 328 | 400 | 31 | 103 | 175 | 247 | 312 | 321 | 393 | 24 | 96 | 168 | 240 | 4725 |
| 161 | 233 | 305 | 377 | 386 | 17 | 89 | 166 | 238 | 310 | 382 | 391 | 22 | 94 | 159 | 231 | 303 | 375 | 384 | 15 | 87 | 4725 |
| 7 | 79 | 151 | 223 | 295 | 367 | 439 | 9 | 81 | 153 | 225 | 297 | 369 | 441 | 11 | 83 | 155 | 227 | 299 | 371 | 443 | 4725 |
| 358 | 430 | 61 | 70 | 142 | 214 | 286 | 360 | 432 | 63 | 72 | 144 | 216 | 288 | 362 | 434 | 65 | 74 | 146 | 218 | 290 | 4725 |
| 205 | 277 | 349 | 421 | 52 | 124 | 133 | 207 | 279 | 351 | 423 | 54 | 126 | 135 | 209 | 281 | 353 | 425 | 56 | 128 | 137 | 4725 |
| 115 | 187 | 196 | 268 | 340 | 412 | 43 | 117 | 189 | 198 | 270 | 342 | 414 | 45 | 119 | 191 | 200 | 272 | 344 | 416 | 47 | 4725 |
| 403 | 34 | 106 | 178 | 250 | 259 | 331 | 405 | 36 | 108 | 180 | 252 | 261 | 333 | 407 | 38 | 110 | 182 | 254 | 263 | 335 | 4725 |
| 313 | 322 | 394 | 25 | 97 | 169 | 241 | 315 | 324 | 396 | 27 | 99 | 171 | 243 | 317 | 326 | 398 | 29 | 101 | 173 | 245 | 4725 |
| 160 | 232 | 304 | 376 | 385 | 16 | 88 | 162 | 234 | 306 | 378 | 387 | 18 | 90 | 164 | 236 | 308 | 380 | 389 | 20 | 92 | 4725 |
| 12 | 84 | 156 | 228 | 300 | 372 | 444 | 5 | 77 | 149 | 221 | 293 | 365 | 437 | 10 | 82 | 154 | 226 | 298 | 370 | 442 | 4725 |
| 363 | 435 | 66 | 75 | 147 | 219 | 291 | 356 | 428 | 59 | 68 | 140 | 212 | 284 | 361 | 433 | 64 | 73 | 145 | 217 | 289 | 4725 |
| 210 | 282 | 354 | 426 | 57 | 129 | 138 | 203 | 275 | 347 | 419 | 50 | 122 | 131 | 208 | 280 | 352 | 424 | 55 | 127 | 136 | 4725 |
| 120 | 192 | 201 | 273 | 345 | 417 | 48 | 113 | 185 | 194 | 266 | 338 | 410 | 41 | 118 | 190 | 199 | 271 | 343 | 415 | 46 | 4725 |
| 408 | 39 | 111 | 183 | 255 | 264 | 336 | 401 | 32 | 104 | 176 | 248 | 257 | 329 | 406 | 37 | 109 | 181 | 253 | 262 | 334 | 4725 |
| 318 | 327 | 399 | 30 | 102 | 174 | 246 | 311 | 320 | 392 | 23 | 95 | 167 | 239 | 316 | 325 | 397 | 28 | 100 | 172 | 244 | 4725 |
| 165 | 237 | 309 | 381 | 390 | 21 | 93 | 158 | 230 | 302 | 374 | 383 | 14 | 86 | 163 | 235 | 307 | 379 | 388 | 19 | 91 | 4725 |
| 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 | 4725 |

The above example 59 is with magic sum $S_{21\times 21} = 4725$, and the sum of all entries is $F_{441} := 21 \times 1815 = 99225 = 315^2$. In this case each $7 \times 7$ block is a **pan diagonal magic square** with a different magic sum, and the combination of these sums forms another magic square of order 3:

**Example 60.** *The following magic square of order 3 results from the combination of the different magic sums of each* $7 \times 7$ ***pan diagonal magic square*** *block of the order 21 magic square:*

| | | | 4725 |
|---|---|---|---|
| 1568 | 1603 | 1554 | 4725 |
| 1561 | 1575 | 1589 | 4725 |
| 1596 | 1547 | 1582 | 4725 |
| 4725 | 4725 | 4725 | 4725 |

$= 7 \times$

| | | | 675 |
|---|---|---|---|
| 224 | 229 | 222 | 675 |
| 223 | 225 | 227 | 675 |
| 228 | 221 | 226 | 675 |
| 675 | 675 | 675 | 675 |

Interestingly, the same happens with the sums of all entries of each $7 \times 7$ block which again form a magic square of order 3.

**Example 61.** *The following magic square of order 3 results from the combination of the different sums of the entries of each* $7 \times 7$ *pan diagonal magic square block of the order 21 magic square in example 59:*

*59 is given by*

| | | | 33075 |
|---|---|---|---|
| 10976 | 11221 | 10878 | 33075 |
| 10927 | 11025 | 11123 | 33075 |
| 11172 | 10829 | 11074 | 33075 |
| 33075 | 33075 | 33075 | 33075 |

$= 7^2 \times$

| | | | 675 |
|---|---|---|---|
| 224 | 229 | 222 | 675 |
| 223 | 225 | 227 | 675 |
| 228 | 221 | 226 | 675 |
| 675 | 675 | 675 | 675 |

The **block-wise** constructions of magic squares of order 21 given in examples 51, 53, 56 and 59 are the compositions of magic squares of order 7 given in subsection 4.5 together with magic squares of order 3 given in subsection 4.1.

## 4.20 Magic Squares of Order 22

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 22 with this property if we use odd number entries. Taking $k = 22$ in equation (5), we get

$$\begin{aligned} & 1+3+5+\ldots+(2\times 22^2-1) = 22^4 \\ \Rightarrow & 1+3+5+\ldots+965+967 = 234256 = 484^2 = 22^4. \end{aligned} \tag{35}$$

**Example 62.** *According to the values given in equation (35), the magic square of order 22 for odd numbers is given by*

| | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | | | | | | | | | | | | 10648 |
| 1 | 187 | 151 | 399 | 227 | 773 | 953 | 913 | 863 | 823 | 49 | 97 | 475 | 523 | 395 | 345 | 305 | 549 | 719 | 683 | 629 | 589 | 10648 |
| 229 | 47 | 11 | 195 | 401 | 869 | 819 | 779 | 949 | 909 | 95 | 133 | 351 | 301 | 481 | 519 | 391 | 639 | 585 | 545 | 725 | 675 | 10648 |
| 403 | 221 | 93 | 55 | 19 | 955 | 905 | 865 | 825 | 775 | 141 | 179 | 525 | 387 | 347 | 307 | 477 | 721 | 681 | 631 | 595 | 541 | 10648 |
| 63 | 405 | 223 | 139 | 99 | 821 | 781 | 951 | 911 | 861 | 177 | 5 | 303 | 483 | 521 | 393 | 343 | 587 | 551 | 717 | 677 | 637 | 10648 |
| 143 | 107 | 397 | 225 | 185 | 907 | 867 | 817 | 777 | 957 | 3 | 51 | 389 | 349 | 299 | 479 | 527 | 673 | 633 | 593 | 543 | 727 | 10648 |
| 563 | 743 | 703 | 653 | 613 | 231 | 499 | 459 | 419 | 365 | 325 | 285 | 749 | 929 | 889 | 839 | 799 | 25 | 205 | 165 | 115 | 75 | 10648 |
| 659 | 609 | 569 | 739 | 699 | 331 | 277 | 237 | 505 | 451 | 411 | 371 | 845 | 795 | 755 | 925 | 885 | 121 | 71 | 31 | 201 | 161 | 10648 |
| 745 | 695 | 655 | 615 | 565 | 417 | 363 | 323 | 283 | 243 | 497 | 457 | 931 | 881 | 841 | 801 | 751 | 207 | 157 | 117 | 77 | 27 | 10648 |
| 611 | 571 | 741 | 701 | 651 | 503 | 463 | 409 | 369 | 329 | 275 | 235 | 797 | 757 | 927 | 887 | 837 | 73 | 33 | 203 | 163 | 113 | 10648 |
| 697 | 657 | 607 | 567 | 747 | 281 | 241 | 495 | 455 | 415 | 375 | 321 | 883 | 843 | 793 | 753 | 933 | 159 | 119 | 69 | 29 | 209 | 10648 |
| 91 | 137 | 183 | 9 | 45 | 367 | 327 | 287 | 233 | 501 | 461 | 407 | 643 | 689 | 735 | 561 | 597 | 873 | 919 | 965 | 791 | 827 | 10648 |
| 181 | 7 | 53 | 89 | 135 | 453 | 413 | 373 | 319 | 279 | 239 | 507 | 733 | 559 | 605 | 641 | 687 | 963 | 789 | 835 | 871 | 917 | 10648 |
| 769 | 939 | 903 | 849 | 809 | 35 | 215 | 175 | 125 | 85 | 601 | 649 | 553 | 715 | 679 | 423 | 251 | 441 | 489 | 361 | 311 | 271 | 10648 |
| 859 | 805 | 765 | 945 | 895 | 131 | 81 | 41 | 211 | 171 | 647 | 685 | 253 | 599 | 539 | 723 | 425 | 317 | 267 | 447 | 485 | 357 | 10648 |
| 941 | 901 | 851 | 815 | 761 | 217 | 167 | 127 | 87 | 37 | 693 | 731 | 427 | 245 | 645 | 583 | 547 | 491 | 353 | 313 | 273 | 443 | 10648 |
| 807 | 771 | 937 | 897 | 857 | 83 | 43 | 213 | 173 | 123 | 729 | 557 | 591 | 429 | 247 | 691 | 627 | 269 | 449 | 487 | 359 | 309 | 10648 |
| 893 | 853 | 813 | 763 | 947 | 169 | 129 | 79 | 39 | 219 | 555 | 603 | 671 | 635 | 421 | 249 | 737 | 355 | 315 | 265 | 445 | 493 | 10648 |
| 465 | 513 | 385 | 335 | 295 | 529 | 709 | 669 | 619 | 579 | 831 | 879 | 21 | 191 | 155 | 101 | 61 | 783 | 935 | 899 | 433 | 261 | 10648 |
| 341 | 291 | 471 | 509 | 381 | 625 | 575 | 535 | 705 | 665 | 877 | 915 | 111 | 57 | 17 | 197 | 147 | 263 | 829 | 759 | 943 | 435 | 10648 |
| 515 | 377 | 337 | 297 | 467 | 711 | 661 | 621 | 581 | 531 | 923 | 961 | 193 | 153 | 103 | 67 | 13 | 437 | 255 | 875 | 803 | 767 | 10648 |
| 293 | 473 | 511 | 383 | 333 | 577 | 537 | 707 | 667 | 617 | 959 | 787 | 59 | 23 | 189 | 149 | 109 | 811 | 439 | 257 | 921 | 847 | 10648 |
| 379 | 339 | 289 | 469 | 517 | 663 | 623 | 573 | 533 | 713 | 785 | 833 | 145 | 105 | 65 | 15 | 199 | 891 | 855 | 431 | 259 | 967 | 10648 |
| 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 | 10648 |

This magic square example has a magic sum $S_{22\times 22} := 10648 = 22^3$ and the sum of all 484 entries is $F_{484} := 234256 = 484^2 = 22^4$. There are 13 blocks of order 4 and 1 of order 7 which are magic squares with their entries written in **bold face letters**. The approach applied to construct this magic square is based on the procedure of H. White [26].

## 4.21 Magic Squares of Order 23

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 23 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

#### 4.21.1 First Approach: Consecutive Odd Numbers

Taking $k = 23$ in equation (5), we get

$$\begin{aligned} &1 + 3 + 5 + \ldots + (2 \times 23^2 - 1) = 23^4 \\ \Rightarrow &1 + 3 + 5 + \ldots + 1055 + 1057 \ = 279841 = 529^2 = 23^4 \end{aligned} \qquad (36)$$

**Example 63.** *According to the 529 values given in equation (36), the* ***pan diagonal magic square*** *of order 23 is given by*

| | | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 89 | 131 | 173 | 215 | 257 | 299 | 341 | 383 | 425 | 467 | 509 | 597 | 639 | 681 | 723 | 765 | 807 | 849 | 891 | 933 | 975 | 1017 | 12167 |
| 12167 | 969 | 1057 | 41 | 83 | 125 | 167 | 209 | 251 | 293 | 335 | 377 | 419 | 461 | 549 | 591 | 633 | 675 | 717 | 759 | 801 | 843 | 885 | 927 | 12167 |
| 12167 | 879 | 921 | 1009 | 1051 | 35 | 77 | 119 | 161 | 203 | 245 | 287 | 329 | 371 | 459 | 501 | 543 | 585 | 627 | 669 | 711 | 753 | 795 | 837 | 12167 |
| 12167 | 789 | 831 | 919 | 961 | 1003 | 1045 | 29 | 71 | 113 | 155 | 197 | 239 | 281 | 323 | 411 | 453 | 495 | 537 | 579 | 621 | 663 | 705 | 747 | 12167 |
| 12167 | 699 | 741 | 783 | 871 | 913 | 955 | 997 | 1039 | 23 | 65 | 107 | 149 | 191 | 233 | 321 | 363 | 405 | 447 | 489 | 531 | 573 | 615 | 657 | 12167 |
| 12167 | 609 | 651 | 693 | 781 | 823 | 865 | 907 | 949 | 991 | 1033 | 17 | 59 | 101 | 143 | 185 | 273 | 315 | 357 | 399 | 441 | 483 | 525 | 567 | 12167 |
| 12167 | 519 | 561 | 603 | 645 | 733 | 775 | 817 | 859 | 901 | 943 | 985 | 1027 | 11 | 53 | 95 | 183 | 225 | 267 | 309 | 351 | 393 | 435 | 477 | 12167 |
| 12167 | 429 | 471 | 513 | 555 | 643 | 685 | 727 | 769 | 811 | 853 | 895 | 937 | 979 | 1021 | 5 | 47 | 135 | 177 | 219 | 261 | 303 | 345 | 387 | 12167 |
| 12167 | 339 | 381 | 423 | 465 | 507 | 595 | 637 | 679 | 721 | 763 | 805 | 847 | 889 | 931 | 973 | 1015 | 45 | 87 | 129 | 171 | 213 | 255 | 297 | 12167 |
| 12167 | 249 | 291 | 333 | 375 | 417 | 505 | 547 | 589 | 631 | 673 | 715 | 757 | 799 | 841 | 883 | 925 | 967 | 1055 | 39 | 81 | 123 | 165 | 207 | 12167 |
| 12167 | 159 | 201 | 243 | 285 | 327 | 369 | 457 | 499 | 541 | 583 | 625 | 667 | 709 | 751 | 793 | 835 | 877 | 965 | 1007 | 1049 | 33 | 75 | 117 | 12167 |
| 12167 | 69 | 111 | 153 | 195 | 237 | 279 | 367 | 409 | 451 | 493 | 535 | 577 | 619 | 661 | 703 | 745 | 787 | 829 | 917 | 959 | 1001 | 1043 | 27 | 12167 |
| 12167 | 1037 | 21 | 63 | 105 | 147 | 189 | 231 | 319 | 361 | 403 | 445 | 487 | 529 | 571 | 613 | 655 | 697 | 739 | 827 | 869 | 911 | 953 | 995 | 12167 |
| 12167 | 947 | 989 | 1031 | 15 | 57 | 99 | 141 | 229 | 271 | 313 | 355 | 397 | 439 | 481 | 523 | 565 | 607 | 649 | 691 | 779 | 821 | 863 | 905 | 12167 |
| 12167 | 857 | 899 | 941 | 983 | 1025 | 9 | 51 | 93 | 181 | 223 | 265 | 307 | 349 | 391 | 433 | 475 | 517 | 559 | 601 | 689 | 731 | 773 | 815 | 12167 |
| 12167 | 767 | 809 | 851 | 893 | 935 | 977 | 1019 | 3 | 91 | 133 | 175 | 217 | 259 | 301 | 343 | 385 | 427 | 469 | 511 | 553 | 641 | 683 | 725 | 12167 |
| 12167 | 677 | 719 | 761 | 803 | 845 | 887 | 929 | 971 | 1013 | 43 | 85 | 127 | 169 | 211 | 253 | 295 | 337 | 379 | 421 | 463 | 551 | 593 | 635 | 12167 |
| 12167 | 587 | 629 | 671 | 713 | 755 | 797 | 839 | 881 | 923 | 1011 | 1053 | 37 | 79 | 121 | 163 | 205 | 247 | 289 | 331 | 373 | 415 | 503 | 545 | 12167 |
| 12167 | 497 | 539 | 581 | 623 | 665 | 707 | 749 | 791 | 833 | 875 | 963 | 1005 | 1047 | 31 | 73 | 115 | 157 | 199 | 241 | 283 | 325 | 413 | 455 | 12167 |
| 12167 | 407 | 449 | 491 | 533 | 575 | 617 | 659 | 701 | 743 | 785 | 873 | 915 | 957 | 999 | 1041 | 25 | 67 | 109 | 151 | 193 | 235 | 277 | 365 | 12167 |
| 12167 | 317 | 359 | 401 | 443 | 485 | 527 | 569 | 611 | 653 | 695 | 737 | 825 | 867 | 909 | 951 | 993 | 1035 | 19 | 61 | 103 | 145 | 187 | 275 | 12167 |
| 12167 | 227 | 269 | 311 | 353 | 395 | 437 | 479 | 521 | 563 | 605 | 647 | 735 | 777 | 819 | 861 | 903 | 945 | 987 | 1029 | 13 | 55 | 97 | 139 | 12167 |
| 12167 | 137 | 179 | 221 | 263 | 305 | 347 | 389 | 431 | 473 | 515 | 557 | 599 | 687 | 729 | 771 | 813 | 855 | 897 | 939 | 981 | 1023 | 7 | 49 | 12167 |
| | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 |

#### 4.21.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-529)(n-528)}{2} = 529(n-264)$$

Taking $n = 793$, we get a perfect square, i.e.,

$$\begin{aligned} G := T(793) - T(264) = \frac{793 \times 794}{2} - \frac{264 \times 265}{2} &= 793 \times 393 - 132 \times 265 \\ &= 16471 - 1830 = 279841 = 529^2 = 23^4. \end{aligned}$$

Simplifying, we get

$$265 + 266 + 267 + \ldots + 792 + 793 = 279841 = 529^2 = 23^4.$$

This gives a perfect square sum for the 529 consecutive natural numbers from to 265 to 793.

**Example 64.** *The* ***pan diagonal magic square*** *of order 23 for the 529 entries from 265 to 793 is given by*

| | | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 265 | 309 | 330 | 351 | 372 | 393 | 414 | 435 | 456 | 477 | 498 | 519 | 563 | 584 | 605 | 626 | 647 | 668 | 689 | 710 | 731 | 752 | 773 | 12167 |
| 12167 | 749 | 793 | 285 | 306 | 327 | 348 | 369 | 390 | 411 | 432 | 453 | 474 | 495 | 539 | 560 | 581 | 602 | 623 | 644 | 665 | 686 | 707 | 728 | 12167 |
| 12167 | 704 | 725 | 769 | 790 | 282 | 303 | 324 | 345 | 366 | 387 | 408 | 429 | 450 | 494 | 515 | 536 | 557 | 578 | 599 | 620 | 641 | 662 | 683 | 12167 |
| 12167 | 659 | 680 | 724 | 745 | 766 | 787 | 279 | 300 | 321 | 342 | 363 | 384 | 405 | 426 | 470 | 491 | 512 | 533 | 554 | 575 | 596 | 617 | 638 | 12167 |
| 12167 | 614 | 635 | 656 | 700 | 721 | 742 | 763 | 784 | 276 | 297 | 318 | 339 | 360 | 381 | 425 | 446 | 467 | 488 | 509 | 530 | 551 | 572 | 593 | 12167 |
| 12167 | 569 | 590 | 611 | 655 | 676 | 697 | 718 | 739 | 760 | 781 | 273 | 294 | 315 | 336 | 357 | 401 | 422 | 443 | 464 | 485 | 506 | 527 | 548 | 12167 |
| 12167 | 524 | 545 | 566 | 587 | 631 | 652 | 673 | 694 | 715 | 736 | 757 | 778 | 270 | 291 | 312 | 356 | 377 | 398 | 419 | 440 | 461 | 482 | 503 | 12167 |
| 12167 | 479 | 500 | 521 | 542 | 586 | 607 | 628 | 649 | 670 | 691 | 712 | 733 | 754 | 775 | 267 | 288 | 332 | 353 | 374 | 395 | 416 | 437 | 458 | 12167 |
| 12167 | 434 | 455 | 476 | 497 | 518 | 562 | 583 | 604 | 625 | 646 | 667 | 688 | 709 | 730 | 751 | 772 | 287 | 308 | 329 | 350 | 371 | 392 | 413 | 12167 |
| 12167 | 389 | 410 | 431 | 452 | 473 | 517 | 538 | 559 | 580 | 601 | 622 | 643 | 664 | 685 | 706 | 727 | 748 | 792 | 284 | 305 | 326 | 347 | 368 | 12167 |
| 12167 | 344 | 365 | 386 | 407 | 428 | 449 | 493 | 514 | 535 | 556 | 577 | 598 | 619 | 640 | 661 | 682 | 703 | 747 | 768 | 789 | 281 | 302 | 323 | 12167 |
| 12167 | 299 | 320 | 341 | 362 | 383 | 404 | 448 | 469 | 490 | 511 | 532 | 553 | 574 | 595 | 616 | 637 | 658 | 679 | 723 | 744 | 765 | 786 | 278 | 12167 |
| 12167 | 783 | 275 | 296 | 317 | 338 | 359 | 380 | 424 | 445 | 466 | 487 | 508 | 529 | 550 | 571 | 592 | 613 | 634 | 678 | 699 | 720 | 741 | 762 | 12167 |
| 12167 | 738 | 759 | 780 | 272 | 293 | 314 | 335 | 379 | 400 | 421 | 442 | 463 | 484 | 505 | 526 | 547 | 568 | 589 | 610 | 654 | 675 | 696 | 717 | 12167 |
| 12167 | 693 | 714 | 735 | 756 | 777 | 269 | 290 | 311 | 355 | 376 | 397 | 418 | 439 | 460 | 481 | 502 | 523 | 544 | 565 | 609 | 630 | 651 | 672 | 12167 |
| 12167 | 648 | 669 | 690 | 711 | 732 | 753 | 774 | 266 | 310 | 331 | 352 | 373 | 394 | 415 | 436 | 457 | 478 | 499 | 520 | 541 | 585 | 606 | 627 | 12167 |
| 12167 | 603 | 624 | 645 | 666 | 687 | 708 | 729 | 750 | 771 | 286 | 307 | 328 | 349 | 370 | 391 | 412 | 433 | 454 | 475 | 496 | 540 | 561 | 582 | 12167 |
| 12167 | 558 | 579 | 600 | 621 | 642 | 663 | 684 | 705 | 726 | 770 | 791 | 283 | 304 | 325 | 346 | 367 | 388 | 409 | 430 | 451 | 472 | 516 | 537 | 12167 |
| 12167 | 513 | 534 | 555 | 576 | 597 | 618 | 639 | 660 | 681 | 702 | 746 | 767 | 788 | 280 | 301 | 322 | 343 | 364 | 385 | 406 | 427 | 471 | 492 | 12167 |
| 12167 | 468 | 489 | 510 | 531 | 552 | 573 | 594 | 615 | 636 | 657 | 701 | 722 | 743 | 764 | 785 | 277 | 298 | 319 | 340 | 361 | 382 | 403 | 447 | 12167 |
| 12167 | 423 | 444 | 465 | 486 | 507 | 528 | 549 | 570 | 591 | 612 | 633 | 677 | 698 | 719 | 740 | 761 | 782 | 274 | 295 | 316 | 337 | 358 | 402 | 12167 |
| 12167 | 378 | 399 | 420 | 441 | 462 | 483 | 504 | 525 | 546 | 567 | 588 | 632 | 653 | 674 | 695 | 716 | 737 | 758 | 779 | 271 | 292 | 313 | 334 | 12167 |
| 12167 | 333 | 354 | 375 | 396 | 417 | 438 | 459 | 480 | 501 | 522 | 543 | 564 | 608 | 629 | 650 | 671 | 692 | 713 | 734 | 755 | 776 | 268 | 289 | 12167 |
| | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 | 12167 |

In both the examples 63 and 64 the magic sum is $S_{23\times 23} = 12167 = 23^3$, and the sum of all numbers is $F_{529} := 279841 = 529^2 = 23^4$.

Both of the examples 63 and 64 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 23, 23^2, 23^3, 23^4 \rangle$.

### 4.21.3 Third Approach: Minimum Perfect Square Sum

The examples given in 63 and 64 satisfy the property (13) of uniformity 2.5.1, i.e., $\langle 23, 23^2, 23^3, 23^4 \rangle$. However, the sum of the magic square entries of 64 is not a minimum perfect square sum. Let's observe the first example of the magic square of order 23 given in section 3:

$$L\left(17^2 + \frac{23^2-1}{2}, 23^2\right) \rightarrow \left(23, 25, 553, 6647, 152881, \sqrt{391}\right)$$

In this case, we have magic square of order 23 with sum of all entries $152881 = 391^2$ which is a perfect square, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries:

**Example 65.** *The* ***pan diagonal magic square*** *of order 23 with 529 entries starting from 25 to 553 is given by*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 |
| | 25 | 69 | 90 | 111 | 132 | 153 | 174 | 195 | 216 | 237 | 258 | 279 | 323 | 344 | 365 | 386 | 407 | 428 | 449 | 470 | 491 | 512 | 533 | 6647 |
| 6647 | 509 | 553 | 45 | 66 | 87 | 108 | 129 | 150 | 171 | 192 | 213 | 234 | 255 | 299 | 320 | 341 | 362 | 383 | 404 | 425 | 446 | 467 | 488 | 6647 |
| 6647 | 464 | 485 | 529 | 550 | 42 | 63 | 84 | 105 | 126 | 147 | 168 | 189 | 210 | 254 | 275 | 296 | 317 | 338 | 359 | 380 | 401 | 422 | 443 | 6647 |
| 6647 | 419 | 440 | 484 | 505 | 526 | 547 | 39 | 60 | 81 | 102 | 123 | 144 | 165 | 186 | 230 | 251 | 272 | 293 | 314 | 335 | 356 | 377 | 398 | 6647 |
| 6647 | 374 | 395 | 416 | 460 | 481 | 502 | 523 | 544 | 36 | 57 | 78 | 99 | 120 | 141 | 185 | 206 | 227 | 248 | 269 | 290 | 311 | 332 | 353 | 6647 |
| 6647 | 329 | 350 | 371 | 415 | 436 | 457 | 478 | 499 | 520 | 541 | 33 | 54 | 75 | 96 | 117 | 161 | 182 | 203 | 224 | 245 | 266 | 287 | 308 | 6647 |
| 6647 | 284 | 305 | 326 | 347 | 391 | 412 | 433 | 454 | 475 | 496 | 517 | 538 | 30 | 51 | 72 | 116 | 137 | 158 | 179 | 200 | 221 | 242 | 263 | 6647 |
| 6647 | 239 | 260 | 281 | 302 | 346 | 367 | 388 | 409 | 430 | 451 | 472 | 493 | 514 | 535 | 27 | 48 | 92 | 113 | 134 | 155 | 176 | 197 | 218 | 6647 |
| 6647 | 194 | 215 | 236 | 257 | 278 | 322 | 343 | 364 | 385 | 406 | 427 | 448 | 469 | 490 | 511 | 532 | 47 | 68 | 89 | 110 | 131 | 152 | 173 | 6647 |
| 6647 | 149 | 170 | 191 | 212 | 233 | 277 | 298 | 319 | 340 | 361 | 382 | 403 | 424 | 445 | 466 | 487 | 508 | 552 | 44 | 65 | 86 | 107 | 128 | 6647 |
| 6647 | 104 | 125 | 146 | 167 | 188 | 209 | 253 | 274 | 295 | 316 | 337 | 358 | 379 | 400 | 421 | 442 | 463 | 507 | 528 | 549 | 41 | 62 | 83 | 6647 |
| 6647 | 59 | 80 | 101 | 122 | 143 | 164 | 208 | 229 | 250 | 271 | 292 | 313 | 334 | 355 | 376 | 397 | 418 | 439 | 483 | 504 | 525 | 546 | 38 | 6647 |
| 6647 | 543 | 35 | 56 | 77 | 98 | 119 | 140 | 184 | 205 | 226 | 247 | 268 | 289 | 310 | 331 | 352 | 373 | 394 | 438 | 459 | 480 | 501 | 522 | 6647 |
| 6647 | 498 | 519 | 540 | 32 | 53 | 74 | 95 | 139 | 160 | 181 | 202 | 223 | 244 | 265 | 286 | 307 | 328 | 349 | 370 | 414 | 435 | 456 | 477 | 6647 |
| 6647 | 453 | 474 | 495 | 516 | 537 | 29 | 50 | 71 | 115 | 136 | 157 | 178 | 199 | 220 | 241 | 262 | 283 | 304 | 325 | 369 | 390 | 411 | 432 | 6647 |
| 6647 | 408 | 429 | 450 | 471 | 492 | 513 | 534 | 26 | 70 | 91 | 112 | 133 | 154 | 175 | 196 | 217 | 238 | 259 | 280 | 301 | 345 | 366 | 387 | 6647 |
| 6647 | 363 | 384 | 405 | 426 | 447 | 468 | 489 | 510 | 531 | 46 | 67 | 88 | 109 | 130 | 151 | 172 | 193 | 214 | 235 | 256 | 300 | 321 | 342 | 6647 |
| 6647 | 318 | 339 | 360 | 381 | 402 | 423 | 444 | 465 | 486 | 530 | 551 | 43 | 64 | 85 | 106 | 127 | 148 | 169 | 190 | 211 | 232 | 276 | 297 | 6647 |
| 6647 | 273 | 294 | 315 | 336 | 357 | 378 | 399 | 420 | 441 | 462 | 506 | 527 | 548 | 40 | 61 | 82 | 103 | 124 | 145 | 166 | 187 | 231 | 252 | 6647 |
| 6647 | 228 | 249 | 270 | 291 | 312 | 333 | 354 | 375 | 396 | 417 | 461 | 482 | 503 | 524 | 545 | 37 | 58 | 79 | 100 | 121 | 142 | 163 | 207 | 6647 |
| 6647 | 183 | 204 | 225 | 246 | 267 | 288 | 309 | 330 | 351 | 372 | 393 | 437 | 458 | 479 | 500 | 521 | 542 | 34 | 55 | 76 | 97 | 118 | 162 | 6647 |
| 6647 | 138 | 159 | 180 | 201 | 222 | 243 | 264 | 285 | 306 | 327 | 348 | 392 | 413 | 434 | 455 | 476 | 497 | 518 | 539 | 31 | 52 | 73 | 94 | 6647 |
| 6647 | 93 | 114 | 135 | 156 | 177 | 198 | 219 | 240 | 261 | 282 | 303 | 324 | 368 | 389 | 410 | 431 | 452 | 473 | 494 | 515 | 536 | 28 | 49 | 6647 |
| | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 | 6647 |

The above example 65 has a magic sum $S_{23\times 23} = 6647$, and the sum of all entries is $F_{529} := 23 \times 6647 = 152881 = 391^2$. Moreover, it is **pan diagonal**.

## 4.22 Magic Squares of Order 24

According to equation (9), we cannot obtain a consecutive natural numbered magic square where the sum of all the entries is a perfect square. According to equation (5), we can only obtain a magic square of order 24 with this property if we use odd number entries. Taking $k = 24$ in equation (5), we get

$$1 + 3 + 5 + \ldots + (2 \times 24^2 - 1) = 24^4$$
$$\Rightarrow 1 + 3 + 5 + \ldots + 1149 + 1151 = 331776 = 576^2 = 24^4. \qquad (37)$$

There are many ways of constructing magic squares of order 24 with odd numbers and these include for example, $8 \times 3$, $3 \times 8$, $4 \times 6$ or $6 \times 4$ Instead of using all of these four ways, we shall use only two. Using the $8 \times 3$ format, we can work with the bimagic square of order 8 as given in subsection 4.6.

### • First Way: $8 \times 3$

**Example 66.** *According to the values given in equation (35), the magic square of order 24 with odd numbers is given by*

| | | | | | | | | | | | | | | | | | | | | | | | | 13824 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 277 | 727 | 637 | 79 | 475 | 1105 | 979 | 313 | 287 | 737 | 647 | 89 | 485 | 1115 | 989 | 323 | 273 | 723 | 633 | 75 | 471 | 1101 | 975 | 309 | 13824 |
| 457 | 1123 | 961 | 331 | 223 | 781 | 583 | 133 | 467 | 1133 | 971 | 341 | 233 | 791 | 593 | 143 | 453 | 1119 | 957 | 327 | 219 | 777 | 579 | 129 | 13824 |
| 7 | 709 | 799 | 205 | 385 | 907 | 1033 | 547 | 17 | 719 | 809 | 215 | 395 | 917 | 1043 | 557 | 3 | 705 | 795 | 201 | 381 | 903 | 1029 | 543 | 13824 |
| 403 | 889 | 1051 | 529 | 61 | 655 | 853 | 151 | 413 | 899 | 1061 | 539 | 71 | 665 | 863 | 161 | 399 | 885 | 1047 | 525 | 57 | 651 | 849 | 147 | 13824 |
| 673 | 43 | 169 | 835 | 871 | 421 | 511 | 1069 | 683 | 53 | 179 | 845 | 881 | 431 | 521 | 1079 | 669 | 39 | 165 | 831 | 867 | 417 | 507 | 1065 | 13824 |
| 925 | 367 | 565 | 1015 | 691 | 25 | 187 | 817 | 935 | 377 | 575 | 1025 | 701 | 35 | 197 | 827 | 921 | 363 | 561 | 1011 | 687 | 21 | 183 | 813 | 13824 |
| 763 | 241 | 115 | 601 | 1141 | 439 | 349 | 943 | 773 | 251 | 125 | 611 | 1151 | 449 | 359 | 953 | 759 | 237 | 111 | 597 | 1137 | 435 | 345 | 939 | 13824 |
| 1087 | 493 | 295 | 997 | 745 | 259 | 97 | 619 | 1097 | 503 | 305 | 1007 | 755 | 269 | 107 | 629 | 1083 | 489 | 291 | 993 | 741 | 255 | 93 | 615 | 13824 |
| 275 | 725 | 635 | 77 | 473 | 1103 | 977 | 311 | 279 | 729 | 639 | 81 | 477 | 1107 | 981 | 315 | 283 | 733 | 643 | 85 | 481 | 1111 | 985 | 319 | 13824 |
| 455 | 1121 | 959 | 329 | 221 | 779 | 581 | 131 | 459 | 1125 | 963 | 333 | 225 | 783 | 585 | 135 | 463 | 1129 | 967 | 337 | 229 | 787 | 589 | 139 | 13824 |
| 5 | 707 | 797 | 203 | 383 | 905 | 1031 | 545 | 9 | 711 | 801 | 207 | 387 | 909 | 1035 | 549 | 13 | 715 | 805 | 211 | 391 | 913 | 1039 | 553 | 13824 |
| 401 | 887 | 1049 | 527 | 59 | 653 | 851 | 149 | 405 | 891 | 1053 | 531 | 63 | 657 | 855 | 153 | 409 | 895 | 1057 | 535 | 67 | 661 | 859 | 157 | 13824 |
| 671 | 41 | 167 | 833 | 869 | 419 | 509 | 1067 | 675 | 45 | 171 | 837 | 873 | 423 | 513 | 1071 | 679 | 49 | 175 | 841 | 877 | 427 | 517 | 1075 | 13824 |
| 923 | 365 | 563 | 1013 | 689 | 23 | 185 | 815 | 927 | 369 | 567 | 1017 | 693 | 27 | 189 | 819 | 931 | 373 | 571 | 1021 | 697 | 31 | 193 | 823 | 13824 |
| 761 | 239 | 113 | 599 | 1139 | 437 | 347 | 941 | 765 | 243 | 117 | 603 | 1143 | 441 | 351 | 945 | 769 | 247 | 121 | 607 | 1147 | 445 | 355 | 949 | 13824 |
| 1085 | 491 | 293 | 995 | 743 | 257 | 95 | 617 | 1089 | 495 | 297 | 999 | 747 | 261 | 99 | 621 | 1093 | 499 | 301 | 1003 | 751 | 265 | 103 | 625 | 13824 |
| 285 | 735 | 645 | 87 | 483 | 1113 | 987 | 321 | 271 | 721 | 631 | 73 | 469 | 1099 | 973 | 307 | 281 | 731 | 641 | 83 | 479 | 1109 | 983 | 317 | 13824 |
| 465 | 1131 | 969 | 339 | 231 | 789 | 591 | 141 | 451 | 1117 | 955 | 325 | 217 | 775 | 577 | 127 | 461 | 1127 | 965 | 335 | 227 | 785 | 587 | 137 | 13824 |
| 15 | 717 | 807 | 213 | 393 | 915 | 1041 | 555 | 1 | 703 | 793 | 199 | 379 | 901 | 1027 | 541 | 11 | 713 | 803 | 209 | 389 | 911 | 1037 | 551 | 13824 |
| 411 | 897 | 1059 | 537 | 69 | 663 | 861 | 159 | 397 | 883 | 1045 | 523 | 55 | 649 | 847 | 145 | 407 | 893 | 1055 | 533 | 65 | 659 | 857 | 155 | 13824 |
| 681 | 51 | 177 | 843 | 879 | 429 | 519 | 1077 | 667 | 37 | 163 | 829 | 865 | 415 | 505 | 1063 | 677 | 47 | 173 | 839 | 875 | 425 | 515 | 1073 | 13824 |
| 933 | 375 | 573 | 1023 | 699 | 33 | 195 | 825 | 919 | 361 | 559 | 1009 | 685 | 19 | 181 | 811 | 929 | 371 | 569 | 1019 | 695 | 29 | 191 | 821 | 13824 |
| 771 | 249 | 123 | 609 | 1149 | 447 | 357 | 951 | 757 | 235 | 109 | 595 | 1135 | 433 | 343 | 937 | 767 | 245 | 119 | 605 | 1145 | 443 | 353 | 947 | 13824 |
| 1095 | 501 | 303 | 1005 | 753 | 267 | 105 | 627 | 1081 | 487 | 289 | 991 | 739 | 253 | 91 | 613 | 1091 | 497 | 299 | 1001 | 749 | 263 | 101 | 623 | 13824 |
| 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 |

The above example 66 is with magic sum $S_{24\times 24} = 13824 = 24^3$ and the sum of all entries is $F_{576} := 24 \times 13824 = 331776 = 576^2 = 24^4$.

The block-wise construction of the magic square of order 24 in example 66 is the composition of **bimagic** squares of order 8 given in subsection 4.6 with a magic square of order 3 given in subsection 4.1.

The magic square of order 3 presented below is formed by the combination of the different **bimagic** squares of order 8 which compose the magic square of order 24

**Example 67.** *Each* $8 \times 8$ *block is a bimagic square forms again a magic square of order 3 given by*

| | | | 13824 |
|---|---|---|---|
| 4592 | 4672 | 4560 | 13824 |
| 4576 | 4608 | 4640 | 13824 |
| 4656 | 4544 | 4624 | 13824 |
| 13824 | 13824 | 13824 | 13824 |

$= 2^4 \times$

| | | | 864 |
|---|---|---|---|
| 287 | 292 | 285 | 864 |
| 286 | 288 | 290 | 864 |
| 291 | 284 | 289 | 864 |
| 864 | 864 | 864 | 864 |

- **Second Way:** $4 \times 6$

**Example 68.** *A magic square of order 24 formed by 24 blocks of* ***perfect square pan diagonal magic squares*** *of order 4:*

| | | | | | | | | | | | | | | | | | | | | | | | | 13824 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 433 | 793 | 1 | 937 | 477 | 837 | 45 | 981 | 487 | 847 | 55 | 991 | 499 | 859 | 67 | 1003 | 465 | 825 | 33 | 969 | 447 | 807 | 15 | 951 | 13824 |
| 73 | 865 | 505 | 721 | 117 | 909 | 549 | 765 | 127 | 919 | 559 | 775 | 139 | 931 | 571 | 787 | 105 | 897 | 537 | 753 | 87 | 879 | 519 | 735 | 13824 |
| 1081 | 145 | 649 | 289 | 1125 | 189 | 693 | 333 | 1135 | 199 | 703 | 343 | 1147 | 211 | 715 | 355 | 1113 | 177 | 681 | 321 | 1095 | 159 | 663 | 303 | 13824 |
| 577 | 361 | 1009 | 217 | 621 | 405 | 1053 | 261 | 631 | 415 | 1063 | 271 | 643 | 427 | 1075 | 283 | 609 | 393 | 1041 | 249 | 591 | 375 | 1023 | 231 | 13824 |
| 489 | 849 | 57 | 993 | 445 | 805 | 13 | 949 | 501 | 861 | 69 | 1005 | 459 | 819 | 27 | 963 | 473 | 833 | 41 | 977 | 441 | 801 | 9 | 945 | 13824 |
| 129 | 921 | 561 | 777 | 85 | 877 | 517 | 733 | 141 | 933 | 573 | 789 | 99 | 891 | 531 | 747 | 113 | 905 | 545 | 761 | 81 | 873 | 513 | 729 | 13824 |
| 1137 | 201 | 705 | 345 | 1093 | 157 | 661 | 301 | 1149 | 213 | 717 | 357 | 1107 | 171 | 675 | 315 | 1121 | 185 | 689 | 329 | 1089 | 153 | 657 | 297 | 13824 |
| 633 | 417 | 1065 | 273 | 589 | 373 | 1021 | 229 | 645 | 429 | 1077 | 285 | 603 | 387 | 1035 | 243 | 617 | 401 | 1049 | 257 | 585 | 369 | 1017 | 225 | 13824 |
| 455 | 815 | 23 | 959 | 443 | 803 | 11 | 947 | 457 | 817 | 25 | 961 | 485 | 845 | 53 | 989 | 493 | 853 | 61 | 997 | 475 | 835 | 43 | 979 | 13824 |
| 95 | 887 | 527 | 743 | 83 | 875 | 515 | 731 | 97 | 889 | 529 | 745 | 125 | 917 | 557 | 773 | 133 | 925 | 565 | 781 | 115 | 907 | 547 | 763 | 13824 |
| 1103 | 167 | 671 | 311 | 1091 | 155 | 659 | 299 | 1105 | 169 | 673 | 313 | 1133 | 197 | 701 | 341 | 1141 | 205 | 709 | 349 | 1123 | 187 | 691 | 331 | 13824 |
| 599 | 383 | 1031 | 239 | 587 | 371 | 1019 | 227 | 601 | 385 | 1033 | 241 | 629 | 413 | 1061 | 269 | 637 | 421 | 1069 | 277 | 619 | 403 | 1051 | 259 | 13824 |
| 495 | 855 | 63 | 999 | 463 | 823 | 31 | 967 | 439 | 799 | 7 | 943 | 479 | 839 | 47 | 983 | 451 | 811 | 19 | 955 | 481 | 841 | 49 | 985 | 13824 |
| 135 | 927 | 567 | 783 | 103 | 895 | 535 | 751 | 79 | 871 | 511 | 727 | 119 | 911 | 551 | 767 | 91 | 883 | 523 | 739 | 121 | 913 | 553 | 769 | 13824 |
| 1143 | 207 | 711 | 351 | 1111 | 175 | 679 | 319 | 1087 | 151 | 655 | 295 | 1127 | 191 | 695 | 335 | 1099 | 163 | 667 | 307 | 1129 | 193 | 697 | 337 | 13824 |
| 639 | 423 | 1071 | 279 | 607 | 391 | 1039 | 247 | 583 | 367 | 1015 | 223 | 623 | 407 | 1055 | 263 | 595 | 379 | 1027 | 235 | 625 | 409 | 1057 | 265 | 13824 |
| 469 | 829 | 37 | 973 | 497 | 857 | 65 | 1001 | 453 | 813 | 21 | 957 | 437 | 797 | 5 | 941 | 491 | 851 | 59 | 995 | 461 | 821 | 29 | 965 | 13824 |
| 109 | 901 | 541 | 757 | 137 | 929 | 569 | 785 | 93 | 885 | 525 | 741 | 77 | 869 | 509 | 725 | 131 | 923 | 563 | 779 | 101 | 893 | 533 | 749 | 13824 |
| 1117 | 181 | 685 | 325 | 1145 | 209 | 713 | 353 | 1101 | 165 | 669 | 309 | 1085 | 149 | 653 | 293 | 1139 | 203 | 707 | 347 | 1109 | 173 | 677 | 317 | 13824 |
| 613 | 397 | 1045 | 253 | 641 | 425 | 1073 | 281 | 597 | 381 | 1029 | 237 | 581 | 365 | 1013 | 221 | 635 | 419 | 1067 | 275 | 605 | 389 | 1037 | 245 | 13824 |
| 467 | 827 | 35 | 971 | 483 | 843 | 51 | 987 | 471 | 831 | 39 | 975 | 449 | 809 | 17 | 953 | 435 | 795 | 3 | 939 | 503 | 863 | 71 | 1007 | 13824 |
| 107 | 899 | 539 | 755 | 123 | 915 | 555 | 771 | 111 | 903 | 543 | 759 | 89 | 881 | 521 | 737 | 75 | 867 | 507 | 723 | 143 | 935 | 575 | 791 | 13824 |
| 1115 | 179 | 683 | 323 | 1131 | 195 | 699 | 339 | 1119 | 183 | 687 | 327 | 1097 | 161 | 665 | 305 | 1083 | 147 | 651 | 291 | 1151 | 215 | 719 | 359 | 13824 |
| 611 | 395 | 1043 | 251 | 627 | 411 | 1059 | 267 | 615 | 399 | 1047 | 255 | 593 | 377 | 1025 | 233 | 579 | 363 | 1011 | 219 | 647 | 431 | 1079 | 287 | 13824 |
| 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 |

The above example 68 has a magic sum $S_{24\times 24} = 13824 = 24^3$, and the sum of all entries is $F_{576} := 24\times 13824 = 331776 = 576^2 = 24^4$.

The block-wise construction of the magic square of order 24 in example 68 is the composition of magic squares of order 4 given in subsection 4.2 with magic square of order 6 given in subsection 4.4. Each $4\times 4$ block is a perfect square pan diagonal magic square of order 4 which has a different magic sum. In turn, when combined, these magic sums make another magic square of order 6:

**Example 69.** *According to the values given in equation (19), the magic square of order 6 is given by*

| | | | | | | 13824 |
|---|---|---|---|---|---|---|
| 2164 | 2340 | 2380 | 2428 | 2292 | 2220 | 13824 |
| 2388 | 2212 | 2436 | 2268 | 2324 | 2196 | 13824 |
| 2252 | 2204 | 2260 | 2372 | 2404 | 2332 | 13824 |
| 2412 | 2284 | 2188 | 2348 | 2236 | 2356 | 13824 |
| 2308 | 2420 | 2244 | 2180 | 2396 | 2276 | 13824 |
| 2300 | 2364 | 2316 | 2228 | 2172 | 2444 | 13824 |
| 13824 | 13824 | 13824 | 13824 | 13824 | 13824 | 13824 |

$= 4 \times$

| | | | | | | 3456 |
|---|---|---|---|---|---|---|
| 541 | 585 | 595 | 607 | 573 | 555 | 3456 |
| 597 | 553 | 609 | 567 | 581 | 549 | 3456 |
| 563 | 551 | 565 | 593 | 601 | 583 | 3456 |
| 603 | 571 | 547 | 587 | 559 | 589 | 3456 |
| 577 | 605 | 561 | 545 | 599 | 569 | 3456 |
| 575 | 591 | 579 | 557 | 543 | 611 | 3456 |
| 3456 | 3456 | 3456 | 3456 | 3456 | 3456 | 3456 |

## 4.23 Bimagic Squares of Order 25

According to equations (5) and (10), one can obtain perfect square sum magic squares of order 25 in two ways: the first of which by using consecutive odd numbers, and the second of which by using consecutive natural numbers. When using consecutive natural numbers there are once again two methods; the first being for uniformity, and the second being for a minimum perfect square sum of all the magic square entries.

### 4.23.1 First Approach: Consecutive Odd Numbers

We will obtain an odd numbered **bimagic square** of order 25. Taking $k = 25$ in equation (5), we get

$$\begin{aligned} &1+3+5+\ldots+(2\times 25^2-1) = 25^4 \\ \Rightarrow &1+3+5+\ldots+1247+1249 \; = 390625 = 625^2 = 25^4. \end{aligned} \tag{38}$$

**Example 70.** *According to values given in (38), the magic square of order 25 is given by*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 337 | 613 | 949 | 1225 | 885 | 1161 | 247 | 273 | 559 | 469 | 545 | 821 | 1107 | 183 | 1093 | 129 | 405 | 731 | 767 | 677 | 953 | 1039 | 65 | 391 |
| 913 | 1249 | 25 | 301 | 637 | 297 | 573 | 859 | 1185 | 211 | 1121 | 157 | 483 | 519 | 845 | 705 | 781 | 1067 | 143 | 429 | 89 | 365 | 691 | 977 | 1003 |
| 325 | 601 | 937 | 1213 | 49 | 1159 | 235 | 261 | 597 | 873 | 533 | 819 | 1145 | 171 | 457 | 117 | 443 | 729 | 755 | 1081 | 991 | 1027 | 53 | 389 | 665 |
| 1237 | 13 | 349 | 625 | 901 | 561 | 897 | 1173 | 209 | 285 | 195 | 471 | 507 | 833 | 1119 | 779 | 1055 | 131 | 417 | 743 | 353 | 689 | 965 | 1041 | 77 |
| 649 | 925 | 1201 | 37 | 313 | 223 | 259 | 585 | 861 | 1197 | 807 | 1133 | 169 | 495 | 521 | 431 | 717 | 793 | 1079 | 105 | 1015 | 91 | 377 | 653 | 989 |
| 1069 | 145 | 421 | 707 | 783 | 693 | 979 | 1005 | 81 | 367 | 27 | 303 | 639 | 915 | 1241 | 851 | 1187 | 213 | 299 | 575 | 485 | 511 | 847 | 1123 | 159 |
| 721 | 757 | 1083 | 119 | 445 | 55 | 381 | 667 | 993 | 1029 | 939 | 1215 | 41 | 327 | 603 | 263 | 599 | 875 | 1151 | 237 | 1147 | 173 | 459 | 535 | 811 |
| 133 | 419 | 745 | 771 | 1057 | 967 | 1043 | 79 | 355 | 681 | 341 | 627 | 903 | 1239 | 15 | 1175 | 201 | 287 | 563 | 899 | 509 | 835 | 1111 | 197 | 473 |
| 795 | 1071 | 107 | 433 | 719 | 379 | 655 | 981 | 1017 | 93 | 1203 | 39 | 315 | 641 | 927 | 587 | 863 | 1199 | 225 | 251 | 161 | 497 | 523 | 809 | 1135 |
| 407 | 733 | 769 | 1095 | 121 | 1031 | 67 | 393 | 679 | 955 | 615 | 941 | 1227 | 3 | 339 | 249 | 275 | 551 | 887 | 1163 | 823 | 1109 | 185 | 461 | 547 |
| 877 | 1153 | 239 | 265 | 591 | 451 | 537 | 813 | 1149 | 175 | 1085 | 111 | 447 | 723 | 759 | 669 | 995 | 1021 | 57 | 383 | 43 | 329 | 605 | 931 | 1217 |
| 289 | 565 | 891 | 1177 | 203 | 1113 | 199 | 475 | 501 | 837 | 747 | 773 | 1059 | 135 | 411 | 71 | 357 | 683 | 969 | 1045 | 905 | 1231 | 17 | 343 | 629 |
| 1191 | 227 | 253 | 589 | 865 | 525 | 801 | 1137 | 163 | 499 | 109 | 435 | 711 | 797 | 1073 | 983 | 1019 | 95 | 371 | 657 | 317 | 643 | 929 | 1205 | 31 |
| 553 | 889 | 1165 | 241 | 277 | 187 | 463 | 549 | 825 | 1101 | 761 | 1097 | 123 | 409 | 735 | 395 | 671 | 957 | 1033 | 69 | 1229 | 5 | 331 | 617 | 943 |
| 215 | 291 | 577 | 853 | 1189 | 849 | 1125 | 151 | 487 | 513 | 423 | 709 | 785 | 1061 | 147 | 1007 | 83 | 369 | 695 | 971 | 631 | 917 | 1243 | 29 | 305 |
| 685 | 961 | 1047 | 73 | 359 | 19 | 345 | 621 | 907 | 1233 | 893 | 1179 | 205 | 281 | 567 | 477 | 503 | 839 | 1115 | 191 | 1051 | 137 | 413 | 749 | 775 |
| 97 | 373 | 659 | 985 | 1011 | 921 | 1207 | 33 | 319 | 645 | 255 | 581 | 867 | 1193 | 229 | 1139 | 165 | 491 | 527 | 803 | 713 | 799 | 1075 | 101 | 437 |
| 959 | 1035 | 61 | 397 | 673 | 333 | 619 | 945 | 1221 | 7 | 1167 | 243 | 279 | 555 | 881 | 541 | 827 | 1103 | 189 | 465 | 125 | 401 | 737 | 763 | 1099 |
| 361 | 697 | 973 | 1009 | 85 | 1245 | 21 | 307 | 633 | 919 | 579 | 855 | 1181 | 217 | 293 | 153 | 489 | 515 | 841 | 1127 | 787 | 1063 | 149 | 425 | 701 |
| 1023 | 59 | 385 | 661 | 997 | 607 | 933 | 1219 | 45 | 321 | 231 | 267 | 593 | 879 | 1155 | 815 | 1141 | 177 | 453 | 539 | 449 | 725 | 751 | 1087 | 113 |
| 493 | 529 | 805 | 1131 | 167 | 1077 | 103 | 439 | 715 | 791 | 651 | 987 | 1013 | 99 | 375 | 35 | 311 | 647 | 923 | 1209 | 869 | 1195 | 221 | 257 | 583 |
| 1105 | 181 | 467 | 543 | 829 | 739 | 765 | 1091 | 127 | 403 | 63 | 399 | 675 | 951 | 1037 | 947 | 1223 | 9 | 335 | 611 | 271 | 557 | 883 | 1169 | 245 |
| 517 | 843 | 1129 | 155 | 481 | 141 | 427 | 703 | 789 | 1065 | 975 | 1001 | 87 | 363 | 699 | 309 | 635 | 911 | 1247 | 23 | 1183 | 219 | 295 | 571 | 857 |
| 179 | 455 | 531 | 817 | 1143 | 753 | 1089 | 115 | 441 | 727 | 387 | 663 | 999 | 1025 | 51 | 1211 | 47 | 323 | 609 | 935 | 595 | 871 | 1157 | 233 | 269 |
| 831 | 1117 | 193 | 479 | 505 | 415 | 741 | 777 | 1053 | 139 | 1049 | 75 | 351 | 687 | 963 | 623 | 909 | 1235 | 11 | 347 | 207 | 283 | 569 | 895 | 1171 |

In this example the magic sum is $S_{25\times 25} := 15625 = 25^3$ and the sum of all 625 entries is $F_{625} := 390625 = 625^2 = 25^4$. The magic square is both **pan diagonal and bimagic** with a bimagic sum $Sb_{25\times 25} := 13020825$. Moreover, each $5\times 5$ block is a **pan diagonal magic square** with a same magic sum $S_{5\times 5} := 3125 = 5^3$ and with a same perfect square sum $S_{25} := 15625 = 5^4$ for their entries.

### 4.23.2 Second Approach: Uniformity

According to equation (10),

$$G := \frac{n(n+1)}{2} - \frac{(n-625)(n-624)}{2} = 625(n-312)$$

Taking $n = 937$, we get a perfect square, i.e.,

$$G := T(937) - T(312) = \frac{937\times 938}{2} - \frac{312\times 313}{2} = 390625 = 625^2 = 25^4.$$

Simplifying, we get

$$313 + 314 + 115 + \ldots + 936 + 937 = 390625 = 625^2 = 25^4.$$

This gives a perfect square sum for the 625 entries of consecutive natural numbers from to 313 to 937.

**Example 71.** *According to the values given above, using consecutive natural numbers we can construct a magic square of order 25 with a perfect square sum of all entries:*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 313 | 481 | 619 | 787 | 925 | 755 | 893 | 436 | 449 | 592 | 547 | 585 | 723 | 866 | 404 | 859 | 377 | 515 | 678 | 696 | 651 | 789 | 832 | 345 | 508 |
| 769 | 937 | 325 | 463 | 631 | 461 | 599 | 742 | 905 | 418 | 873 | 391 | 554 | 572 | 735 | 665 | 703 | 846 | 384 | 527 | 357 | 495 | 658 | 801 | 814 |
| 475 | 613 | 781 | 919 | 337 | 892 | 430 | 443 | 611 | 749 | 579 | 722 | 885 | 398 | 541 | 371 | 534 | 677 | 690 | 853 | 808 | 826 | 339 | 507 | 645 |
| 931 | 319 | 487 | 625 | 763 | 593 | 761 | 899 | 417 | 455 | 410 | 548 | 566 | 729 | 872 | 702 | 840 | 378 | 521 | 684 | 489 | 657 | 795 | 833 | 351 |
| 637 | 775 | 913 | 331 | 469 | 424 | 442 | 605 | 743 | 911 | 716 | 879 | 397 | 560 | 573 | 528 | 671 | 709 | 852 | 365 | 820 | 358 | 501 | 639 | 807 |
| 847 | 385 | 523 | 666 | 704 | 659 | 802 | 815 | 353 | 496 | 326 | 464 | 632 | 770 | 933 | 738 | 906 | 419 | 462 | 600 | 555 | 568 | 736 | 874 | 392 |
| 673 | 691 | 854 | 372 | 535 | 340 | 503 | 646 | 809 | 827 | 782 | 920 | 333 | 476 | 614 | 444 | 612 | 750 | 888 | 431 | 886 | 399 | 542 | 580 | 718 |
| 379 | 522 | 685 | 698 | 841 | 796 | 834 | 352 | 490 | 653 | 483 | 626 | 764 | 932 | 320 | 900 | 413 | 456 | 594 | 762 | 567 | 730 | 868 | 411 | 549 |
| 710 | 848 | 366 | 529 | 672 | 502 | 640 | 803 | 821 | 359 | 914 | 332 | 470 | 633 | 776 | 606 | 744 | 912 | 425 | 438 | 393 | 561 | 574 | 717 | 880 |
| 516 | 679 | 697 | 860 | 373 | 828 | 346 | 509 | 652 | 790 | 620 | 783 | 926 | 314 | 482 | 437 | 450 | 588 | 756 | 894 | 724 | 867 | 405 | 543 | 586 |
| 751 | 889 | 432 | 445 | 608 | 538 | 581 | 719 | 887 | 400 | 855 | 368 | 536 | 674 | 692 | 647 | 810 | 823 | 341 | 504 | 334 | 477 | 615 | 778 | 921 |
| 457 | 595 | 758 | 901 | 414 | 869 | 412 | 550 | 563 | 731 | 686 | 699 | 842 | 380 | 518 | 348 | 491 | 654 | 797 | 835 | 765 | 928 | 321 | 484 | 627 |
| 908 | 426 | 439 | 607 | 745 | 575 | 713 | 881 | 394 | 562 | 367 | 530 | 668 | 711 | 849 | 804 | 822 | 360 | 498 | 641 | 471 | 634 | 777 | 915 | 328 |
| 589 | 757 | 895 | 433 | 451 | 406 | 544 | 587 | 725 | 863 | 693 | 861 | 374 | 517 | 680 | 510 | 648 | 791 | 829 | 347 | 927 | 315 | 478 | 621 | 784 |
| 420 | 458 | 601 | 739 | 907 | 737 | 875 | 388 | 556 | 569 | 524 | 667 | 705 | 843 | 386 | 816 | 354 | 497 | 660 | 798 | 628 | 771 | 934 | 327 | 465 |
| 655 | 793 | 836 | 349 | 492 | 322 | 485 | 623 | 766 | 929 | 759 | 902 | 415 | 453 | 596 | 551 | 564 | 732 | 870 | 408 | 838 | 381 | 519 | 687 | 700 |
| 361 | 499 | 642 | 805 | 818 | 773 | 916 | 329 | 472 | 635 | 440 | 603 | 746 | 909 | 427 | 882 | 395 | 558 | 576 | 714 | 669 | 712 | 850 | 363 | 531 |
| 792 | 830 | 343 | 511 | 649 | 479 | 622 | 785 | 923 | 316 | 896 | 434 | 452 | 590 | 753 | 583 | 726 | 864 | 407 | 545 | 375 | 513 | 681 | 694 | 862 |
| 493 | 661 | 799 | 817 | 355 | 935 | 323 | 466 | 629 | 772 | 602 | 740 | 903 | 421 | 459 | 389 | 557 | 570 | 733 | 876 | 706 | 844 | 387 | 525 | 663 |
| 824 | 342 | 505 | 643 | 811 | 616 | 779 | 922 | 335 | 473 | 428 | 446 | 609 | 752 | 890 | 720 | 883 | 401 | 539 | 582 | 537 | 675 | 688 | 856 | 369 |
| 559 | 577 | 715 | 878 | 396 | 851 | 364 | 532 | 670 | 708 | 638 | 806 | 819 | 362 | 500 | 330 | 468 | 636 | 774 | 917 | 747 | 910 | 423 | 441 | 604 |
| 865 | 403 | 546 | 584 | 727 | 682 | 695 | 858 | 376 | 514 | 344 | 512 | 650 | 788 | 831 | 786 | 924 | 317 | 480 | 618 | 448 | 591 | 754 | 897 | 435 |
| 571 | 734 | 877 | 390 | 553 | 383 | 526 | 664 | 707 | 845 | 800 | 813 | 356 | 494 | 662 | 467 | 630 | 768 | 936 | 324 | 904 | 422 | 460 | 598 | 741 |
| 402 | 540 | 578 | 721 | 884 | 689 | 857 | 370 | 533 | 676 | 506 | 644 | 812 | 825 | 338 | 918 | 336 | 474 | 617 | 780 | 610 | 748 | 891 | 429 | 447 |
| 728 | 871 | 409 | 552 | 565 | 520 | 683 | 701 | 839 | 382 | 837 | 350 | 488 | 656 | 794 | 624 | 767 | 930 | 318 | 486 | 416 | 454 | 597 | 760 | 898 |

In this example the magic sum is $S_{25\times 25} := 15625 = 25^3$ and the sum of all 625 entries is $F_{625} := 390625 = 625^2 = 25^4$. The magic square is **pan diagonal and bimagic** with a imagic sum $Sb_{25\times 25} := 10579425$. Moreover, each of the 25 $5\times 5$ blocks is a pan diagonal magic square with the same magic sum $S_{5\times 5} := 3125 = 5^3$ and with the same perfect square sum $S_{25} := 15625 = 5^4$ for their entries. Apart from their contrasting consecutive odd or natural number constructions, the main difference between the examples 70 and 71 is the **bimagic sum**. What is more, both 70 and 71 satisfy the property (13) of uniformity 2.5.1, i.e., $\left\langle 25, 25^2, 25^3, 25^4 \right\rangle$.

### 4.23.3 Third Approach: Minimum Perfect Square Sum

The examples given in 70 and 71 satisfy the property (13) of uniformity 2.5.1, i.e., $\left\langle 25, 25^2, 25^3, 25^4 \right\rangle$. Moreover each block of order 5 also has the same property, $\left\langle 5, 5^2, 5^3, 5^4 \right\rangle$. However, the example 71 does not have a minimum perfect square sum for its entries. Let's consider the first example of magic square order 25 given in section 3:

$$L\left(18^2 + \frac{25^2-1}{2},\, 25^2\right) \rightarrow \left(25, 12, 636, 8100, 202500, 15\sqrt{2}\right)$$

In this case, we obtain a magic square of order 25 with a perfect square sum $202500 = 450^2$ of all entries, but it does not satisfy the property (13) of uniformity given in 2.5.1. The magic square shown below has a minimum perfect square sum of all entries.

**Example 72.** *According to the values given above, we can construct magic square of order 25 of consecutive natural numbers from 12 to 636 with a perfect square sum of all entries:*

| | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 12 | 180 | 318 | 486 | 624 | 454 | 592 | 135 | 148 | 291 | 246 | 284 | 422 | 565 | 103 | 558 | 76 | 214 | 377 | 395 | 350 | 488 | 531 | 44 | 207 |
| 468 | 636 | 24 | 162 | 330 | 160 | 298 | 441 | 604 | 117 | 572 | 90 | 253 | 271 | 434 | 364 | 402 | 545 | 83 | 226 | 56 | 194 | 357 | 500 | 513 |
| 174 | 312 | 480 | 618 | 36 | 591 | 129 | 142 | 310 | 448 | 278 | 421 | 584 | 97 | 240 | 70 | 233 | 376 | 389 | 552 | 507 | 525 | 38 | 206 | 344 |
| 630 | 18 | 186 | 324 | 462 | 292 | 460 | 598 | 116 | 154 | 109 | 247 | 265 | 428 | 571 | 401 | 539 | 77 | 220 | 383 | 188 | 356 | 494 | 532 | 50 |
| 336 | 474 | 612 | 30 | 168 | 123 | 141 | 304 | 442 | 610 | 415 | 578 | 96 | 259 | 272 | 227 | 370 | 408 | 551 | 64 | 519 | 57 | 200 | 338 | 506 |
| 546 | 84 | 222 | 365 | 403 | 358 | 501 | 514 | 52 | 195 | 25 | 163 | 331 | 469 | 632 | 437 | 605 | 118 | 161 | 299 | 254 | 267 | 435 | 573 | 91 |
| 372 | 390 | 553 | 71 | 234 | 39 | 202 | 345 | 508 | 526 | 481 | 619 | 32 | 175 | 313 | 143 | 311 | 449 | 587 | 130 | 585 | 98 | 241 | 279 | 417 |
| 78 | 221 | 384 | 397 | 540 | 495 | 533 | 51 | 189 | 352 | 182 | 325 | 463 | 631 | 19 | 599 | 112 | 155 | 293 | 461 | 266 | 429 | 567 | 110 | 248 |
| 409 | 547 | 65 | 228 | 371 | 201 | 339 | 502 | 520 | 58 | 613 | 31 | 169 | 332 | 475 | 305 | 443 | 611 | 124 | 137 | 92 | 260 | 273 | 416 | 579 |
| 215 | 378 | 396 | 559 | 72 | 527 | 45 | 208 | 351 | 489 | 319 | 482 | 625 | 13 | 181 | 136 | 149 | 287 | 455 | 593 | 423 | 566 | 104 | 242 | 285 |
| 450 | 588 | 131 | 144 | 307 | 237 | 280 | 418 | 586 | 99 | 554 | 67 | 235 | 373 | 391 | 346 | 509 | 522 | 40 | 203 | 33 | 176 | 314 | 477 | 620 |
| 156 | 294 | 457 | 600 | 113 | 568 | 111 | 249 | 262 | 430 | 385 | 398 | 541 | 79 | 217 | 47 | 190 | 353 | 496 | 534 | 464 | 627 | 20 | 183 | 326 |
| 607 | 125 | 138 | 306 | 444 | 274 | 412 | 580 | 93 | 261 | 66 | 229 | 367 | 410 | 548 | 503 | 521 | 59 | 197 | 340 | 170 | 333 | 476 | 614 | 27 |
| 288 | 456 | 594 | 132 | 150 | 105 | 243 | 286 | 424 | 562 | 392 | 560 | 73 | 216 | 379 | 209 | 347 | 490 | 528 | 46 | 626 | 14 | 177 | 320 | 483 |
| 119 | 157 | 300 | 438 | 606 | 436 | 574 | 87 | 255 | 268 | 223 | 366 | 404 | 542 | 85 | 515 | 53 | 196 | 359 | 497 | 327 | 470 | 633 | 26 | 164 |
| 354 | 492 | 535 | 48 | 191 | 21 | 184 | 322 | 465 | 628 | 458 | 601 | 114 | 152 | 295 | 250 | 263 | 431 | 569 | 107 | 537 | 80 | 218 | 386 | 399 |
| 60 | 198 | 341 | 504 | 517 | 472 | 615 | 28 | 171 | 334 | 139 | 302 | 445 | 608 | 126 | 581 | 94 | 257 | 275 | 413 | 368 | 411 | 549 | 62 | 230 |
| 491 | 529 | 42 | 210 | 348 | 178 | 321 | 484 | 622 | 15 | 595 | 133 | 151 | 289 | 452 | 282 | 425 | 563 | 106 | 244 | 74 | 212 | 380 | 393 | 561 |
| 192 | 360 | 498 | 516 | 54 | 634 | 22 | 165 | 328 | 471 | 301 | 439 | 602 | 120 | 158 | 88 | 256 | 269 | 432 | 575 | 405 | 543 | 86 | 224 | 362 |
| 523 | 41 | 204 | 342 | 510 | 315 | 478 | 621 | 34 | 172 | 127 | 145 | 308 | 451 | 589 | 419 | 582 | 100 | 238 | 281 | 236 | 374 | 387 | 555 | 68 |
| 258 | 276 | 414 | 577 | 95 | 550 | 63 | 231 | 369 | 407 | 337 | 505 | 518 | 61 | 199 | 29 | 167 | 335 | 473 | 616 | 446 | 609 | 122 | 140 | 303 |
| 564 | 102 | 245 | 283 | 426 | 381 | 394 | 557 | 75 | 213 | 43 | 211 | 349 | 487 | 530 | 485 | 623 | 16 | 179 | 317 | 147 | 290 | 453 | 596 | 134 |
| 270 | 433 | 576 | 89 | 252 | 82 | 225 | 363 | 406 | 544 | 499 | 512 | 55 | 193 | 361 | 166 | 329 | 467 | 635 | 23 | 603 | 121 | 159 | 297 | 440 |
| 101 | 239 | 277 | 420 | 583 | 388 | 556 | 69 | 232 | 375 | 205 | 343 | 511 | 524 | 37 | 617 | 35 | 173 | 316 | 479 | 309 | 447 | 590 | 128 | 146 |
| 427 | 570 | 108 | 251 | 264 | 219 | 382 | 400 | 538 | 81 | 536 | 49 | 187 | 355 | 493 | 323 | 466 | 629 | 17 | 185 | 115 | 153 | 296 | 459 | 597 |

In this example the magic sum is $S_{25\times 25} := 8100$ and the sum of all 625 entries is $F_{625} := 202500 = 450^2$. The magic square is **pan diagonal and bimagic** with a bimagic sum $Sb_{25\times 25} := 3438200$. Moreover, each of the 25 $5\times 5$ blocks is a **pan diagonal magic square** with the same magic sum, $S_{5\times 5} := 1620$ and with the same perfect square sum $S_{25} := 8100 = 90^2$ for their entries.

The block-wise constructions of the three examples 70, 71 and 72 of magic squares of order 25 are the composition of 25 magic squares of order 5 with another magic square of order 5 from subsection 4.3. These constructions are based on mutually orthogonal diagonal Latin squares. For details, refer [13], [18].

# 5 Final Comments

In this work we constructed magic squares of orders 3 to 25 in such a way that all of them have a perfect square sum of their entries. This has been done in two ways, firstly by using consecutive odd numbers starting from 1, and secondly by using consecutive natural numbers which have a minimum perfect square sum for their entries and/or satisfy the uniformity property given in subsection 2.5.1. This work is a **revised and generalized** version of author's previous work [20]. The examples of magic squares considered above are the adaptations of the author's work [16, 17, 18] on **intervally distributed magic squares**. All the magic square in this work are constructed without using computer programming, except two given in examples 44 and 62 for magic square of orders 18 and 22 respectively. These two are based on White's [26] software. The **block-wise** construction of examples **bimagic squares** of orders 8, 9, 16 and 25 given in in subsections 4.6, 4.7, 4.14, 4.23, are based on author's work [14]. The idea of **mutual orthogonal diagonal Latin squares** [25, 27] is applied to magic squares of orders 10 and 14. White's [26] approach, based on **self-orthogonal diagonal Latin squares**, is applied to magic squares of orders 18 and 22 (see examples 44 and 62). Moreover, the **software** created by White [26] allows us to construct doubles of prime order magic squares, in orders 10, 14, 22, 26, etc. The **block-wise** constructions given in [16, 17, 18] are helpful for constructing magic squares of orders, such as, 12, 15, 18, 21, 24, etc. More on magic and bimagic squares refers to [2, 3, 4].

The item-wise details of author's work on magic squares:

(i) **Digital numbers** magic squares – [7, 8, 9, 10, 11, 12];

(ii) **Block-wise construction** of **bimagic squares** – [13];

(iii) Connections with **genetic tables** and **Shannon's entropy** – [14];

(iv) **Selfie** and **palindromic-type** magic squares – [15];

(v) **Intervally distributed** and **block-wise** magic squares – [16, 17, 18];

(vi) **Multi-digits** magic squares – [19];

(vii) **Perfect square sum** magic squares with uniformity and minimum sum – [20] and this work;

(viii) **Pythagorean triples** to generate **perfect square sum** magic squares – [21].

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