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Hermite-Hadamard inequality involving conformable fractional integrals for twice differential convex functions

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Abstract

In this paper, we give an integral identity involving twice differentiable functions and conformable fractional integral. Then, we utilize the convexity and s-convexity of a twice differentiable function to this new established integral identity to obtain some new Hermite-Hadamard's inequalities.

Keywords: Fractional Hermite-Hadamard inequality, Conformable fractional integral, Convex functions.

1. Introduction

It is well known that the chain rule for Caputo and Riemann-Liouville fractional (global) derivatives do not like the symmetrical results for the classical integer (local) derivative. Recently, a local fractional derivative involving a stand limit process instead of a global singular integral called the conformable fractional derivative appeared in [1, 2, 3, 4]. We also remark that there are many basic properties of conformable fractional derivative in [2], which can be used to find a solution of differential equations with conformable fractional derivative [5].

The Hermite-Hadamard inequality was firstly discovered by Hermite in 1881, which provides a lower and an upper estimations for the integral average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain. Due to the widely application of Hermite-Hadamard inequality, there are many generalized, improved, and extended work on this fields, one can see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references therein.

Next, we note that Anderson [18] investigated the following conformable integral version of Hermite-Hadamard inequality.

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Theorem 1.1. *(see [18, Corollary 5]) Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ be α-integrable function, i.e., $J_\alpha(f)$ exists. Then the following inequalities hold:*

$$J_\alpha(f)(b) \leq \frac{f(a)+f(b)}{2}$$

provided that f is α-differentiable on (a,b), i.e., $D_\alpha(f)$ exists on (a,b), and it is increasing on $[a,b]$;

$$f\left(\frac{a+b}{2}\right) \leq J_\alpha(f)(b)$$

provided that $D_\alpha(f)$ exists on (a,b) and it is decreasing on $[a,b]$, where $J_\alpha(f)$ and $D_\alpha(f)$ are defined by

$$\begin{aligned} J_\alpha(f)(x) &:= \frac{\alpha}{x^\alpha - a^\alpha}\int_a^x f(t)d_\alpha t, \quad d_\alpha t := t^{1-\alpha}dt, \\ D_\alpha(f)(x) &:= \lim_{\epsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha})}{\varepsilon}. \end{aligned}$$

Here we note that $D_\alpha(f)$ and $J_\alpha(f)$ are called conformable fractional derivative and conformable fractional integral of f of the order α, respectively.

Sarikaya et al. [19] present the following Hermite-Hadamard's inequalities for conformable fractional integral.

Theorem 1.2. *(see [19, Theorem 11]) Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ be a convex function and $J_\alpha f$ exists on $[a,b]$. Then one has*

$$f\left(\frac{a^\alpha+b^\alpha}{2}\right) \leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t^\alpha)d_\alpha t \leq \frac{f(a^\alpha)+f(b^\alpha)}{2}.$$

Further, the following integral identity involving conformable fractional integral is presented.

Lemma 1.3. *(see [19, Lemma 3]) Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and $D_\alpha(f)$ exists on on (a,b). If $J_\alpha D_\alpha f$ exists on $[a,b]$, i.e., $D_\alpha f$ is an α-integrable function on $[a,b]$, then*

$$\begin{aligned} &\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(t^\alpha)d_\alpha t - \frac{f(a^\alpha)+f(b^\alpha)}{2} \\ = \ &\frac{1}{2}\int_0^1 (1-2t^\alpha)D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)d_\alpha t. \end{aligned} \tag{1}$$

We also emphasize that Wang et al. [15] give two fundamental integral identities involving Riemann-Liouville fractional integrals for second order differentiable functions, then, establish many interesting Hermite-Hadamard's inequalities involving left-sided and right-sided Riemann-Liouville fractional integrals.

Motivated by [15, 19], we generzlize (1) to be a fundamental integral identity involving twice differentiable functions and conformable fractional integral (see Theorem 2.1). Based on this new integral identity, we use convexity and s-convexity of a twice differentiable function to establish some new Hermite-Hadamard's inequalities.

2. Integral identities involving conformable fractional integral

In this section, we give two useful integral identities involving conformable fractional integral, which will be used to establish the Hermite-Hadamard's inequalities in the next section.

Theorem 2.1. *Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and f'' exists on (a,b). Then we have*

$$\begin{aligned} & \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x - \frac{f(a^\alpha) + f(b^\alpha)}{2} \\ = & -\frac{1}{2\alpha} \int_0^1 (1-t^\alpha) t^\alpha D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha) d_\alpha t, \end{aligned} \tag{2}$$

where

$$D_{\alpha,\alpha}(f)(t) := t^{1-\alpha}[t^{1-\alpha} f'(t)]' = t^{1-2\alpha}[(1-\alpha) f'(t) + t f''(t)]. \tag{3}$$

Proof. The proof is straightway, one can use the integration by parts to derive

$$\begin{aligned} & \int_0^1 (1-2t^\alpha) D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha) d_\alpha t \\ = & \int_0^1 \frac{1}{\alpha} D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha) d(t^\alpha - t^{2\alpha}) \\ = & \left(\frac{1}{\alpha} (t^\alpha - t^{2\alpha}) D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha) \right) \Big|_0^1 \\ & -\frac{1}{\alpha} \int_0^1 (t^\alpha - t^{2\alpha}) d(D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)) dt \\ = & -\frac{1}{\alpha} \int_0^1 (1-t^\alpha) t^\alpha D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha) d_\alpha t. \end{aligned} \tag{4}$$

Now linking (1) and (4), the identity (2) holds. This complete the proof. □

Remark 2.2. *If $D_{\alpha,\alpha}(f)$ in (3) exists, then we can call f is a twice α-differentiable function. We make some examples for special functions as follows:*

(i) $D_{\alpha,\alpha}(1) = 0$;

(ii) $D_{\alpha,\alpha}(e^{ax}) = [(1-\alpha) x^{1-2\alpha} + a x^{2(1-\alpha)}] a e^{ax}$;

(iii) $D_{\alpha,\alpha}(e^{\frac{x^\alpha}{a}}) = e^{\frac{x^\alpha}{a}}$.

The following lemma will be used in the sequel.

Lemma 2.3. *Let $0 < a < b$, $\alpha \in (0,1)$, $f, g : [a,b] \to \mathbb{R}$ and f'' exists on on (a,b). Then we have*

$$D_{\alpha,\alpha}(f \circ g)(t) = D_{\alpha,\alpha}(g(t))(t)f'(g(t)) + f''(g)D_{\alpha}^2(g)(t).$$

Proof. Note that (3),

$$\begin{aligned} D_{\alpha,\alpha}(f \circ g)(t) &= t^{1-2\alpha}((1-\alpha)f'(g(t))g'(t) + tf''(g(t))(g'(t))^2 + tf'(g(t))g''(t)) \\ &= t^{1-2\alpha}((1-\alpha)g'(t) + tg''(t))f'(g(t)) + f''(g(t))(t^{1-\alpha}g'(t))^2 \\ &= D_{\alpha,\alpha}(g(t))(t)f'(g(t)) + f''(g)D_{\alpha}^2(g)(t). \end{aligned}$$

This complete the proof. □

3. Hermite-Hadamard's inequalities for twice differentiable convex functions

In this section, we recall the basic definition of convex and s-convex functions.

Definition 3.1. *The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if for every $x, y \in I$ and $\lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.*

Definition 3.2. *The function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}$ is said to be s-convex (in the second sense), where $s \in (0,1]$, if for every $x, y \in I$ and $\lambda \in [0,1]$, we have $f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$.*

Now we are ready to apply the above convex functions via Theorem 2.1 to present some possible Hermite-Hadamard's inequalities.

Theorem 3.3. *Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and f'' exists on on (a,b). If $|f''|$ is convex function, then we have*

$$\begin{aligned} &\left|\frac{f(a^\alpha) + f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\ \leq\ & \frac{(b^\alpha - a^\alpha)^2}{12}\left[\frac{a^{2\alpha(\alpha-1)}\left|D_{\alpha,\alpha}(f)(a^\alpha)\right| + b^{2\alpha(\alpha-1)}\left|D_{\alpha,\alpha}(f)(b^\alpha)\right|}{2}\right]. \end{aligned}$$

Proof. According to the Theorem 2.1, we have

$$\begin{aligned} &\left|\frac{f(a^\alpha) + f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\ \leq\ & \frac{1}{2\alpha}\int_0^1 |(1-t^\alpha)t^\alpha||D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|d_\alpha t. \end{aligned} \tag{5}$$

According to (3) and Lemma 2.3 via the fact of $|f''|$ is convex function, we have

$$\left|D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)\right|$$

$$
\begin{aligned}
&= \Big| D_{\alpha,\alpha}(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha) f'(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha) \\
&\quad + f''(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)[D_\alpha(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)]^2 \Big| \\
&= \Big| t^{1-2\alpha}[(1-\alpha)(\alpha(a^\alpha - b^\alpha)t^{\alpha-1}) + t(\alpha(a^\alpha - b^\alpha)(\alpha-1)t^{\alpha-2})] f'(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha) \\
&\quad + f''(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)[t^{1-\alpha}(\alpha a^\alpha t^{\alpha-1} - \alpha b^\alpha t^{\alpha-1})] \Big| \\
&= \Big| \alpha^2 (b^\alpha - a^\alpha)^2 f''(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha) \Big| \\
&\le \alpha^2 (b^\alpha - a^\alpha)^2 \Big[t^\alpha |f''(a^\alpha)| + (1-t^\alpha)|f''(b^\alpha)| \Big] \\
&= \alpha^2 (b^\alpha - a^\alpha)^2 \Big[t^\alpha a^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)| + (1-t^\alpha) b^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)| \Big]. \qquad (6)
\end{aligned}
$$

Submitting (6) into (5), we have

$$
\begin{aligned}
&\left| \frac{f(a^\alpha) + f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x \right| \\
\le\ & \frac{\alpha^2 (b^\alpha - a^\alpha)^2}{2\alpha} \int_1^0 |(1-t^\alpha)t^\alpha| \Big[t^\alpha a^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)| \\
& + (1-t^\alpha) b^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)| \Big] d_\alpha t \\
=\ & \frac{\alpha^2 (b^\alpha - a^\alpha)^2}{2\alpha} \Big[a^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)| \int_0^1 |(1-t^\alpha)t^\alpha| t^\alpha d_\alpha t \\
& + b^{2\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)| \int_0^1 |(1-t^\alpha)t^\alpha|(1-t^\alpha) d_\alpha t,
\end{aligned}
$$

where we use the fact

$$
\int_0^1 |(1-t^\alpha)t^\alpha| t^\alpha d_\alpha t = \int_0^1 |(1-t^\alpha)t^\alpha|(1-t^\alpha) d_\alpha t = \frac{1}{12\alpha}.
$$

This complete the proof. □

Theorem 3.4. *Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and f'' exists on on (a,b). If $|f''|^q$ is convex function, then we have*

$$
\begin{aligned}
&\left| \frac{f(a^\alpha) + f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x \right| \\
\le\ & \frac{\alpha (b^\alpha - a^\alpha)^2}{2} (A(\alpha))^{\frac{1}{p}} \left[\frac{a^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)|^q + b^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)|^q}{2} \right]^{\frac{1}{q}}.
\end{aligned}
$$

where

$$A(\alpha)=\frac{1}{\alpha}\bigg(\sum_{k=0}^{\infty}\binom{p}{k}\frac{(-1)^k}{p+k+1}\bigg), \tag{7}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By Theorem 2.1 and Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{f(a^\alpha)+f(b^\alpha)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right|\\
\leq\ & \frac{1}{2\alpha}\int_0^1 |(1-t^\alpha)t^\alpha||D_{\alpha,\alpha}(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)|d_\alpha t\\
\leq\ & \frac{1}{2\alpha}\bigg(\int_0^1((1-t^\alpha)t^\alpha)^p d_\alpha t\bigg)^{\frac{1}{p}}\bigg(\int_0^1 |D_{\alpha,\alpha}(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)|^q d_\alpha t\bigg)^{\frac{1}{q}}.
\end{aligned}$$

According to (3) and Lemma 2.3 via the fact of $|f''|^q$ is convex function, we have

$$\begin{aligned}
&|D_{\alpha,\alpha}(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)|^q\\
=\ & |\alpha^2(b^\alpha-a^\alpha)^2 f''(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)|^q\\
\leq\ & \alpha^{2q}(b^\alpha-a^\alpha)^{2q}(t^\alpha|f''(a^\alpha)|^q+(1-t^\alpha)|f''(b^\alpha)|^q)\\
\leq\ & \alpha^{2q}(b^\alpha-a^\alpha)^{2q}(t^\alpha a^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)|^q+(1-t^\alpha)b^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|^q).
\end{aligned} \tag{8}$$

By using (8), we have

$$\begin{aligned}
&\left|\frac{fa^\alpha)+fb^\alpha)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right|\\
\leq\ & \frac{\alpha^2(b^\alpha-a^\alpha)^2}{2\alpha}\bigg(\int_0^1((1-t^\alpha)t^\alpha)^p d_\alpha t\bigg)^{\frac{1}{p}}\\
&\times\bigg(\int_0^1(t^\alpha a^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)|^q+(1-t^\alpha)b^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|^q)d_\alpha t\bigg)^{\frac{1}{q}}\\
\leq\ & \frac{\alpha^2(b^\alpha-a^\alpha)^2}{2\alpha}\bigg(\int_0^1((1-t^\alpha)t^\alpha)^p d_\alpha t\bigg)^{\frac{1}{p}}\\
&\times\bigg(\frac{a^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)|^q+b^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|^q)}{2\alpha}\bigg)^{\frac{1}{q}},
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^1((1-t^\alpha)t^\alpha)^p d_\alpha t\\
=\ & \frac{1}{\alpha}\int_0^1(1-z)^p z^p dz=\frac{1}{\alpha}\int_0^1 z^p\bigg(\sum_{k=0}^{\infty}\binom{p}{k}1^{p-k}(-z)^k\bigg)dz
\end{aligned}$$

$$= \frac{1}{\alpha}\left(\sum_{k=0}^{\infty}\binom{p}{k}(-1)^k\right)\int_0^1 z^{p+k}dz = \frac{1}{\alpha}\left(\sum_{k=0}^{\infty}\binom{p}{k}\frac{(-1)^k}{p+k+1}\right).$$

This complete the proof. □

Theorem 3.5. *Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and f'' exists on on (a,b). If $|f''|$ is s-convex function, then we have*

$$\begin{aligned}
&\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\
\leq\ & \frac{(b^\alpha - a^\alpha)^2}{(s+2)(s+3)}\left(\frac{a^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)| + b^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|}{2}\right).
\end{aligned}$$

Proof. By Theorem 2.1, we have

$$\begin{aligned}
&\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\
\leq\ & \frac{1}{2\alpha}\int_0^1 |(1-t^\alpha)t^\alpha||D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|d_\alpha x.
\end{aligned}$$

According to (3) and Lemma 2.3 via the fact of $|f''|$ is s-convex function, we have

$$\begin{aligned}
& |D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)| \\
=\ & |\alpha^2(b^\alpha - a^\alpha)^2 f''(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)| \\
\leq\ & \alpha^2(b^\alpha - a^\alpha)^2(t^{\alpha s}|f''(a^\alpha)| + (1-t^\alpha)^s|f''(b^\alpha)|) \\
\leq\ & \alpha^2(b^\alpha - a^\alpha)^2(t^{\alpha s}a^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)| + (1-t^\alpha)^s b^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|).
\end{aligned} \tag{9}$$

By using (9), we have

$$\begin{aligned}
&\left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\
\leq\ & \frac{1}{2\alpha}\int_0^1 |(1-t^\alpha)t^\alpha||D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|d_\alpha t \\
\leq\ & \frac{\alpha^2(b^\alpha - a^\alpha)^2}{2\alpha}\int_0^1 |(1-t^\alpha)t^\alpha|\left[t^{\alpha s}a^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)| + (1-t^\alpha)^s b^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|\right]d_\alpha t \\
=\ & \frac{\alpha(b^\alpha - a^\alpha)^2}{2}\left[a^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)|\int_0^1 |(1-t^\alpha)t^\alpha|t^{\alpha s}d_\alpha t\right. \\
& \left. + b^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|\int_0^1 |(1-t^\alpha)t^\alpha|(1-t^\alpha)^s d_\alpha t\right].
\end{aligned}$$

Since

$$\int_0^1 |(1-t^\alpha)t^\alpha|t^{\alpha s}d_\alpha t = \int_0^1 (1-t^\alpha)t^{(s+2)\alpha-1}dt = \frac{1}{\alpha(s+2)(s+3)},$$

and

$$\int_0^1 |(1-t^\alpha)t^\alpha|(1-t^\alpha)^s d_\alpha t = \int_0^1 (1-t^\alpha)^{(s+1)} t^{2\alpha-1} dt = \frac{1}{\alpha(s+2)(s+3)},$$

we have

$$\int_0^1 |(1-t^\alpha)t^\alpha|(1-t^\alpha)^s d_\alpha t = \int_0^1 |(1-t^\alpha)t^\alpha| t^{\alpha s} d_\alpha t.$$

This complete the proof. □

Theorem 3.6. *Let $0 < a < b$, $\alpha \in (0,1)$, $f : [a,b] \to \mathbb{R}$ and f'' exists on on (a,b). If $|f''|^q$ is s-convex function, then we have*

$$\begin{aligned} & \left| \frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x \right| \\ \leq \quad & \frac{\alpha(b^\alpha - a^\alpha)^2}{2} (A(\alpha))^{\frac{1}{p}} \left(\frac{a^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)|^q + b^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)|^q}{(s+1)\alpha} \right)^{\frac{1}{q}}, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, and $A(\alpha)$ is given in (7).

Proof. By Theorem 2.1 and Hölder's integral inequality, we have

$$\begin{aligned} & \left| \frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x \right| \\ \leq \quad & \frac{1}{2\alpha} \int_0^1 |(1-t^\alpha)t^\alpha| |D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)| d_\alpha t \\ \leq \quad & \frac{1}{2\alpha} \left(\int_0^1 ((1-t^\alpha)t^\alpha)^p d_\alpha t \right)^{\frac{1}{p}} \left(\int_0^1 |D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|^q d_\alpha t \right)^{\frac{1}{q}}. \end{aligned}$$

According to (3) and Lemma 2.3 via the fact of $|f''|^q$ is a s-convex function, we have

$$\begin{aligned} & |D_{\alpha,\alpha}(f)(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|^q = |\alpha^2 (b^\alpha - a^\alpha)^2 f''(t^\alpha a^\alpha + (1-t^\alpha)b^\alpha)|^q \\ \leq \quad & \alpha^{2q}(b^\alpha - a^\alpha)^{2q} \Big[t^{\alpha s} a^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)|^q + (1-t^\alpha)^s b^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)|^q \Big]. \end{aligned} \tag{10}$$

By using (10), then

$$\begin{aligned} & \left| \frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha} \int_a^b f(x^\alpha) d_\alpha x \right| \\ \leq \quad & \frac{\alpha^2(b^\alpha - a^\alpha)^2}{2\alpha} \left(\int_0^1 ((1-t^\alpha)t^\alpha)^p d_\alpha t \right)^{\frac{1}{p}} \\ & \times \left(\int_0^1 (t^{\alpha s} a^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(a^\alpha)|^q + (1-t^\alpha)^s b^{2q\alpha(\alpha-1)} |D_{\alpha,\alpha}(f)(b^\alpha)|^q) d_\alpha t \right)^{\frac{1}{q}}. \end{aligned}$$

Finally, note that

$$\int_0^1 ((1-t^\alpha)t^\alpha)^p d_\alpha t = \frac{1}{\alpha}\sum_{k=0}^{\infty}\left(\binom{p}{k}\frac{(-1)^k}{p+k+1}\right).$$

This complete the proof. □

4. Numerical examples

Let $\alpha = \frac{1}{2}$, $f(x) = x^4$. Obviously, $|f''| = 12x^2$, $|f''|$ is convex function on $(1,4)$, $|f''|$ is a s-convex function on $[1,4]$ via $s = 0.5$, and $|f''|^2$ is a s-convex function on $[1,4]$ via $s = 0.5$.

Example 4.1. *Using Theorem 3.3, the following comfortable integral inequality holds:*

$$\begin{aligned}
\frac{23}{10} &= \left|\frac{1+16}{2} - \frac{\frac{1}{2}}{2-1}\int_1^4 x^2 d_{\frac{1}{2}}x\right| \\
&= \left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\
&\leq \frac{(b^\alpha-a^\alpha)^2}{12}\left[\frac{a^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)| + b^{2\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|}{2}\right] \\
&\leq \frac{(2-1)^2}{12}\left[\frac{|D_{\frac{1}{2},\frac{1}{2}}(f)(1)| + |D_{\frac{1}{2},\frac{1}{2}}(f)(2)|}{2}\right] \\
&= \frac{35}{12}.
\end{aligned}$$

Example 4.2. *Using Theorem 3.4, the following comfortable integral inequality holds:*

$$\begin{aligned}
\frac{23}{10} &= \left|\frac{1+16}{2} - \frac{\frac{1}{2}}{2-1}\int_1^4 x^2 d_{\frac{1}{2}}x\right| \\
&= \left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)d_\alpha x\right| \\
&\leq \frac{\alpha(b^\alpha-a^\alpha)^2}{2}(A(\alpha))^{\frac{1}{p}}\left(\frac{a^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(a^\alpha)|^q + b^{2q\alpha(\alpha-1)}|D_{\alpha,\alpha}(f)(b^\alpha)|^q}{2\alpha}\right)^{\frac{1}{q}} \\
&\leq \frac{\frac{1}{2}(2-1)^2}{2}(A(\frac{1}{2}))^{\frac{1}{2}}\left(|D_{\frac{1}{2},\frac{1}{2}}(f)(1)|^2 + |D_{\frac{1}{2},\frac{1}{2}}(f)(2)|^2\right)^{\frac{1}{2}} \\
&= \frac{7\sqrt{255}}{30},
\end{aligned}$$

where we set $p = q = 2$, $A(\alpha) = \frac{1}{\alpha}\sum_{k=1}^{\infty}\left[\binom{p}{k}\frac{(-1)^k}{p+k+1}\right] \Longrightarrow A(\frac{1}{2}) = 2\sum_{k=1}^{2}\left[\binom{2}{k}\frac{(-1)^k}{2+k+1}\right]$.

Example 4.3. *Using Theorem 3.5, the following inequality is satisfied:*

$$\frac{23}{10} = \left|\frac{1+16}{2} - 0.5\int_1^4 x^2 d_{0.5}x\right| \leq \frac{4}{35}\frac{|D_{0.5,0.5}(f)(1)| + 0.5|D_{0.5,0.5}(f)(2)|}{2} = 4$$

Example 4.4. *Using Theorem 3.6, the following inequality is satisfied:*

$$\frac{23}{10} = \left| \frac{1+16}{2} - 0.5 \int_1^4 x^2 d_{0.5}x \right| \leq \frac{1}{4} \sqrt{A(0.5) \frac{|D_{0.5,0.5}(f)(1)|^2 + 0.25|D_{0.5,0.5}(f)(2)|^2}{0.5(0.5+1)}} = \frac{7\sqrt{85}}{15}.$$

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