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SOME GENERALIZATIONS OF SCHWARZ INEQUALITY IN INNER PRODUCT SPACES

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Abstract. Some two complex parameters generalizations of the celebrated Schwarz inequality in inner product spaces are given. Applications for n-tuples of complex numbers are provided.

1. Introduction

Let $(H,\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality*

$$\|x\|^2\|y\|^2\geq|\langle x,y\rangle|^2 \tag{1.1}$$

for any $x,\ y\in H$. The equality case holds in (1.1) if and only if there exists a constant $\lambda\in\mathbb{K}$ such that $x=\lambda y$. This inequality can be written in an equivalent form as

$$\|x\|\|y\|\geq|\langle x,y\rangle|\,. \tag{1.2}$$

Assume that $P:H\to H$ is an orthogonal projection on H, namely, it satisfies the condition $P^2=P=P^*$. We obviously have in the operator order of $B(H)$, the Banach algebra of all linear bounded operators on H, that $0\leq P\leq 1_H$.

In the recent paper [8, Eq. (2.6)] we established among others that

$$\|x\|\|y\|\geq\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}+|\langle x,y\rangle-\langle Px,y\rangle| \tag{1.3}$$

for any $x,\ y\in H$. Since by the triangle inequality we have

$$|\langle x,y\rangle-\langle Px,y\rangle|\geq|\langle x,y\rangle|-|\langle Px,y\rangle|$$

and by the Schwarz inequality for nonnegative selfadjoint operators we have

$$\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}\geq|\langle Px,y\rangle|$$

for any $x,\ y\in H$, then we get from (1.3) the following refinement of (1.2)

$$\|x\|\|y\|-|\langle x,y\rangle|\geq\langle Px,x\rangle^{1/2}\langle Py,y\rangle^{1/2}-|\langle Px,y\rangle|\geq 0 \tag{1.4}$$

for any $x,\ y\in H$.

In 1985 the author [2] (see also [3] or [6, p. 36]) established the following inequality related to Schwarz inequality

$$\left(\|x\|^2\|z\|^2-|\langle x,z\rangle|^2\right)\left(\|y\|^2\|z\|^2-|\langle y,z\rangle|^2\right)\geq\left|\langle x,y\rangle\|z\|^2-\langle x,z\rangle\langle z,y\rangle\right|^2 \tag{1.5}$$

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for any x, y, $z \in H$ and obtained, as a consequence, the following refinement of (1.2):

$$\|x\| \|y\| \geq |\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| + |\langle x, e\rangle \langle e, y\rangle| \geq |\langle x, y\rangle| \tag{1.6}$$

for any x, y, $e \in H$ with $\|e\| = 1$.

If we take the square root in (1.5) and use the triangle inequality, we get for x, y, $z \in H \setminus \{0\}$ that

$$\begin{aligned} &\left(\|x\|^2 \|z\|^2 - |\langle x, z\rangle|^2\right)^{1/2} \left(\|y\|^2 \|z\|^2 - |\langle y, z\rangle|^2\right)^{1/2} \\ &\geq \left|\langle x, y\rangle \|z\|^2 - \langle x, z\rangle \langle z, y\rangle\right| \geq |\langle x, z\rangle \langle z, y\rangle| - |\langle x, y\rangle| \|z\|^2 \end{aligned}$$

which by division with $\|x\|^2 \|y\|^2 \|z\|^2 \neq 0$ produces

$$\frac{|\langle x, y\rangle|}{\|x\| \|y\|} \geq \frac{|\langle x, z\rangle|}{\|x\| \|z\|} \frac{|\langle z, y\rangle|}{\|z\| \|y\|} - \sqrt{1 - \frac{|\langle x, z\rangle|^2}{\|x\|^2 \|z\|^2}} \sqrt{1 - \frac{|\langle y, z\rangle|^2}{\|y\|^2 \|z\|^2}}. \tag{1.7}$$

For other results connected with Schwarz inequality in inner product spaces see the monographs [5] and [6]. For various results related to Cauchy-Bunyakovsky-Schwarz inequality for the n-tuples of numbers see the survey [7] and the monograph [4]. For recent results in connection to Schwarz inequality, see [1], [11] and [13]-[15].

In this paper we obtain some two complex parameters generalizations of the celebrated Schwarz inequality in inner product spaces. Applications for n-tuples of complex numbers are provided.

2. Main Results

We can state the following two complex parameters generalization of Schwarz's inequality:

Theorem 1. *Let x, y, $e \in H$ with $\|e\| = 1$ and α, $\beta \in C$. Then*

$$\begin{aligned} &\left|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle \left(1 - \alpha\overline{\beta}\right)\right|^2 \\ &\leq \left[\|x\|^2 + \left(|\alpha|^2 - 1\right) |\langle x, e\rangle|^2\right] \left[\|y\|^2 + \left(|\beta|^2 - 1\right) |\langle e, y\rangle|^2\right]. \end{aligned} \tag{2.1}$$

Proof. We start with Schwarz inequality for the vectors $x - \gamma e$ and $y - \delta e$ with γ, $\delta \in C$, namely

$$|\langle x - \gamma e, y - \delta e\rangle|^2 \leq \|x - \gamma e\|^2 \|y - \delta e\|^2. \tag{2.2}$$

Observe that

$$\begin{aligned} \|x - \gamma e\|^2 &= \|x\|^2 - 2 \operatorname{Re}(\overline{\gamma} \langle x, e\rangle) + |\gamma|^2 \\ &= \|x\|^2 - |\langle x, e\rangle|^2 + |\langle x, e\rangle|^2 - 2 \operatorname{Re}(\overline{\gamma} \langle x, e\rangle) + |\gamma|^2 \\ &= \|x\|^2 - |\langle x, e\rangle|^2 + |\gamma - \langle x, e\rangle|^2 \end{aligned}$$

and, similarly

$$\|y - \delta e\|^2 = \|y\|^2 - |\langle y, e\rangle|^2 + |\delta - \langle y, e\rangle|^2.$$

We also have

$$\begin{aligned}\langle x-\gamma e, y-\delta e\rangle &= \langle x,y\rangle - \gamma\langle e,y\rangle - \overline{\delta}\langle x,e\rangle + \gamma\overline{\delta} \\ &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle + \gamma\overline{\delta} + \langle x,e\rangle\langle e,y\rangle - \gamma\langle e,y\rangle - \overline{\delta}\langle x,e\rangle \\ &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle + (\gamma - \langle x,e\rangle)\left(\overline{\delta} - \langle e,y\rangle\right) \\ &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle + (\gamma - \langle x,e\rangle)\overline{(\delta - \langle y,e\rangle)}.\end{aligned}$$

Now, if we take

$$\gamma = \langle x,e\rangle + \langle x,e\rangle\alpha$$

and

$$\delta = \langle y,e\rangle + \langle y,e\rangle\beta,$$

then we get

$$\|x-\gamma e\|^2 = \|x\|^2 + \left(|\alpha|^2 - 1\right)|\langle x,e\rangle|^2,$$
$$\|y-\delta e\|^2 = \|y\|^2 + \left(|\beta|^2 - 1\right)|\langle y,e\rangle|^2$$

and

$$\begin{aligned}\langle x-\gamma e, y-\delta e\rangle &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle + \langle x,e\rangle\alpha\overline{(\langle y,e\rangle\beta)} \\ &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle + \langle x,e\rangle\langle e,y\rangle\alpha\overline{\beta} \\ &= \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-\alpha\overline{\beta}\right)\end{aligned}$$

and by (2.2) we get (2.1). □

Remark 1. *In particular, for $\beta = \alpha$ we get*

$$\begin{aligned}&\left|\langle x,y\rangle + \langle x,e\rangle\langle e,y\rangle\left(|\alpha|^2-1\right)\right|^2 \\ &\qquad \le \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]\left[\|y\|^2 + \left(|\alpha|^2-1\right)|\langle y,e\rangle|^2\right]\end{aligned}$$

and by taking $t = |\alpha|^2 - 1 \in [-1,\infty)$ then we get the simpler inequality of interest

$$|\langle x,y\rangle + t\langle x,e\rangle\langle e,y\rangle|^2 \le \left[\|x\|^2 + t|\langle x,e\rangle|^2\right]\left[\|y\|^2 + t|\langle y,e\rangle|^2\right], \tag{2.3}$$

for any $x,\ y,\ e\in H$ with $\|e\| = 1$ and $t\in[-1,\infty)$.

We observe that if $\alpha,\ \beta\in C$ with $|\alpha|,\ |\beta|\le 1$, then by (2.1) we get the family of Schwarz inequalities

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-\alpha\overline{\beta}\right)\right| \le \|x\|\,\|y\|, \tag{2.4}$$

where $x,\ y,\ e\in H$ with $\|e\| = 1$.

If we take in (2.4) $\alpha = \beta = 1$, then we get the classical Schwarz inequality, while for $\alpha = 1$, $\beta = -1$ we get

$$|\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \le \|x\|\,\|y\|. \tag{2.5}$$

By the triangle inequality we have

$$2|\langle x,e\rangle\langle e,y\rangle| - |\langle x,y\rangle| \le |\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle| \tag{2.6}$$

and then by (2.5) and (2.6) we get the celebrated Buzano's inequality

$$|\langle x,e\rangle\langle e,y\rangle| \le \frac{1}{2}\left[|\langle x,y\rangle| + \|x\|\,\|y\|\right] \tag{2.7}$$

where $x,\ y,\ e\in H$ with $\|e\| = 1$.

We observe that if $|\alpha|, |\beta| \geq 1$ then by (2.1) we also get

$$\left|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle \left(1-\alpha\overline{\beta}\right)\right| \leq |\alpha| |\beta| \|x\| \|y\| \tag{2.8}$$

where $x, y, e \in H$ with $\|e\| = 1$.

Corollary 1. *With the assumptions of Theorem 1 we have*

$$\begin{aligned} |\langle x,y\rangle| &\leq |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right| \\ &+ \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + \left(|\beta|^2-1\right)|\langle y,e\rangle|^2\right]^{1/2} \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right| &\leq |\langle x,y\rangle| \\ &+ \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + \left(|\beta|^2-1\right)|\langle y,e\rangle|^2\right]^{1/2}. \end{aligned} \tag{2.10}$$

Proof. By taking the square root in (2.1) we get

$$\begin{aligned} &\left|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle \left(1-\alpha\overline{\beta}\right)\right| \\ &\leq \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + \left(|\beta|^2-1\right)|\langle y,e\rangle|^2\right]^{1/2}. \end{aligned} \tag{2.11}$$

By the continuity of the modulus inequality we also have

$$\left||\langle x,y\rangle| - |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right|\right| \leq \left|\langle x,y\rangle - \langle x,e\rangle \langle e,y\rangle \left(1-\alpha\overline{\beta}\right)\right|. \tag{2.12}$$

Since, obviously

$$|\langle x,y\rangle| - |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right| \leq \left||\langle x,y\rangle| - |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right|\right|,$$

then by (2.11) and (2.12) we get (2.9).

Also, since

$$|\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right| - |\langle x,y\rangle| \leq \left||\langle x,y\rangle| - |\langle x,e\rangle \langle e,y\rangle| \left|1-\alpha\overline{\beta}\right|\right|$$

then by (2.11) and (2.12) we get (2.10). □

Remark 2. *In particular, if we take $\beta = \alpha$ above, we get*

$$\begin{aligned} |\langle x,y\rangle| &\leq |\langle x,e\rangle \langle e,y\rangle| \left||\alpha|^2-1\right| \\ &+ \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + \left(|\alpha|^2-1\right)|\langle y,e\rangle|^2\right]^{1/2} \end{aligned}$$

and

$$\begin{aligned} |\langle x,e\rangle \langle e,y\rangle| \left||\alpha|^2-1\right| &\leq |\langle x,y\rangle| \\ &+ \left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + \left(|\alpha|^2-1\right)|\langle y,e\rangle|^2\right]^{1/2} \end{aligned}$$

and by taking $t = |\alpha|^2 - 1 \in [-1, \infty)$ then we get the simpler inequalities of interest

$$|\langle x,y\rangle| \leq |\langle x,e\rangle \langle e,y\rangle| |t| + \left[\|x\|^2 + t|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + t|\langle y,e\rangle|^2\right]^{1/2} \tag{2.13}$$

and

$$|\langle x,e\rangle \langle e,y\rangle| |t| \leq |\langle x,y\rangle| + \left[\|x\|^2 + t|\langle x,e\rangle|^2\right]^{1/2} \left[\|y\|^2 + t|\langle y,e\rangle|^2\right]^{1/2}, \tag{2.14}$$

where $x, y, e \in H$ with $\|e\| = 1$ and $t \in [-1, \infty)$.

Corollary 2. *Let x, y, $e \in H$ with $\|e\| = 1$ and α, $\beta \in C$ with $|\alpha| \le 1$ and $|\beta| \le 1$. Then we have*

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left[\left(1-\alpha\overline{\beta}\right) \pm \left(1-|\alpha|^2\right)^{1/2}\left(1-|\beta|^2\right)^{1/2}\right]\right| \le \left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-\alpha\overline{\beta}\right)\right| + |\langle x,e\rangle\langle e,y\rangle|\left(1-|\alpha|^2\right)^{1/2}\left(1-|\beta|^2\right)^{1/2} \le \|x\|\,\|y\|. \tag{2.15}$$

Proof. The first inequality follows by the triangle inequality for modulus.

We use the following elementary inequality

$$\left(a^2-b^2\right)\left(c^2-d^2\right) \le (ac-bd)^2$$

for any real numbers a, b, c, d.

Therefore

$$\begin{aligned}&\left[\|x\|^2 + \left(|\alpha|^2-1\right)|\langle x,e\rangle|^2\right]\left[\|y\|^2 + \left(|\beta|^2-1\right)|\langle y,e\rangle|^2\right]\\ &= \left[\|x\|^2 - \left(1-|\alpha|^2\right)|\langle x,e\rangle|^2\right]\left[\|y\|^2 - \left(1-|\beta|^2\right)|\langle y,e\rangle|^2\right]\\ &\le \left[\|x\|\,\|y\| - |\langle x,e\rangle\langle e,y\rangle|\left(1-|\alpha|^2\right)^{1/2}\left(1-|\beta|^2\right)^{1/2}\right]^2.\end{aligned}$$

Since by Schwarz's inequality and the fact that $|\alpha| \le 1$ and $|\beta| \le 1$, we have

$$\|x\| \ge |\langle x,e\rangle| \ge |\langle x,e\rangle|\left(1-|\alpha|^2\right)^{1/2}$$

and

$$\|y\| \ge |\langle e,y\rangle| \ge \left(1-|\beta|^2\right)^{1/2}|\langle e,y\rangle|$$

implying that

$$\left[\|x\|^2 - \left(1-|\alpha|^2\right)|\langle x,e\rangle|^2\right]^{1/2}\left[\|y\|^2 - \left(1-|\beta|^2\right)|\langle y,e\rangle|^2\right]^{1/2} \le \|x\|\,\|y\| - |\langle x,e\rangle\langle e,y\rangle|\left(1-|\alpha|^2\right)^{1/2}\left(1-|\beta|^2\right)^{1/2}. \tag{2.16}$$

From (2.1) we also have

$$\left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-\alpha\overline{\beta}\right)\right| \le \left[\|x\|^2 - \left(1-|\alpha|^2\right)|\langle x,e\rangle|^2\right]^{1/2}\left[\|y\|^2 - \left(1-|\beta|^2\right)|\langle y,e\rangle|^2\right]^{1/2}. \tag{2.17}$$

Now, by making use of (2.16) and (2.17) we deduce the second inequality in (2.15). □

Remark 3. *If we take $\beta = \alpha$ above, then we get*

$$\left|\langle x,y\rangle - 2\langle x,e\rangle\langle e,y\rangle\left(1-|\alpha|^2\right)\right| \le \left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-|\alpha|^2\right)\right| + |\langle x,e\rangle\langle e,y\rangle|\left(1-|\alpha|^2\right) \le \|x\|\,\|y\|$$

and

$$|\langle x,y\rangle| \le \left|\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle\left(1-|\alpha|^2\right)\right| + |\langle x,e\rangle\langle e,y\rangle|\left(1-|\alpha|^2\right) \le \|x\|\,\|y\|,$$

and by taking $s=1-|\alpha|^2\in[0,1]$ *we get the following inequalities of interest*

$$|\langle x,y\rangle-2s\langle x,e\rangle\langle e,y\rangle|\le|\langle x,y\rangle-s\langle x,e\rangle\langle e,y\rangle|+s|\langle x,e\rangle\langle e,y\rangle|\le\|x\|\|y\| \tag{2.18}$$

and

$$|\langle x,y\rangle|\le|\langle x,y\rangle-s\langle x,e\rangle\langle e,y\rangle|+s|\langle x,e\rangle\langle e,y\rangle|\le\|x\|\|y\|, \tag{2.19}$$

where $x,\ y,\ e\in H$ *with* $\|e\|=1$ *and* $s\in[0,1]$.

If we take $s=\frac{1}{2}$ *in (2.18) and (2.19), then we get*

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\le\left|\langle x,y\rangle-\frac{1}{2}\langle x,e\rangle\langle e,y\rangle\right|+\frac{1}{2}|\langle x,e\rangle\langle e,y\rangle|\le\|x\|\|y\| \tag{2.20}$$

and

$$|\langle x,y\rangle|\le\left|\langle x,y\rangle-\frac{1}{2}\langle x,e\rangle\langle e,y\rangle\right|+\frac{1}{2}|\langle x,e\rangle\langle e,y\rangle|\le\|x\|\|y\|, \tag{2.21}$$

where $x,\ y,\ e\in H$ *with* $\|e\|=1$.

If we take $s=1$ *in (2.19) then we recapture the inequality (1.6) from the introduction.*

3. Applications for n-Tuples of Complex Numbers

Let $x=(x_1,...,x_n)\,,\ y=(y_1,...,y_n)\,,\ e=(e_1,...,e_n)\in\mathbb{C}^n$ with $\sum_{k=1}^n|e_k|^2=1$. Then by writing the above inequality (2.1) for the inner product $\langle x,y\rangle:=\sum_{k=1}^n x_k\overline{y}_k$ we get

$$\begin{aligned}\left|\sum_{k=1}^n x_k\overline{y}_k-\left(1-\alpha\overline{\beta}\right)\sum_{k=1}^n x_k\overline{e}_k\sum_{k=1}^n e_k\overline{y}_k\right|^2 \\ \le\left[\sum_{k=1}^n|x_k|^2+\left(|\alpha|^2-1\right)\left|\sum_{k=1}^n x_k\overline{e}_k\right|^2\right]\left[\sum_{k=1}^n|y_k|^2+\left(|\beta|^2-1\right)\left|\sum_{k=1}^n e_k\overline{y}_k\right|^2\right],\end{aligned} \tag{3.1}$$

for any $\alpha,\ \beta\in\mathbb{C}$.

If we take $e_m=1$ for $m\in\{1,...,n\}$ and $e_k=0$ for any $k\in\{1,...,n\}\,,\ k\neq m$, then $\sum_{k=1}^n|e_k|^2=1$ and by (3.1) we get

$$\begin{aligned}\left|\sum_{k=1}^n x_k\overline{y}_k-\left(1-\alpha\overline{\beta}\right)x_m\overline{y}_m\right|^2 \\ \le\left[\sum_{k=1}^n|x_k|^2+\left(|\alpha|^2-1\right)|x_m|^2\right]\left[\sum_{k=1}^n|y_k|^2+\left(|\beta|^2-1\right)|y_m|^2\right],\end{aligned} \tag{3.2}$$

for any $\alpha,\ \beta\in\mathbb{C}$.

If we take $e_k=\frac{1}{\sqrt{n}}$ for $k\in\{1,...,n\}$, then $\sum_{k=1}^n|e_k|^2=1$ and by (3.1) we get

$$\begin{aligned}\left|\frac{1}{n}\sum_{k=1}^n x_k\overline{y}_k-\left(1-\alpha\overline{\beta}\right)\frac{1}{n}\sum_{k=1}^n x_k\frac{1}{n}\sum_{k=1}^n\overline{y}_k\right|^2 \\ \le\left[\frac{1}{n}\sum_{k=1}^n|x_k|^2+\left(|\alpha|^2-1\right)\left|\frac{1}{n}\sum_{k=1}^n x_k\right|^2\right]\left[\frac{1}{n}\sum_{k=1}^n|y_k|^2+\left(|\beta|^2-1\right)\left|\frac{1}{n}\sum_{k=1}^n\overline{y}_k\right|^2\right],\end{aligned} \tag{3.3}$$

for any $\alpha,\ \beta\in\mathbb{C}$.

From (3.1) we have the family of Cauchy-Bunyakovsky-Schwarz inequalities

$$\left|\sum_{k=1}^{n} x_k\overline{y}_k - \left(1-\alpha\overline{\beta}\right)\sum_{k=1}^{n} x_k\overline{e}_k \sum_{k=1}^{n} e_k\overline{y}_k\right|^2 \leq \sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2, \tag{3.4}$$

for any $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|,\ |\beta|\leq 1$.

From (3.4) we get

$$\max_{m\in\{1,\ldots,n\}}\left|\sum_{k=1}^{n} x_k\overline{y}_k - \left(1-\alpha\overline{\beta}\right)x_m\overline{y}_m\right|^2 \leq \sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2, \tag{3.5}$$

and

$$\left|\frac{1}{n}\sum_{k=1}^{n} x_k\overline{y}_k - \left(1-\alpha\overline{\beta}\right)\frac{1}{n}\sum_{k=1}^{n} x_k\frac{1}{n}\sum_{k=1}^{n}\overline{y}_k\right|^2 \leq \frac{1}{n}\sum_{k=1}^{n}|x_k|^2\frac{1}{n}\sum_{k=1}^{n}|y_k|^2, \tag{3.6}$$

for any $\alpha,\ \beta\in\mathbb{C}$ with $|\alpha|,\ |\beta|\leq 1$.

If, for instance we take $\alpha=\beta=\frac{\sqrt{2}}{2}$ in (3.4), then we get

$$\left|\sum_{k=1}^{n} x_k\overline{y}_k - \frac{1}{2}\sum_{k=1}^{n} x_k\overline{e}_k \sum_{k=1}^{n} e_k\overline{y}_k\right|^2 \leq \sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2, \tag{3.7}$$

while for $\alpha=-\beta=\frac{\sqrt{2}}{2}$, then we get

$$\left|\sum_{k=1}^{n} x_k\overline{y}_k - \frac{3}{2}\sum_{k=1}^{n} x_k\overline{e}_k \sum_{k=1}^{n} e_k\overline{y}_k\right|^2 \leq \sum_{k=1}^{n}|x_k|^2\sum_{k=1}^{n}|y_k|^2, \tag{3.8}$$

for $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$, $e=(e_1,\ldots,e_n)\in\mathbb{C}^n$ with $\sum_{k=1}^{n}|e_k|^2=1$.

From the inequality (2.19) we get the following refinement of Cauchy-Bunyakovsky-Schwarz inequality

$$\begin{aligned}\left|\sum_{k=1}^{n} x_k\overline{y}_k\right| &\leq \left|\sum_{k=1}^{n} x_k\overline{y}_k - s\sum_{k=1}^{n} x_k\overline{e}_k \sum_{k=1}^{n} e_k\overline{y}_k\right| + s\left|\sum_{k=1}^{n} x_k\overline{e}_k \sum_{k=1}^{n} e_k\overline{y}_k\right| \\ &\leq \left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2},\end{aligned} \tag{3.9}$$

for any $s\in[0,1]$, where $x=(x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)$, $e=(e_1,\ldots,e_n)\in\mathbb{C}^n$ with $\sum_{k=1}^{n}|e_k|^2=1$.

This implies the following inequalities

$$\begin{aligned}\left|\sum_{k=1}^{n} x_k\overline{y}_k\right| &\leq \left|\sum_{k=1}^{n} x_k\overline{y}_k - s x_m\overline{y}_m\right| + s\left|x_m\overline{y}_m\right| \\ &\leq \left(\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\sum_{k=1}^{n}|y_k|^2\right)^{1/2},\end{aligned} \tag{3.10}$$

and

$$\left|\frac{1}{n}\sum_{k=1}^{n}x_k\overline{y}_k\right| \leq \left|\frac{1}{n}\sum_{k=1}^{n}x_k\overline{y}_k - s\frac{1}{n}\sum_{k=1}^{n}x_k\frac{1}{n}\sum_{k=1}^{n}\overline{y}_k\right| + s\left|\frac{1}{n}\sum_{k=1}^{n}x_k\frac{1}{n}\sum_{k=1}^{n}\overline{y}_k\right| \leq \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|^2\right)^{1/2}\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|^2\right)^{1/2}, \tag{3.11}$$

for any $s \in [0,1]$, where $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{C}^n$.

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