Received 13/09/17

OPERATOR REFINEMENTS OF SCHWARZ INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some improvements of the celebrated Schwarz inequality in complex inner product spaces in terms of selfadjoint operators $0 \leq U \leq 1_H$ are given. Applications for orthonormal families of vectors are also provided.

1. Introduction

In the recent paper [13], S. G. Walker has obtained the following refinement of Cauchy-Bunyakovsky-Schwarz inequality for the n-tuples of real numbers $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n) \in \mathbb{R}^n$

$$\begin{aligned}(1.1)\quad & \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \\ & \geq n^2 \left(\det \begin{bmatrix} \left|\frac{1}{n}\sum_{k=1}^{n} a_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n} a_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^2\right)^{1/2} \\ \left|\frac{1}{n}\sum_{k=1}^{n} b_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n} b_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)^2\right)^{1/2} \end{bmatrix}\right)^2 \geq 0\end{aligned}$$

and provided some interesting applications for the celebrated Cramer-Rao inequality.

In order to extend this result for the case of complex inner product spaces $(H, \langle \cdot, \cdot \rangle)$ we obtained in [6] the following result

$$(1.2)\quad \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \left(\det \begin{bmatrix} |\langle x, e\rangle| & \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} \\ |\langle y, e\rangle| & \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} \end{bmatrix}\right)^2 \geq 0$$

for any x, y, $e \in H$ with $\|e\| = 1$.

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $e = (e_1, ..., e_n) \in \mathbb{C}^n$ with $\sum_{k=1}^{n} |e_k|^2 = 1$. Then by writing the above inequality (1.2) for the inner product $\langle x, y\rangle :=$

1991 *Mathematics Subject Classification.* 46C05; 26D15.
Key words and phrases. Inner product spaces, Schwarz's inequality.

RGMIA Res. Rep. Coll. 20 (2017), Art. 139, 16 pp

$\sum_{k=1}^{n} x_k \overline{y}_k$ we have [6]

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq \left(\det \begin{bmatrix} \left|\sum_{k=1}^{n} x_k \overline{e}_k\right| & \left(\sum_{k=1}^{n} |x_k|^2 - \left|\sum_{k=1}^{n} x_k \overline{e}_k\right|^2 \right)^{1/2} \\ \left|\sum_{k=1}^{n} y_k \overline{e}_k\right| & \left(\sum_{k=1}^{n} |y_k|^2 - \left|\sum_{k=1}^{n} y_k \overline{e}_k\right|^2 \right)^{1/2} \end{bmatrix} \right)^2 \geq 0. \tag{1.3}$$

If we take $e_m = 1$ for $m \in \{1, ..., n\}$ and $e_k = 0$ for any $k \in \{1, ..., n\}$, $k \neq m$, then $\sum_{k=1}^{n} |e_k|^2 = 1$ and by (1.3) we get [6]

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq \max_{m \in \{1,...,n\}} \left(\det \begin{bmatrix} |x_m| & \left(\sum_{1 \leq k \neq m \leq n} |x_k|^2 \right)^{1/2} \\ |y_m| & \left(\sum_{1 \leq k \neq m \leq n} |y_k|^2 \right)^{1/2} \end{bmatrix} \right)^2. \tag{1.4}$$

If we take $e_k = \frac{1}{\sqrt{n}}$ for $k \in \{1, ..., n\}$, then $\sum_{k=1}^{n} |e_k|^2 = 1$ and by (1.3) we obtain the following complex version of Walker's inequality (1.1)

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq n^2 \left(\det \begin{bmatrix} \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|^2 - \left|\frac{1}{n}\sum_{k=1}^{n} x_k\right|^2 \right)^{1/2} \\ \left|\frac{1}{n}\sum_{k=1}^{n} y_k\right| & \left(\frac{1}{n}\sum_{k=1}^{n} |y_k|^2 - \left|\frac{1}{n}\sum_{k=1}^{n} y_k\right|^2 \right)^{1/2} \end{bmatrix} \right)^2. \tag{1.5}$$

For various results related to Cauchy-Bunyakovsky-Schwarz inequality for the n-tuples of numbers see the survey [2] and the monograph [3]. For recent results in connection to Schwarz inequality, see [1], [8] and [10]-[12].

In order to provide an extension and a refinement of (1.2) for operators we need the following preparations.

2. The Main Results

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X, i.e., the mapping $(\cdot, \cdot) : X \times X \to \mathbb{K}$ belongs to $\mathcal{H}(X)$ if it satisfies the conditions

(i) $(x, x) \geq 0$ for all x in X;
(ii) $(\alpha x + \beta y, z) = \alpha (x, z) + \beta (y, z)$ for all $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$
(iii) $(y, x) = \overline{(x, y)}$ for all $x, y \in X$.

If $(\cdot,\cdot)\in\mathcal{H}(X)$, then the functional $\|\cdot\|=(\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2\|y\|^2\geq|(x,y)|^2 \quad \text{or} \quad \|x\|\|y\|\geq|(x,y)| \tag{2.1}$$

for any $x,y\in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1,(\cdot,\cdot)_2\in\mathcal{H}(X)$ implies that $(\cdot,\cdot)_1+(\cdot,\cdot)_2\in\mathcal{H}(X)$;
(ee) $\alpha\geq 0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot)\in\mathcal{H}(X)$.

We can introduce on $\mathcal{H}(X)$ the following binary relation [7]:

$$(\cdot,\cdot)_2\geq(\cdot,\cdot)_1 \quad \text{if and only if} \quad \|x\|_2\geq\|x\|_1 \quad \text{for all} \ \ x\in X. \tag{2.2}$$

We observe that the following properties hold:

(b) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ for all $(\cdot,\cdot)\in\mathcal{H}(X)$;
(bb) $(\cdot,\cdot)_3\geq(\cdot,\cdot)_2$ and $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3\geq(\cdot,\cdot)_1$;
(bbb) $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$ implies that $(\cdot,\cdot)_2=(\cdot,\cdot)_1$;

i.e., the binary " $\geq$ " relation defined by (2.2) is *an order relation* on $\mathcal{H}(X)$.

While (b) and (bb) are obvious from the definition, we should remark, for (bbb), that if $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$ and $(\cdot,\cdot)_1\geq(\cdot,\cdot)_2$, then obviously $\|x\|_2=\|x\|_1$ for all $x\in X$, which implies, by the following well known identity:

$$(x,y)_k:=\frac{1}{4}\left[\|x+y\|_k^2-\|x-y\|_k^2+i\left(\|x+iy\|_k^2-\|x-iy\|_k^2\right)\right]$$

with $x,\,y\in X$ and $k\in\{1,\ 2\}$, that $(x,y)_2=(x,y)_1$ for all $x,\,y\in X$.Now consider the following mapping naturally associated to Schwarz's inequality, namely [7]

$$\delta:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\quad \delta((\cdot,\cdot);x,y):=\|x\|^2\|y\|^2-|(x,y)|^2.$$

It is obvious that the following properties are valid:

(i) $\delta((\cdot,\cdot);x,y)\geq 0$ (Schwarz's inequality);
(ii) $\delta((\cdot,\cdot);x,y)=\delta((\cdot,\cdot);y,x)$;
(iii) $\delta(\alpha(\cdot,\cdot);x,y)=\alpha^2\delta((\cdot,\cdot);x,y)$

for all $x,y\in X$, $\alpha\geq 0$ and $(\cdot,\cdot)\in\mathcal{H}(X)$.

The following lemma incorporates some further properties of this functional [7] (see also [5, p. 10]):

Lemma 1. *With the above assumptions, we have:*

(i) *If* $(\cdot,\cdot)_k\in\mathcal{H}(X)$ $(k=1,2)$, *then*

$$\begin{aligned}\delta((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y)-\delta((\cdot,\cdot)_1;x,y)-\delta((\cdot,\cdot)_2;x,y)&\\ \geq\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2\geq 0;&\end{aligned} \tag{2.3}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong superadditive on* $\mathcal{H}(X)$.

(ii) *If* $(\cdot,\cdot)_k\in\mathcal{H}(X)$ $(k=1,2)$, *with* $(\cdot,\cdot)_2\geq(\cdot,\cdot)_1$, *then*

$$\begin{aligned}&\delta((\cdot,\cdot)_2;x,y)-\delta((\cdot,\cdot)_1;x,y)\\ &\geq\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \left(\|x\|_2^2-\|x\|_1^2\right)^{\frac{1}{2}} & \left(\|y\|_2^2-\|y\|_1^2\right)^{\frac{1}{2}}\end{bmatrix}\right)^2\geq 0;\end{aligned} \tag{2.4}$$

i.e., the mapping $\delta(\cdot;x,y)$ *is strong nondecreasing on* $\mathcal{H}(X)$.

Proof. For the sake of completeness we give here a simple proof.

(i) For all $(\cdot,\cdot)_k \in \mathcal{H}(X)$ $(k=1,2)$ and $x,\ y\in X$ we have

$$\begin{aligned}
&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right) \\
&= \left(\|x\|_2^2+\|x\|_1^2\right)\left(\|y\|_2^2+\|y\|_1^2\right)-\left|(x,y)_2+(x,y)_1\right|^2 \\
&\geq \|x\|_2^2\|y\|_2^2+\|x\|_1^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2 \\
&\qquad -\left(\left|(x,y)_2\right|+\left|(x,y)_1\right|\right)^2 \\
&= \delta\left((\cdot,\cdot)_2;x,y\right)+\delta\left((\cdot,\cdot)_1;x,y\right) \\
&\qquad +\|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2-2\left|(x,y)_2(x,y)_1\right|.
\end{aligned} \tag{2.5}$$

By Schwarz's inequality we have

$$\left|(x,y)_2(x,y)_1\right| \leq \|x\|_1\|y\|_1\|x\|_2\|y\|_2, \tag{2.6}$$

therefore, by (2.5) and (2.6), we can state that

$$\begin{aligned}
&\delta\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_2;x,y\right) \\
&\geq \|x\|_1^2\|y\|_2^2+\|x\|_2^2\|y\|_1^2-2\|x\|_1\|y\|_1\|x\|_2\|y\|_2 \\
&= \left(\|x\|_1\|y\|_2-\|x\|_2\|y\|_1\right)^2
\end{aligned}$$

and the inequality (2.3) is proved.

(ii) Suppose that $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ and define $(\cdot,\cdot)_{2,1} := (\cdot,\cdot)_2-(\cdot,\cdot)_1$. Then $(\cdot,\cdot)_{2,1}$ is a nonnegative hermitian form and by (i) we have

$$\begin{aligned}
\delta\left((\cdot,\cdot)_{2,1};x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right) &= \delta\left((\cdot,\cdot)_{2,1}+(\cdot,\cdot)_1;x,y\right)-\delta\left((\cdot,\cdot)_1;x,y\right) \\
&\geq \delta\left((\cdot,\cdot)_{2,1};x,y\right)+\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2 \\
&\geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_{2,1} & \|y\|_{2,1}\end{bmatrix}\right)^2.
\end{aligned}$$

Since $\|z\|_{2,1}=\left(\|z\|_2^2-\|z\|_1^2\right)^{\frac{1}{2}}$ for $z\in X$, hence the inequality (2.4) is proved. □

Remark 1. *If we consider the functionals*

$$\delta_r:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\ \ \delta_r\left((\cdot,\cdot);x,y\right):=\|x\|^2\|y\|^2-\left[\operatorname{Re}(x,y)\right]^2$$

and

$$\delta_i:\mathcal{H}(X)\times X^2\to\mathbb{R}_+,\ \ \delta_i\left((\cdot,\cdot);x,y\right):=\|x\|^2\|y\|^2-\left[\operatorname{Im}(x,y)\right]^2,$$

then we can prove in a similar way the following properties:

(a) *If* $(\cdot,\cdot)_k\in\mathcal{H}(X)$ $(k=1,2)$ *and* $\ell\in\{r,i\}$ *then*

$$\begin{aligned}
&\delta_\ell\left((\cdot,\cdot)_1+(\cdot,\cdot)_2;x,y\right)-\delta_\ell\left((\cdot,\cdot)_1;x,y\right)-\delta_\ell\left((\cdot,\cdot)_2;x,y\right) \\
&\geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2\geq 0;
\end{aligned} \tag{2.7}$$

i.e., the mapping $\delta_\ell(\cdot;x,y)$ *is strong superadditive on* $\mathcal{H}(X)$.

(aa) *If* $(\cdot,\cdot)_k \in \mathcal{H}(X)$ $(k=1,2)$, *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$ *and* $\ell \in \{r,i\}$, *then*

$$\delta_\ell((\cdot,\cdot)_2;x,y) - \delta_\ell((\cdot,\cdot)_1;x,y) \geq \left(\det\begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \left(\|x\|_2^2 - \|x\|_1^2\right)^{\frac{1}{2}} & \left(\|y\|_2^2 - \|y\|_1^2\right)^{\frac{1}{2}} \end{bmatrix}\right)^2 \geq 0; \tag{2.8}$$

i.e., the mapping $\delta_\ell(\cdot;x,y)$ *is strong nondecreasing on* $\mathcal{H}(X)$.

We have the following refinement of Schwarz inequality:

Theorem 1. *Let* $(H,\langle\cdot,\cdot\rangle)$ *be a complex Hilbert space and* $U: H \to H$ *a selfadjoint operator such that* $0 \leq U \leq 1_H$. *Then for any* $x,\ y \in H$ *we have*

$$\begin{aligned} \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 &\geq \langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2 \\ &+ \left[\|x\|^2 - \langle Ux,x\rangle\right]\left[\|y\|^2 - \langle Uy,y\rangle\right] - |\langle x,y\rangle - \langle Ux,y\rangle|^2 \\ &+ \left(\det\begin{bmatrix} \langle Ux,x\rangle^{1/2} & \langle Uy,y\rangle^{1/2} \\ \left[\|x\|^2 - \langle Ux,x\rangle\right]^{1/2} & \left[\|y\|^2 - \langle Uy,y\rangle\right]^{1/2} \end{bmatrix}\right)^2. \end{aligned} \tag{2.9}$$

Proof. Consider the nonnegative forms $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ on H defined by

$$(x,y)_1 := \langle Ux,y\rangle \text{ and } (x,y)_2 := \langle (1_H - U)x,y\rangle,\ x,\ y \in H.$$

Then

$$(x,y)_1 + (x,y)_2 = \langle x,y\rangle,\ \|x\|_1^2 = \langle Ux,x\rangle$$

and

$$\|x\|_2^2 = \langle (1_H - U)x,x\rangle = \|x\|^2 - \langle Ux,x\rangle$$

for $x,\ y \in H$.

Since

$$\delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) = \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2,$$
$$\delta((\cdot,\cdot)_1;x,y) = \langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2 \geq 0$$

and

$$\delta((\cdot,\cdot)_2;x,y) = \left[\|x\|^2 - \langle Ux,x\rangle\right]\left[\|y\|^2 - \langle Uy,y\rangle\right] - |\langle x,y\rangle - \langle Ux,y\rangle|^2 \geq 0$$

where the last two inequalities follow by Schwarz's inequality for nonnegative operators, then by (2.3) we get (2.9). □

Remark 2. *Let* $U: H \to H$ *be a selfadjoint operator such that* $0 \leq U \leq 1_H$. *We observe that from (2.9) we get the simpler inequalities that are coarser but may be more useful for applications such as*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2 \geq 0 \tag{2.10}$$

and

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \left(\det\begin{bmatrix} \langle Ux,x\rangle^{1/2} & \langle Uy,y\rangle^{1/2} \\ \left[\|x\|^2 - \langle Ux,x\rangle\right]^{1/2} & \left[\|y\|^2 - \langle Uy,y\rangle\right]^{1/2} \end{bmatrix}\right)^2 \geq 0 \tag{2.11}$$

for any $x,\ y \in H$.

For other results connected with Schwarz inequality in inner product spaces see the monographs [4] and [5].

Assume that $P : H \to H$ is an *orthogonal projection* on H, namely it satisfies the condition $P^2 = P = P^*$. We obviously have in the operator order of $\mathcal{B}(H)$, the Banach algebra of all linear bounded operators on H, that $0 \le P \le 1_H$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j,k \in J \text{ with } j \neq k \text{ and } \|e_j\| = 1 \text{ for any } j,k \in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

For an othonormal family $\mathcal{E} = \{e_j\}_{j\in J}$ we define the operator $P_{\mathcal{E}} : H \to H$ by

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j \ , \ x \in H. \tag{2.12}$$

We know that $P_{\mathcal{E}}$ is an *orthogonal projection* and

$$\langle P_{\mathcal{E}} x, y\rangle = \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle \ , \ x, y \in H \text{ and } \langle P_{\mathcal{E}} x, x\rangle = \sum_{j\in J} |\langle x, e_j\rangle|^2 \ , \ x \in H.$$

The particular case when the family reduces to one vector, namely $\mathcal{E} = \{e\}$, $\|e\| = 1$, is of interest since in this case $P_e x := \langle x, e\rangle e$, $x \in H$,

$$\langle P_e x, y\rangle = \langle x, e\rangle \langle e, y\rangle \ , \ x, y \in H \text{ and } \langle P_e x, x\rangle = |\langle x, e\rangle|^2 \ , \ x \in H.$$

Corollary 1. *Let $\mathcal{E} = \{e_j\}_{j\in J}$ be an othonormal family in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Then for any x, $y \in H$ we have*

$$\begin{aligned} \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 &\ge \sum_{j\in J} |\langle x, e_j\rangle|^2 \sum_{j\in J} |\langle y, e_j\rangle|^2 - \left|\sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right|^2 \\ &+ \left[\|x\|^2 - \sum_{j\in J} |\langle x, e_j\rangle|^2\right]\left[\|y\|^2 - \sum_{j\in J} |\langle y, e_j\rangle|^2\right] - \left|\langle x, y\rangle - \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right|^2 \\ &+ \left(\det \begin{bmatrix} \left[\sum_{j\in J} |\langle x, e_j\rangle|^2\right]^{1/2} & \left[\sum_{j\in J} |\langle y, e_j\rangle|^2\right]^{1/2} \\ \left[\|x\|^2 - \sum_{j\in J} |\langle x, e_j\rangle|^2\right]^{1/2} & \left[\|y\|^2 - \sum_{j\in J} |\langle y, e_j\rangle|^2\right]^{1/2} \end{bmatrix}\right)^2 . \end{aligned} \tag{2.13}$$

In particular, if $e \in H$, $\|e\| = 1$, then for any x, $y \in H$ we have

$$\begin{aligned} &\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \\ &\ge \left[\|x\|^2 - |\langle x, e\rangle|^2\right]\left[\|y\|^2 - |\langle y, e\rangle|^2\right] - |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle|^2 \\ &\quad + \left(\det \begin{bmatrix} |\langle x, e\rangle| & |\langle y, e\rangle| \\ \left[\|x\|^2 - |\langle x, e\rangle|^2\right]^{1/2} & \left[\|y\|^2 - |\langle y, e\rangle|^2\right]^{1/2} \end{bmatrix}\right)^2 . \end{aligned} \tag{2.14}$$

Remark 3. *Let $\mathcal{E} = \{e_j\}_{j\in J}$ be an othonormal family in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. We observe that, from (2.13) we get the inequalities*

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \sum_{j\in J}|\langle x,e_j\rangle|^2\sum_{j\in J}|\langle y,e_j\rangle|^2 - \left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|^2 \geq 0, \tag{2.15}$$

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \left[\|x\|^2 - \sum_{j\in J}|\langle x,e_j\rangle|^2\right]\left[\|y\|^2 - \sum_{j\in J}|\langle y,e_j\rangle|^2\right] - \left|\langle x,y\rangle - \sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|^2 \geq 0 \tag{2.16}$$

and

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \left(\det\begin{bmatrix}\left[\sum_{j\in J}|\langle x,e_j\rangle|^2\right]^{1/2} & \left[\sum_{j\in J}|\langle y,e_j\rangle|^2\right]^{1/2}\\ \left[\|x\|^2-\sum_{j\in J}|\langle x,e_j\rangle|^2\right]^{1/2} & \left[\|y\|^2-\sum_{j\in J}|\langle y,e_j\rangle|^2\right]^{1/2}\end{bmatrix}\right)^2 \geq 0 \tag{2.17}$$

for any $x, y \in H$.

From (2.14) we have

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \left[\|x\|^2 - |\langle x,e\rangle|^2\right]\left[\|y\|^2 - |\langle y,e\rangle|^2\right] - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \tag{2.18}$$

and

$$\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 \geq \left(\det\begin{bmatrix}|\langle x,e\rangle| & |\langle y,e\rangle|\\ \left[\|x\|^2-|\langle x,e\rangle|^2\right]^{1/2} & \left[\|y\|^2-|\langle y,e\rangle|^2\right]^{1/2}\end{bmatrix}\right)^2$$

for any x, y, $e \in H$ with $\|e\| = 1$ and thus recapture, in a simple way - compare with the proof in [6]*, the inequality (1.2) from the introduction.*

Corollary 2. *Assume that the bounded linear operator $A : H \to H$ satisfies the condition $\|Au\| \leq \|u\|$ for any $u \in H$. Then we have the inequality*

$$\begin{aligned}\|x\|^2\|y\|^2 - |\langle x,y\rangle|^2 &\geq \|Ax\|^2\|Ay\|^2 - |\langle Ax,Ay\rangle|^2\\ &+ \left[\|x\|^2 - \|Ax\|^2\right]\left[\|y\|^2 - \|Ay\|^2\right] - |\langle x,y\rangle - \langle Ax,Ay\rangle|^2\\ &+ \left(\det\begin{bmatrix}\|Ax\| & \|Ay\|\\ \left[\|x\|^2-\|Ax\|^2\right]^{1/2} & \left[\|y\|^2-\|Ay\|^2\right]^{1/2}\end{bmatrix}\right)^2\end{aligned} \tag{2.19}$$

for any x, $y \in H$.

Proof. We observe that the condition $\|Au\| \leq \|u\|$ for any $u \in H$ is equivalent to the fact that $0 \leq A^*A \leq 1_H$ and by writing the inequality (2.9) for $U = A^*A$ we get (2.19). □

Remark 4. *From (2.19) we have the simpler inequalities*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \|Ax\|^2 \|Ay\|^2 - |\langle Ax, Ay\rangle|^2 \geq 0 \tag{2.20}$$

and

$$\begin{aligned} &\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \\ &\geq \left(\det \begin{bmatrix} \|Ax\| & \|Ay\| \\ \left[\|x\|^2 - \|Ax\|^2\right]^{1/2} & \left[\|y\|^2 - \|Ay\|^2\right]^{1/2} \end{bmatrix}\right)^2 \geq 0 \end{aligned} \tag{2.21}$$

for any $x,\ y \in H$.

We have:

Lemma 2. *If* $(\cdot,\cdot)_k \in \mathcal{H}(X)$ $(k = 1, 2)$, *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, *then for any* $x,\ y \in H$

$$\delta\left((\cdot,\cdot)_2 ; x, y\right) + \delta\left((\cdot,\cdot)_1 ; x, y\right) \geq \begin{cases} 2\det \begin{bmatrix} \|x\|_1 \|y\|_1 & |\langle x, y\rangle_1| \\ |\langle x, y\rangle_2| & \|x\|_2 \|y\|_2 \end{bmatrix} \\ \left(\det \begin{bmatrix} \|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2 \end{bmatrix}\right)^2 \end{cases} \geq 0. \tag{2.22}$$

Proof. Since $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, then $(\cdot,\cdot)_{2,1} := (\cdot,\cdot)_2 - (\cdot,\cdot)_1$ is a nonnegative hermitian form and by Schwarz inequality for $(\cdot,\cdot)_{2,1}$ we have

$$\begin{aligned} \left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) &\geq |\langle x, y\rangle_2 - \langle x, y\rangle_1|^2 \\ &\geq \left||\langle x, y\rangle_2| - |\langle x, y\rangle_1|\right|^2 \end{aligned}$$

for any $x,\ y \in H$, where for the last inequality we used the continuity of the modulus property.

This inequality is equivalent to

$$\begin{aligned} &\|x\|_2^2 \|y\|_2^2 + \|x\|_1^2 \|y\|_1^2 - \|x\|_1^2 \|y\|_2^2 - \|x\|_2^2 \|y\|_1^2 \\ &\geq |\langle x, y\rangle_2|^2 - 2\,|\langle x, y\rangle_2|\,|\langle x, y\rangle_1| + |\langle x, y\rangle_1|^2 \end{aligned}$$

or to

$$\delta\left((\cdot,\cdot)_2 ; x, y\right) + \delta\left((\cdot,\cdot)_1 ; x, y\right) \geq \|x\|_1^2 \|y\|_2^2 - 2\,|\langle x, y\rangle_2|\,|\langle x, y\rangle_1| + \|x\|_2^2 \|y\|_1^2 \tag{2.23}$$

for any $x,\ y \in H$.

By Schwarz inequality for $(\cdot,\cdot)_2$ and $(\cdot,\cdot)_1$ we have

$$\|x\|_1 \|y\|_1 \geq |\langle x, y\rangle_1| \text{ and } \|x\|_2 \|y\|_2 \geq |\langle x, y\rangle_2|$$

which by multiplication gives

$$\|x\|_1 \|y\|_1 \|x\|_2 \|y\|_2 \geq |\langle x, y\rangle_1|\,|\langle x, y\rangle_2|$$

for any $x,\ y \in H$.

Therefore

$$\begin{aligned}(2.24)\qquad & \|x\|_1^2\|y\|_2^2 - 2|\langle x,y\rangle_2||\langle x,y\rangle_1| + \|x\|_2^2\|y\|_1^2 \\ & \geq \|x\|_1^2\|y\|_2^2 - 2\|x\|_1\|y\|_1\|x\|_2\|y\|_2 + \|x\|_2^2\|y\|_1^2 \\ & = (\|x\|_1\|y\|_2 - \|x\|_2\|y\|_1)^2\end{aligned}$$

for any $x,\, y \in H$.

By utilising (2.23) and (2.24) we get the second branch in the inequality (2.22).

By the elementary inequality $a^2+b^2 \geq 2ab,\ a,b\in\mathbb{R}$ we have

$$\|x\|_1^2\|y\|_2^2 + \|x\|_2^2\|y\|_1^2 \geq 2\|x\|_1\|y\|_1\|x\|_2\|y\|_2$$

for any $x,\, y \in H$.

This implies that

$$\begin{aligned}(2.25)\qquad & \|x\|_1^2\|y\|_2^2 - 2|\langle x,y\rangle_2||\langle x,y\rangle_1| + \|x\|_2^2\|y\|_1^2 \\ & \geq 2(\|x\|_1\|y\|_1\|x\|_2\|y\|_2 - |\langle x,y\rangle_2||\langle x,y\rangle_1|) \geq 0\end{aligned}$$

for any $x,\, y \in H$.

By making use of (2.23) and (2.25) we get the first branch of (2.22). □

Corollary 3. *If* $(\cdot,\cdot)_k \in \mathcal{H}(X)\ (k=1,2)$, *with* $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1$, *then for any* x, $y \in H$ *we have*

$$(2.26)\qquad \delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) \geq 2\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2 \geq 0$$

and

$$\begin{aligned}(2.27)\quad & \delta((\cdot,\cdot)_2;x,y) \\ & \geq \frac{1}{2}\left[\left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \left(\|x\|_2^2-\|x\|_1^2\right)^{1/2} & \left(\|y\|_2^2-\|y\|_1^2\right)^{1/2}\end{bmatrix}\right)^2\right. \\ & \qquad\qquad \left. + \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2\right] \geq 0.\end{aligned}$$

Proof. From the inequality (2.3) we have

$$\begin{aligned}(2.28)\qquad & \delta((\cdot,\cdot)_1 + (\cdot,\cdot)_2;x,y) \\ & \geq \delta((\cdot,\cdot)_1;x,y) + \delta((\cdot,\cdot)_2;x,y) + \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2 \geq 0\end{aligned}$$

while from (2.22) we have

$$(2.29)\qquad \delta((\cdot,\cdot)_2;x,y) + \delta((\cdot,\cdot)_1;x,y) \geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2\end{bmatrix}\right)^2$$

for any $x,\, y \in H$, which imply the desired result (2.26).

From (2.4) we also have

$$\begin{aligned}(2.30)\qquad & \delta((\cdot,\cdot)_2;x,y) - \delta((\cdot,\cdot)_1;x,y) \\ & \geq \left(\det\begin{bmatrix}\|x\|_1 & \|y\|_1 \\ \left(\|x\|_2^2-\|x\|_1^2\right)^{\frac{1}{2}} & \left(\|y\|_2^2-\|y\|_1^2\right)^{\frac{1}{2}}\end{bmatrix}\right)^2 \geq 0\end{aligned}$$

for any $x, y \in H$. If we add the inequality (2.29) with (2.30) and divide by 2 we get (2.27). $\square$

Theorem 2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $U : H \to H$ a selfadjoint operator such that $0 \leq U \leq 1_H$. Then for any $x, y \in H$ we have*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 + \langle Ux, x\rangle \langle Uy, y\rangle - |\langle Ux, y\rangle|^2 \geq \begin{cases} 2\det \begin{bmatrix} \langle Ux, x\rangle^{1/2} \langle Uy, y\rangle^{1/2} & |\langle Ux, y\rangle| \\ |\langle x, y\rangle| & \|x\| \|y\| \end{bmatrix} \\ \left(\det \begin{bmatrix} \langle Ux, x\rangle^{1/2} & \langle Uy, y\rangle^{1/2} \\ \|x\| & \|y\| \end{bmatrix}\right)^2 \end{cases} \geq 0, \tag{2.31}$$

$$\left(\|x\|^2 + \langle Ux, x\rangle\right)\left(\|y\|^2 + \langle Uy, y\rangle\right) - |\langle x, y\rangle + \langle Ux, y\rangle|^2 \geq 2\left(\det \begin{bmatrix} \langle Ux, x\rangle^{1/2} & \langle Uy, y\rangle^{1/2} \\ \|x\| & \|y\| \end{bmatrix}\right)^2 \geq 0 \tag{2.32}$$

and

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \frac{1}{2}\left[\left(\det \begin{bmatrix} \langle Ux, x\rangle^{1/2} & \langle Uy, y\rangle^{1/2} \\ \left(\|x\|^2 - \langle Ux, x\rangle\right)^{1/2} & \left(\|y\|^2 - \langle Uy, y\rangle\right)^{1/2} \end{bmatrix}\right)^2 + \left(\det \begin{bmatrix} \langle Ux, x\rangle^{1/2} & \langle Uy, y\rangle^{1/2} \\ \|x\| & \|y\| \end{bmatrix}\right)^2\right] \geq 0. \tag{2.33}$$

Proof. Consider the nonnegative Hermitian forms

$$(x, y)_2 := \langle x, y\rangle \text{ and } (x, y)_1 := \langle Ux, y\rangle \text{ for } x, \ y \in H.$$

Then we have

$$\|x\|_2^2 = \|x\|^2 \geq \langle Ux, x\rangle = \|x\|_1^2 \text{ for } x \in H.$$

Then by (2.22), (2.26) and (2.27) we deduce the desired results (2.31)-(2.33). $\square$

Remark 5. *Let $\mathcal{E} = \{e_j\}_{j\in J}$ be an othonormal family in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If we take $U = P_{\mathcal{E}}$, where*

$$P_{\mathcal{E}} x := \sum_{j\in J} \langle x, e_j\rangle e_j \ , \ x \in H$$

in (2.31), then we get

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2+\sum_{j\in J}|\langle x,e_j\rangle|^2\sum_{j\in J}|\langle y,e_j\rangle|^2-\left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|^2 \tag{2.34}$$

$$\geq\begin{cases}2\det\begin{bmatrix}\left(\sum_{j\in J}|\langle x,e_j\rangle|^2\right)^{1/2}\left(\sum_{j\in J}|\langle y,e_j\rangle|^2\right)^{1/2} & \left|\sum_{j\in J}\langle x,e_j\rangle\langle e_j,y\rangle\right|\\ |\langle x,y\rangle| & \|x\|\|y\|\end{bmatrix}\\ \left(\det\begin{bmatrix}\left(\sum_{j\in J}|\langle x,e_j\rangle|^2\right)^{1/2} & \left(\sum_{j\in J}|\langle y,e_j\rangle|^2\right)^{1/2}\\ \|x\| & \|y\|\end{bmatrix}\right)^2\end{cases}$$

$$\geq 0.$$

If we take $\mathcal{E}=\{e\}$, $\|e\|=1$ *in (2.34) then we get*

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \geq\begin{cases}2\det\begin{bmatrix}|\langle x,e\rangle\langle e,y\rangle| & |\langle x,e\rangle\langle e,y\rangle|\\ |\langle x,y\rangle| & \|x\|\|y\|\end{bmatrix};\\ \left(\det\begin{bmatrix}|\langle x,e\rangle| & |\langle y,e\rangle|\\ \|x\| & \|y\|\end{bmatrix}\right)^2\end{cases}\geq 0, \tag{2.35}$$

for any $x, y\in H$.

The first inequality can be written as

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq 2|\langle x,e\rangle\langle e,y\rangle|(\|x\|\|y\|-|\langle x,y\rangle|),$$

which produces the Buzano's inequality

$$\frac{1}{2}(\|x\|\|y\|+|\langle x,y\rangle|)\geq|\langle x,e\rangle\langle e,y\rangle| \tag{2.36}$$

for any $x, y\in H$.

From the second branch of (2.35) we obtain the following refinement of Schwarz inequality

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq(\|x\||\langle y,e\rangle|-|\langle x,e\rangle|\|y\|)^2\geq 0 \tag{2.37}$$

for any $x, y\in H$ *and* $e\in H$, $\|e\|=1$.

Let $\mathcal{E} = \{e_j\}_{j\in J}$ be an othonormal family in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$. If we write the inequality (2.32) for $U = P_{\mathcal{E}}$, then we have

$$\left(\|x\|^2 + \sum_{j\in J} |\langle x, e_j\rangle|^2\right)\left(\|y\|^2 + \sum_{j\in J} |\langle y, e_j\rangle|^2\right) - \left|\langle x, y\rangle + \sum_{j\in J} \langle x, e_j\rangle \langle e_j, y\rangle\right|^2 \geq 2\left(\det\begin{bmatrix} \left(\sum_{j\in J} |\langle x, e_j\rangle|^2\right)^{1/2} & \left(\sum_{j\in J} |\langle y, e_j\rangle|^2\right)^{1/2} \\ \|x\| & \|y\| \end{bmatrix}\right)^2 \geq 0 \tag{2.38}$$

for any $x, y \in H$.

In particular, for $\mathcal{E} = \{e\}$, $\|e\| = 1$ we get

$$\left(\|x\|^2 + |\langle x, e\rangle|^2\right)\left(\|y\|^2 + |\langle y, e\rangle|^2\right) - |\langle x, y\rangle + \langle x, e\rangle \langle e, y\rangle|^2 \geq 2\left(\det\begin{bmatrix} |\langle x, e\rangle| & |\langle y, e\rangle| \\ \|x\| & \|y\| \end{bmatrix}\right)^2 \geq 0 \tag{2.39}$$

for any $x, y \in H$.

From (2.33) we have for $U = P_{\mathcal{E}}$

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \frac{1}{2}\left[\left(\det\begin{bmatrix} \left(\sum_{j\in J} |\langle x, e_j\rangle|^2\right)^{1/2} & \left(\sum_{j\in J} |\langle y, e_j\rangle|^2\right)^{1/2} \\ \left(\|x\|^2 - \sum_{j\in J} |\langle x, e_j\rangle|^2\right)^{1/2} & \left(\|y\|^2 - \sum_{j\in J} |\langle y, e_j\rangle|^2\right)^{1/2} \end{bmatrix}\right)^2 + \left(\det\begin{bmatrix} \left(\sum_{j\in J} |\langle x, e_j\rangle|^2\right)^{1/2} & \left(\sum_{j\in J} |\langle y, e_j\rangle|^2\right)^{1/2} \\ \|x\| & \|y\| \end{bmatrix}\right)^2\right] \geq 0 \tag{2.40}$$

for any $x, y \in H$.

In particular, for $\mathcal{E} = \{e\}$, $\|e\| = 1$ we get

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \frac{1}{2}\left[\left(\det\begin{bmatrix} |\langle x, e\rangle| & |\langle y, e\rangle| \\ \left(\|x\|^2 - |\langle x, e\rangle|^2\right)^{1/2} & \left(\|y\|^2 - |\langle y, e\rangle|^2\right)^{1/2} \end{bmatrix}\right)^2 + \left(\det\begin{bmatrix} |\langle x, e\rangle| & |\langle y, e\rangle| \\ \|x\| & \|y\| \end{bmatrix}\right)^2\right] \geq 0 \tag{2.41}$$

for any $x, y \in H$.

Now, consider the operator $U : H \to H$ with the property that $0 \leq U \leq \frac{1}{2}1_H$. This is equivalent to

$$0 \leq \langle Ux, x\rangle \leq \frac{1}{2}\|x\|^2 \text{ for any } x \in H \tag{2.42}$$

or, equivalently

$$\|x\|^2 - \langle Ux, x\rangle \geq \langle Ux, x\rangle \geq 0 \text{ for any } x \in H. \tag{2.43}$$

Consider the hermitian forms $(x,y)_2 := \langle x,y\rangle - \langle Ux,y\rangle$ and $(x,y)_1 := \langle Ux,y\rangle$ for x, $y \in H$. Then by (2.43) we have that $(\cdot,\cdot)_2 \geq (\cdot,\cdot)_1 \geq 0$, $(\cdot,\cdot)_2 + (\cdot,\cdot)_1 = \langle\cdot,\cdot\rangle$ and by (2.22) we get

$$\begin{aligned}
&\left(\|x\|^2 - \langle Ux,x\rangle\right)\left(\|y\|^2 - \langle Uy,y\rangle\right) - |\langle x,y\rangle - \langle Ux,y\rangle|^2 \\
&\qquad + \langle Ux,x\rangle\langle Uy,y\rangle - |\langle Ux,y\rangle|^2 \\
&\geq \begin{cases} 2\det\begin{bmatrix} \langle Ux,x\rangle^{1/2}\langle Uy,y\rangle^{1/2} & |\langle Ux,y\rangle| \\ |\langle x,y\rangle - \langle Ux,y\rangle| & \left(\|x\|^2 - \langle Ux,x\rangle\right)^{1/2}\left(\|y\|^2 - \langle Uy,y\rangle\right)^{1/2} \end{bmatrix} \\ \left(\det\begin{bmatrix} \langle Ux,x\rangle^{1/2} & \langle Uy,y\rangle^{1/2} \\ \left(\|x\|^2 - \langle Ux,x\rangle\right)^{1/2} & \left(\|y\|^2 - \langle Uy,y\rangle\right)^{1/2} \end{bmatrix}\right)^2 \end{cases} \\
&\geq 0.
\end{aligned} \tag{2.44}$$

Now, let $e \in H$, $e \neq 0$ with $\|e\| \leq \frac{\sqrt{2}}{2}$. Consider the operator $Ux = \langle x,e\rangle e$, $x \in H$. Then

$$0 \leq \langle Ux,x\rangle = |\langle x,e\rangle|^2 \leq \|x\|^2\|e\|^2 \leq \frac{1}{2}\|x\|^2, \ x \in H,$$

which shows that U satisfies the condition (2.42). By utilising (2.44) for this operator, we get

$$\begin{aligned}
&\left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,e\rangle|^2\right) - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \\
&\geq \begin{cases} 2\det\begin{bmatrix} |\langle x,e\rangle\langle e,y\rangle| & |\langle x,e\rangle\langle e,y\rangle| \\ |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| & \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \end{bmatrix} \\ \left(\det\begin{bmatrix} |\langle x,e\rangle| & |\langle y,e\rangle| \\ \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2} & \left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} \end{bmatrix}\right)^2 \end{cases} \\
&\geq 0,
\end{aligned} \tag{2.45}$$

for any x, y, $e \in H$ with $\|e\| \leq \frac{\sqrt{2}}{2}$.

From the first branch of (2.45) we get

$$\begin{aligned}
&\left(\|x\|^2 - |\langle x,e\rangle|^2\right)\left(\|y\|^2 - |\langle y,e\rangle|^2\right) - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|^2 \\
&\geq 2|\langle x,e\rangle\langle e,y\rangle|\left(\left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle|\right),
\end{aligned}$$

which implies that

$$\frac{1}{2}\left[\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}+|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\right] \geq |\langle x,e\rangle\langle e,y\rangle| \tag{2.46}$$

for any x, y, $e\in H$ with $\|e\|\leq\frac{\sqrt{2}}{2}$.

From the second branch of (2.45) we have

$$\left(\|x\|^2-|\langle x,e\rangle|^2\right)\left(\|y\|^2-|\langle y,e\rangle|^2\right)-|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|^2 \geq \left(\det\begin{bmatrix} |\langle x,e\rangle| & |\langle y,e\rangle| \\ \left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2} & \left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}\end{bmatrix}\right)^2 \tag{2.47}$$

for any x, y, $e\in H$ with $\|e\|\leq\frac{\sqrt{2}}{2}$.

From the inequality (2.26) for the hermitian forms $(x,y)_2 := \langle x,y\rangle-\langle Ux,y\rangle$ and $(x,y)_1 := \langle Ux,y\rangle$ with U satisfying (2.42) we have the simple refinement of Schwarz's inequality

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \geq 2\left(\det\begin{bmatrix} \langle Ux,x\rangle^{1/2} & \langle Uy,y\rangle^{1/2} \\ \left(\|x\|^2-\langle Ux,x\rangle\right)^{1/2} & \left(\|y\|^2-\langle Uy,y\rangle\right)^{1/2}\end{bmatrix}\right)^2 \geq 0 \tag{2.48}$$

for any x, $y\in H$.

If we write the inequality (2.48) for the operator $Ux=\langle x,e\rangle e$, $x\in H$ with $\|e\|\leq\frac{\sqrt{2}}{2}$, then we get

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 \geq 2\left(\det\begin{bmatrix} |\langle x,e\rangle| & |\langle y,e\rangle| \\ \left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2} & \left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}\end{bmatrix}\right)^2 \geq 0 \tag{2.49}$$

for any x, $y\in H$.

3. Applications for n-Tuples of Complex Numbers

Let $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$, $e=(e_1,...,e_n)\in\mathbb{C}^n$ with $\sum_{k=1}^n|e_k|^2=1$. Then by writing the above inequality (2.37) for the inner product $\langle x,y\rangle := \sum_{k=1}^n x_k\overline{y}_k$ we get

$$\sum_{k=1}^n|x_k|^2\sum_{k=1}^n|y_k|^2-\left|\sum_{k=1}^n x_k\overline{y}_k\right|^2 \geq \left(\left(\sum_{k=1}^n|x_k|^2\right)^{1/2}\left|\sum_{k=1}^n y_k\overline{e}_k\right|-\left|\sum_{k=1}^n x_k\overline{e}_k\right|\left(\sum_{k=1}^n|y_k|^2\right)^{1/2}\right)^2 \geq 0. \tag{3.1}$$

If we take $e_m = 1$ for $m \in \{1, ..., n\}$ and $e_k = 0$ for any $k \in \{1, ..., n\}$, $k \neq m$, then $\sum_{k=1}^{n} |e_k|^2 = 1$ and by (3.1) we get

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq \max_{m \in \{1,...,n\}} \left(\left(\sum_{k=1}^{n} |x_k|^2 \right)^{1/2} |y_m| - |x_m| \left(\sum_{k=1}^{n} |y_k|^2 \right)^{1/2} \right)^2 \geq 0. \tag{3.2}$$

If we take $e_k = \frac{1}{\sqrt{n}}$ for $k \in \{1, ..., n\}$, then $\sum_{k=1}^{n} |e_k|^2 = 1$ and by (3.1) we get

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq n^2 \left(\left(\frac{1}{n} \sum_{k=1}^{n} |x_k|^2 \right)^{1/2} \left| \frac{1}{n} \sum_{k=1}^{n} y_k \right| - \left| \frac{1}{n} \sum_{k=1}^{n} x_k \right| \left(\frac{1}{n} \sum_{k=1}^{n} |y_k|^2 \right)^{1/2} \right)^2 \geq 0. \tag{3.3}$$

Let $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $e = (e_1, ..., e_n) \in \mathbb{C}^n$ with $\sum_{k=1}^{n} |e_k|^2 = \frac{1}{2}$. Then by writing the above inequality (2.49) for the inner product $\langle x, y \rangle := \sum_{k=1}^{n} x_k \overline{y}_k$ we have

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq 2 \left(\det \begin{bmatrix} \left| \sum_{k=1}^{n} x_k \overline{e}_k \right| & \left| \sum_{k=1}^{n} y_k \overline{e}_k \right| \\ \left(\sum_{k=1}^{n} |x_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{e}_k \right|^2 \right)^{1/2} & \left(\sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} y_k \overline{e}_k \right|^2 \right)^{1/2} \end{bmatrix} \right)^2 \geq 0. \tag{3.4}$$

If we take $e_m = \frac{\sqrt{2}}{2}$ for $m \in \{1, ..., n\}$ and $e_k = 0$ for any $k \in \{1, ..., n\}$, $k \neq m$, then $\sum_{k=1}^{n} |e_k|^2 = \frac{1}{2}$ and by (3.4) we have

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \geq \max_{m \in \{1,...,n\}} \left(\det \begin{bmatrix} |x_m| & |y_m| \\ \left(\sum_{k=1}^{n} |x_k|^2 - \frac{1}{2} |x_m|^2 \right)^{1/2} & \left(\sum_{k=1}^{n} |y_k|^2 - \frac{1}{2} |y_m|^2 \right)^{1/2} \end{bmatrix} \right)^2 \geq 0. \tag{3.5}$$

If we take $e_k = \frac{\sqrt{2}}{2\sqrt{n}}$ for $k \in \{1, ..., n\}$, then $\sum_{k=1}^{n} |e_k|^2 = 1$ and by (3.4) we get

$$\sum_{k=1}^{n} |x_k|^2 \sum_{k=1}^{n} |y_k|^2 - \left| \sum_{k=1}^{n} x_k \overline{y}_k \right|^2 \tag{3.6}$$
$$\geq n^2 \left(\det \begin{bmatrix} \left| \frac{\sum_{k=1}^{n} x_k}{n} \right| & \left| \frac{\sum_{k=1}^{n} y_k}{n} \right| \\ \left(\frac{\sum_{k=1}^{n} |x_k|^2}{n} - \left| \frac{\sum_{k=1}^{n} x_k}{2n} \right|^2 \right)^{1/2} & \left(\frac{\sum_{k=1}^{n} |y_k|^2}{n} - \left| \frac{\sum_{k=1}^{n} y_k}{2n} \right|^2 \right)^{1/2} \end{bmatrix} \right)^2$$
$$\geq 0.$$

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