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REFINEMENTS OF SCHWARZ INEQUALITY IN INNER PRODUCT SPACES WITH APPLICATIONS TO INTEGRALS

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Abstract. Some refinements of the celebrated Schwarz inequality in complex inner product spaces are given. Applications for integrals of complex-valued functions are provided as well.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex numbers field $\mathbb{K}$. The following inequality is well known in literature as the *Schwarz inequality* in quadratic form

$$\|x\|^2 \|y\|^2 \geq |\langle x, y \rangle|^2 \tag{1.1}$$

for any x, $y \in H$. The equality case holds in (1.1) if and only if there exists a constant $\lambda \in \mathbb{K}$ such that $x = \lambda y$. This inequality can be written in an equivalent form as

$$\|x\| \|y\| \geq |\langle x, y \rangle| . \tag{1.2}$$

For various inequalities in inner product spaces related to this famous result see the monographs [3] and [4]. For recent results related to Schwarz inequality in various settings, see [1] and [9]-[14].

In 1985 the author [2] (see also [8] or [4, p. 36]) established the following inequality related to Schwarz inequality

$$\left(\|x\|^2 \|z\|^2 - |\langle x, z \rangle|^2\right) \left(\|y\|^2 \|z\|^2 - |\langle y, z \rangle|^2\right) \geq \left|\langle x, y \rangle \|z\|^2 - \langle x, z \rangle \langle z, y \rangle\right|^2 \tag{1.3}$$

for any x, y, $z \in H$ and obtained, as a consequence, the following refinement of (1.2):

$$\|x\| \|y\| \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| + |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| \tag{1.4}$$

for any x, y, $e \in H$ with $\|e\| = 1$.

Since, by the triangle inequality for modulus we have

$$|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle| \geq |\langle x, e \rangle \langle e, y \rangle| - |\langle x, y \rangle| ,$$

then by the first inequality in (1.4) we also get the celebrated *Buzano's inequality*

$$\frac{1}{2} (\|x\| \|y\| + |\langle x, y \rangle|) \geq |\langle x, e \rangle \langle e, y \rangle| \tag{1.5}$$

for any x, y, $e \in H$ with $\|e\| = 1$.

In the recent paper [6] we obtained the following generalization of (1.4) as follows

$$\|x\| \|y\| \geq |\langle x, y \rangle - s \langle x, e \rangle \langle e, y \rangle| + s |\langle x, e \rangle \langle e, y \rangle| \geq |\langle x, y \rangle| , \tag{1.6}$$

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where $x, y, e \in H$ with $\|e\| = 1$ and $s \in [0,1]$. This inequality gives for $s = 1$ the inequality (1.4) while for $s = \frac{1}{2}$ we get

$$\|x\| \|y\| \geq \left| \langle x, y \rangle - \frac{1}{2} \langle x, e \rangle \langle e, y \rangle \right| + \frac{1}{2} \left| \langle x, e \rangle \langle e, y \rangle \right| \geq \left| \langle x, y \rangle \right|, \tag{1.7}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

In [5] we obtained the following two refinements of Schwarz inequality in the quadratic form

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq \left(\det \begin{bmatrix} \|x\| & \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \\ \|y\| & \left(\|y\|^2 - |\langle y, e \rangle|^2 \right)^{1/2} \end{bmatrix} \right)^2 \tag{1.8}$$

and

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq \left(\det \begin{bmatrix} |\langle x, e \rangle| & \left(\|x\|^2 - |\langle x, e \rangle|^2 \right)^{1/2} \\ |\langle y, e \rangle| & \left(\|y\|^2 - |\langle y, e \rangle|^2 \right)^{1/2} \end{bmatrix} \right)^2 \tag{1.9}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

We notice that the above inequalities also hold for $\operatorname{Re} \langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle$, however the details are not presented here.

Motivated by the above results, in this paper we establish some new refinements of the celebrated Schwarz inequality in complex inner product. Applications for integrals of complex-valued functions are provided as well.

2. The Main Results

We have:

Theorem 1. *For any $x, y, e \in H$ with $\|e\| = 1$, we have*

$$\|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq \left(\det \begin{bmatrix} \|x\| & |\langle x, e \rangle| \\ \|y\| & |\langle y, e \rangle| \end{bmatrix} \right)^2. \tag{2.1}$$

Proof. By Schwarz's inequality we have

$$\|x - \langle x, e \rangle e\|^2 \|y - \langle y, e \rangle e\|^2 \geq |\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle|^2 \tag{2.2}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

Since

$$\|x - \langle x, e \rangle e\|^2 = \|x\|^2 - |\langle x, e \rangle|^2, \quad \|y - \langle y, e \rangle e\|^2 = \|y\|^2 - |\langle y, e \rangle|^2$$

and

$$\langle x - \langle x, e \rangle e, y - \langle y, e \rangle e \rangle = \langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle$$

for any $x, y, e \in H$ with $\|e\| = 1$, then by (2.2) we get

$$\left(\|x\|^2 - |\langle x, e \rangle|^2 \right) \left(\|y\|^2 - |\langle y, e \rangle|^2 \right) \geq |\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle|^2 \tag{2.3}$$

for any $x, y, e \in H$ with $\|e\| = 1$.

By performing the required calculations, the inequality (2.3) can be written as

$$\begin{aligned}&\|x\|^2\|y\|^2-|\langle x,e\rangle|^2\|y\|^2-\|x\|^2|\langle y,e\rangle|^2+|\langle x,e\rangle|^2|\langle y,e\rangle|^2\\&\ge|\langle x,y\rangle|^2-2\operatorname{Re}\left(\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right)+|\langle x,e\rangle\langle e,y\rangle|^2\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\\&\ge|\langle x,e\rangle|^2\|y\|^2+\|x\|^2|\langle y,e\rangle|^2-2\operatorname{Re}\left(\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right)\\&=|\langle x,e\rangle|^2\|y\|^2+\|x\|^2|\langle y,e\rangle|^2-2\|x\|\,|\langle x,e\rangle|\,\|y\|\,|\langle y,e\rangle|\\&+2\left(\|x\|\,\|y\|\,|\langle x,e\rangle\langle e,y\rangle|-\operatorname{Re}\left(\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right)\right)\\&=\left(\det\begin{bmatrix}\|x\| & |\langle x,e\rangle|\\ \|y\| & |\langle y,e\rangle|\end{bmatrix}\right)^2\\&+2\left(\|x\|\,\|y\|\,|\langle x,e\rangle\langle e,y\rangle|-\operatorname{Re}\left(\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right)\right)\end{aligned}\tag{2.4}$$

for any $x,\ y,\ e\in H$ with $\|e\|=1$.

By the properties of complex numbers we have

$$\operatorname{Re}\left(\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right)\le\left|\langle x,y\rangle\overline{\langle x,e\rangle\langle e,y\rangle}\right|=|\langle x,y\rangle|\,|\langle x,e\rangle\langle e,y\rangle|$$

and then by (2.4) we get the following inequality of interest:

$$\begin{aligned}&\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\\&\ge\left(\det\begin{bmatrix}\|x\| & |\langle x,e\rangle|\\ \|y\| & |\langle y,e\rangle|\end{bmatrix}\right)^2+2\,|\langle x,e\rangle\langle e,y\rangle|\,(\|x\|\,\|y\|-|\langle x,y\rangle|)\end{aligned}\tag{2.5}$$

for any $x,\ y,\ e\in H$ with $\|e\|=1$.

Since, by Schwarz inequality we have $\|x\|\,\|y\|-|\langle x,y\rangle|\ge 0$, then by (2.5) we obtain the desired result (2.1). □

Remark 1. *Since*

$$\begin{aligned}&\|x\|^2\|y\|^2-|\langle x,y\rangle|^2-2\,|\langle x,e\rangle\langle e,y\rangle|\,(\|x\|\,\|y\|-|\langle x,y\rangle|)\\&=2\,(\|x\|\,\|y\|-|\langle x,y\rangle|)\left[\frac{1}{2}(\|x\|\,\|y\|+|\langle x,y\rangle|)-|\langle x,e\rangle\langle e,y\rangle|\right],\end{aligned}$$

hence from (2.5) we get the "Schwarz-Buzano product" type inequality

$$\begin{aligned}&(\|x\|\,\|y\|-|\langle x,y\rangle|)\left[\frac{1}{2}(\|x\|\,\|y\|+|\langle x,y\rangle|)-|\langle x,e\rangle\langle e,y\rangle|\right]\\&\ge\frac{1}{2}\left(\det\begin{bmatrix}\|x\| & |\langle x,e\rangle|\\ \|y\| & |\langle y,e\rangle|\end{bmatrix}\right)^2\end{aligned}\tag{2.6}$$

for any $x,\ y,\ e\in H$ with $\|e\|=1$.

A family $\{e_j\}_{j\in J}$ of vectors in H is called *orthonormal* if

$$e_j \perp e_k \text{ for any } j,k\in J \text{ with } j\neq k \text{ and } \|e_j\|=1 \text{ for any } j,k\in J.$$

If the *linear span* of the family $\{e_j\}_{j\in J}$ is *dense* in H, then we call it an *orthonormal basis* in H.

In [7], Dragomir and Mond (see also [4, p. 17]) obtained the following refinement of Schwarz inequality:

Lemma 1. *Let $\{e_j\}_{j\in J}$ be an orthonormal family in H. Then*

$$\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq\sum_{j\in J}\left|\langle x,e\rangle_j\right|^2\sum_{j\in J}\left|\langle y,e\rangle_j\right|^2-\left|\sum_{j\in J}\langle x,e\rangle_j\langle e,y\rangle_j\right|^2\geq 0 \tag{2.7}$$

for any $x,\ y\in H$.

We have:

Theorem 2. *Let $\{e_j\}_{j\in\{1,\dots,n\}}$ be an orthonormal family of n vectors in H, then*

$$\begin{aligned}\|x\|^2\|y\|^2-|\langle x,y\rangle|^2 &\geq\sum_{1\leq k<j\leq n}\left|\det\begin{bmatrix}\langle x,e_j\rangle & \langle x,e_k\rangle\\ \langle y,e_j\rangle & \langle y,e_k\rangle\end{bmatrix}\right|^2\\ &\geq\sum_{1\leq k<j\leq n}\left(\det\begin{bmatrix}|\langle x,e_j\rangle| & |\langle x,e_k\rangle|\\ |\langle y,e_j\rangle| & |\langle y,e_k\rangle|\end{bmatrix}\right)^2.\end{aligned}\tag{2.8}$$

Proof. Using the symmetry of the indices in the double sum, we have

$$\sum_{1\leq k<j\leq n}\left|\det\begin{bmatrix}\langle x,e_j\rangle & \langle x,e_k\rangle\\ \langle y,e_j\rangle & \langle y,e_k\rangle\end{bmatrix}\right|^2=\frac{1}{2}\sum_{k,j=1}^{n}\left|\det\begin{bmatrix}\langle x,e_j\rangle & \langle x,e_k\rangle\\ \langle y,e_j\rangle & \langle y,e_k\rangle\end{bmatrix}\right|^2$$

for any $x,\ y\in H$.

Also

$$\begin{aligned}&\sum_{k,j=1}^{n}\left|\det\begin{bmatrix}\langle x,e_j\rangle & \langle x,e_k\rangle\\ \langle y,e_j\rangle & \langle y,e_k\rangle\end{bmatrix}\right|^2\\ &=\sum_{k,j=1}^{n}|\langle x,e_j\rangle\langle y,e_k\rangle-\langle x,e_k\rangle\langle y,e_j\rangle|^2\\ &=\sum_{k,j=1}^{n}\Big[|\langle x,e_j\rangle|^2|\langle y,e_k\rangle|^2-2\operatorname{Re}\left(\langle x,e_j\rangle\langle y,e_k\rangle\overline{\langle x,e_k\rangle\langle y,e_j\rangle}\right)\\ &+|\langle x,e_k\rangle|^2|\langle y,e_j\rangle|^2\Big]\\ &=\sum_{k,j=1}^{n}|\langle x,e_j\rangle|^2|\langle y,e_k\rangle|^2-2\sum_{k,j=1}^{n}\operatorname{Re}\left(\langle x,e_j\rangle\langle e_j,y\rangle\overline{\langle x,e_k\rangle\langle e_k,y\rangle}\right)\\ &+\sum_{k,j=1}^{n}|\langle x,e_k\rangle|^2|\langle y,e_j\rangle|^2\end{aligned}$$

$$= \sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \sum_{k=1}^{n} |\langle y, e_k\rangle|^2 - 2\operatorname{Re}\left(\sum_{j=1}^{n} \langle x, e_j\rangle \langle e_j, y\rangle \sum_{k=1}^{n} \overline{\langle x, e_k\rangle \langle e_k, y\rangle}\right)$$
$$+ \sum_{k=1}^{n} |\langle x, e_k\rangle|^2 \sum_{j=1}^{n} |\langle y, e_j\rangle|^2$$
$$= 2\left[\sum_{j=1}^{n} |\langle x, e_j\rangle|^2 \sum_{j=1}^{n} |\langle y, e_j\rangle|^2 - \left|\sum_{j=1}^{n} \langle x, e_j\rangle \langle e_j, y\rangle\right|^2\right]$$

for any $x,\ y \in H$.

By using the inequality (2.7) for $J = \{1, ..., n\}$, we get the first inequality in (2.8).

By the continuity property of modulus, $|u - v| \geq ||u| - |v||$ for $u,\ v \in \mathbb{C}$, we have

$$|\langle x, e_j\rangle \langle y, e_k\rangle - \langle x, e_k\rangle \langle y, e_j\rangle| \geq ||\langle x, e_j\rangle| |\langle y, e_k\rangle| - |\langle x, e_k\rangle| |\langle y, e_j\rangle||$$

for any $j,\ k \in \{1, ..., n\}$, which proves the second inequality in (2.8). □

The following particular case is of interest since it provides a simple refinement of Schwarz inequality in terms of a pair of orthonormal vectors.

Corollary 1. *Let $e,\ f \in H$ with $e \perp f$ and $\|e\| = \|f\| = 1$. Then for any $x,\ y \in H$ we have*

$$\|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \geq \left|\det \begin{bmatrix} \langle x, e\rangle & \langle x, f\rangle \\ \langle y, e\rangle & \langle y, f\rangle \end{bmatrix}\right|^2 \geq \left(\det \begin{bmatrix} |\langle x, e\rangle| & |\langle x, f\rangle| \\ |\langle y, e\rangle| & |\langle y, f\rangle| \end{bmatrix}\right)^2. \tag{2.9}$$

3. Applications for Integrals

Let (S, Σ, μ) be a positive measure space and $L_2(S, \Sigma, \mu)$, the Hilbert space of complex-valued 2-μ-integrable functions defined on S, namely $z \in L_2(S, \Sigma, \mu)$ if and only if $\|z\|_2 := \left(\int_S |z(t)|^2 d\mu(t)\right)^{1/2} < \infty$. If $x,\ y \in L_2(S, \Sigma, \mu)$ then the inner product $\langle x, y\rangle_2 := \int_S x(t)\overline{y(t)} d\mu(t)$ generates the norm $\|\cdot\|_2$.

If $x,\ y,\ e \in L_2(S, \Sigma, \mu)$ with $\left(\int_S |e(t)|^2 d\mu(t)\right)^{1/2} = 1$, then by (1.6), (1.8), (1.9) and (2.1) we have

$$\begin{aligned} &\left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} \\ &\geq \left|\int_S x(t)\overline{y(t)} d\mu(t) - s\int_S x(t)\overline{e(t)} d\mu(t) \int_S e(t)\overline{y(t)} d\mu(t)\right| \\ &\quad + s\left|\int_S x(t)\overline{e(t)} d\mu(t) \int_S e(t)\overline{y(t)} d\mu(t)\right| \\ &\geq \left|\int_S x(t)\overline{y(t)} d\mu(t)\right|, \text{ for } s \in [0, 1], \end{aligned} \tag{3.1}$$

$$\int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \tag{3.2}$$
$$\geq \left(\det \begin{bmatrix} \left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left(\int_S |x(t)|^2 d\mu(t) - \left|\int_S x(t) \overline{e(t)} d\mu(t)\right|^2\right)^{1/2} \\ \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left(\int_S |y(t)|^2 d\mu(t) - \left|\int_S y(t) \overline{e(t)} d\mu(t)\right|^2\right)^{1/2} \end{bmatrix} \right)^2,$$

$$\int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \tag{3.3}$$
$$\geq \left(\det \begin{bmatrix} \left|\int_S x(t) \overline{e(t)} d\mu(t)\right| & \left(\int_S |x(t)|^2 d\mu(t) - \left|\int_S x(t) \overline{e(t)} d\mu(t)\right|^2\right)^{1/2} \\ \left|\int_S y(t) \overline{e(t)} d\mu(t)\right| & \left(\int_S |y(t)|^2 d\mu(t) - \left|\int_S y(t) \overline{e(t)} d\mu(t)\right|^2\right)^{1/2} \end{bmatrix} \right)^2$$

and

$$\int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \tag{3.4}$$
$$\geq \left(\det \begin{bmatrix} \left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left|\int_S x(t) \overline{e(t)} d\mu(t)\right| \\ \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left|\int_S y(t) \overline{e(t)} d\mu(t)\right| \end{bmatrix} \right)^2.$$

If x, y, e, $f \in L_2(S, \Sigma, \mu)$ with $\left(\int_S |e(t)|^2 d\mu(t)\right)^{1/2} = \left(\int_S |f(t)|^2 d\mu(t)\right)^{1/2} = 1$ and $\int_S e(t) \overline{f(t)} d\mu(t) = 0$, then by (2.9) we get

$$\int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \tag{3.5}$$
$$\geq \left| \det \begin{bmatrix} \int_S x(t) \overline{e(t)} d\mu(t) & \int_S x(t) \overline{f(t)} d\mu(t) \\ \int_S y(t) \overline{e(t)} d\mu(t) & \int_S y(t) \overline{f(t)} d\mu(t) \end{bmatrix} \right|^2$$
$$\geq \left(\det \begin{bmatrix} \left|\int_S x(t) \overline{e(t)} d\mu(t)\right| & \left|\int_S x(t) \overline{f(t)} d\mu(t)\right| \\ \left|\int_S y(t) \overline{e(t)} d\mu(t)\right| & \left|\int_S y(t) \overline{f(t)} d\mu(t)\right| \end{bmatrix} \right)^2.$$

Let E be a μ-measurable subset of S with $0 < \mu(E) < \infty$. We define the function $e : S \to \mathbb{R}$ by

$$e(t) := \begin{cases} \frac{1}{[\mu(E)]^{1/2}} \text{ if } t \in E, \\ 0 \text{ if } t \in S \setminus E. \end{cases}$$

Then $e \in L_2(S, \Sigma, \mu)$ and

$$\int_S |e(t)|^2 d\mu(t) = \frac{1}{\mu(E)} \int_E 1 d\mu(t) = 1.$$

We also have

$$\int_S x(t) \overline{e(t)} d\mu(t) = \frac{1}{[\mu(E)]^{1/2}} \int_E x(t) d\mu(t)$$

and

$$\int_S e(t) \overline{y(t)} d\mu(t) = \frac{1}{[\mu(E)]^{1/2}} \int_E \overline{y(t)} d\mu(t)$$

for any $x, y \in L_2(S, \Sigma, \mu)$.

From the inequalities (3.1)-(3.4) we get the inequalities

$$\begin{aligned}(3.6)\quad & \left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} \\ & \geq \left| \int_S x(t) \overline{y(t)} d\mu(t) - s\frac{1}{\mu(E)} \int_E x(t) d\mu(t) \int_E \overline{y(t)} d\mu(t) \right| \\ & \quad + s\frac{1}{\mu(E)} \left| \int_E x(t) d\mu(t) \int_E \overline{y(t)} d\mu(t) \right| \\ & \geq \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|, \text{ for } s \in [0,1],\end{aligned}$$

$$\begin{aligned}(3.7)\quad & \int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \\ & \geq \left(\det \begin{bmatrix} \left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left(\int_S |x(t)|^2 d\mu(t) - \frac{1}{\mu(E)} \left|\int_E x(t) d\mu(t)\right|^2\right)^{1/2} \\ \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left(\int_S |y(t)|^2 d\mu(t) - \frac{1}{\mu(E)} \left|\int_E y(t) d\mu(t)\right|^2\right)^{1/2} \end{bmatrix} \right)^2,\end{aligned}$$

$$\begin{aligned}(3.8)\quad & \int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \\ & \geq \frac{1}{\mu(E)} \left(\det \begin{bmatrix} \left|\int_E x(t) d\mu(t)\right| & \left(\int_S |x(t)|^2 d\mu(t) - \frac{1}{\mu(E)} \left|\int_E x(t) d\mu(t)\right|^2\right)^{1/2} \\ \left|\int_E y(t) d\mu(t)\right| & \left(\int_S |y(t)|^2 d\mu(t) - \frac{1}{\mu(E)} \left|\int_E y(t) d\mu(t)\right|^2\right)^{1/2} \end{bmatrix} \right)^2\end{aligned}$$

and

$$\begin{aligned}(3.9)\quad & \int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \\ & \geq \frac{1}{\mu(E)} \left(\det \begin{bmatrix} \left(\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left|\int_E x(t) d\mu(t)\right| \\ \left(\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left|\int_E y(t) d\mu(t)\right| \end{bmatrix} \right)^2.\end{aligned}$$

If $\mu(S)<\infty$ then by taking $E=S$ in (3.6)-(3.9), we get the following inequalities for integral means

$$\left(\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\right)^{1/2}\left(\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t)\right)^{1/2} \tag{3.10}$$
$$\geq \left|\frac{1}{\mu(S)}\int_S x(t)\overline{y(t)}d\mu(t) - s\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\,\frac{1}{\mu(S)}\int_S \overline{y(t)}d\mu(t)\right|$$
$$+ s\left|\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\,\frac{1}{\mu(S)}\int_S \overline{y(t)}d\mu(t)\right|$$
$$\geq \left|\frac{1}{\mu(S)}\int_S x(t)\overline{y(t)}d\mu(t)\right|,\ \text{ for } s\in[0,1]$$

$$\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\,\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S x(t)\overline{y(t)}d\mu(t)\right|^2 \tag{3.11}$$
$$\geq\left(\det\begin{bmatrix} \left(\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left(\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\right|^2\right)^{1/2} \\ \left(\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left(\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S y(t)\,d\mu(t)\right|^2\right)^{1/2}\end{bmatrix}\right)^2,$$

$$\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\,\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S x(t)\overline{y(t)}d\mu(t)\right|^2 \tag{3.12}$$
$$\geq\left(\det\begin{bmatrix} \left|\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\right| & \left(\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\right|^2\right)^{1/2} \\ \left|\frac{1}{\mu(S)}\int_S y(t)\,d\mu(t)\right| & \left(\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S y(t)\,d\mu(t)\right|^2\right)^{1/2}\end{bmatrix}\right)^2$$

and

$$\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\,\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t) - \left|\frac{1}{\mu(S)}\int_S x(t)\overline{y(t)}d\mu(t)\right|^2 \tag{3.13}$$
$$\geq\left(\det\begin{bmatrix} \left(\frac{1}{\mu(S)}\int_S |x(t)|^2 d\mu(t)\right)^{1/2} & \left|\frac{1}{\mu(S)}\int_S x(t)\,d\mu(t)\right| \\ \left(\frac{1}{\mu(S)}\int_S |y(t)|^2 d\mu(t)\right)^{1/2} & \left|\frac{1}{\mu(S)}\int_S y(t)\,d\mu(t)\right|\end{bmatrix}\right)^2.$$

Now, consider the μ-measurable subsets E, F of S such that $0<\mu(E)$, $\mu(F)<\infty$ and $\mu(E\cap F)=0$. We consider the functions $e,\ f:S\to\mathbb{R}$ defined by

$$e(t):=\begin{cases}\frac{1}{[\mu(E)]^{1/2}} \text{ if } t\in E,\\ 0 \text{ if } t\in S\setminus E.\end{cases},\quad f(t):=\begin{cases}\frac{1}{[\mu(F)]^{1/2}} \text{ if } t\in F,\\ 0 \text{ if } t\in S\setminus E.\end{cases}$$

Then

$$\int_S |e(t)|^2 d\mu(t) = \int_S |f(t)|^2 d\mu(t) = 1 \text{ and } \int_S e(t)\overline{f(t)}d\mu(t) = 0.$$

By the inequality (3.5) we get

$$\begin{aligned} (3.14) \quad & \int_S |x(t)|^2 d\mu(t) \int_S |y(t)|^2 d\mu(t) - \left| \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \\ & \geq \frac{1}{\mu(E)\mu(F)} \left| \det \begin{bmatrix} \int_E x(t) d\mu(t) & \int_F x(t) d\mu(t) \\ \int_E y(t) d\mu(t) & \int_F y(t) d\mu(t) \end{bmatrix} \right|^2 \\ & \geq \frac{1}{\mu(E)\mu(F)} \left(\det \begin{bmatrix} \left|\int_E x(t) d\mu(t)\right| & \left|\int_F x(t) d\mu(t)\right| \\ \left|\int_E y(t) d\mu(t)\right| & \left|\int_F y(t) d\mu(t)\right| \end{bmatrix} \right)^2 . \end{aligned}$$

If $\mu(S) < \infty$ and E, F are μ-measurable subsets of S such that $\mu(E \cap F) = 0$, then by (3.14) we get the following inequality for integral means

$$\begin{aligned} (3.15) \quad & \frac{1}{\mu(S)} \int_S |x(t)|^2 d\mu(t) \frac{1}{\mu(S)} \int_S |y(t)|^2 d\mu(t) - \left| \frac{1}{\mu(S)} \int_S x(t) \overline{y(t)} d\mu(t) \right|^2 \\ & \geq \frac{\mu(E)\mu(F)}{[\mu(S)]^2} \left| \det \begin{bmatrix} \frac{1}{\mu(E)} \int_E x(t) d\mu(t) & \frac{1}{\mu(F)} \int_F x(t) d\mu(t) \\ \frac{1}{\mu(E)} \int_E y(t) d\mu(t) & \frac{1}{\mu(F)} \int_F y(t) d\mu(t) \end{bmatrix} \right|^2 \\ & \geq \frac{\mu(E)\mu(F)}{[\mu(S)]^2} \left(\det \begin{bmatrix} \left|\frac{1}{\mu(E)} \int_E x(t) d\mu(t)\right| & \left|\frac{1}{\mu(F)} \int_F x(t) d\mu(t)\right| \\ \left|\frac{1}{\mu(E)} \int_E y(t) d\mu(t)\right| & \left|\frac{1}{\mu(F)} \int_F y(t) d\mu(t)\right| \end{bmatrix} \right)^2 . \end{aligned}$$

If $S = \{1, ..., n\}$ and μ is the counting measure on $S = \{1, ..., n\}$, then all the above inequalities may be written for n-tuples of complex numbers in $\mathbb{C}^n$. As an example we use the inequality (3.5) for the n-tuples of complex numbers $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n)$, $e = (e_1, ..., e_n)$ and $f = (f_1, ..., f_n)$ with

$$\sum_{k=1}^n |e_k|^2 = \sum_{k=1}^n |f_k|^2 = 1 \text{ and } \sum_{k=1}^n e_k \overline{f_k} = 0$$

to get

$$\begin{aligned} (3.16) \quad & \sum_{k=1}^n |x_k|^2 \sum_{k=1}^n |y_k|^2 - \left| \sum_{k=1}^n x_k \overline{y_k} \right|^2 \geq \left| \det \begin{bmatrix} \sum_{k=1}^n x_k \overline{e_k} & \sum_{k=1}^n x_k \overline{f_k} \\ \sum_{k=1}^n y_k \overline{e_k} & \sum_{k=1}^n y_k \overline{f_k} \end{bmatrix} \right|^2 \\ & \geq \left(\det \begin{bmatrix} \left|\sum_{k=1}^n x_k \overline{e_k}\right| & \left|\sum_{k=1}^n x_k \overline{f_k}\right| \\ \left|\sum_{k=1}^n y_k \overline{e_k}\right| & \left|\sum_{k=1}^n y_k \overline{f_k}\right| \end{bmatrix} \right)^2 . \end{aligned}$$

If $m\in\{1,...,n\}$ with $n\geq 2$, then from inequality (3.15) we get for $n>m$ that

$$\frac{1}{n}\sum_{k=1}^{n}|x_k|^2\frac{1}{n}\sum_{k=1}^{n}|y_k|^2-\left|\frac{1}{n}\sum_{k=1}^{n}x_k\overline{y_k}\right|^2 \tag{3.17}$$
$$\geq\frac{m(n-m)}{n^2}\left|\det\begin{bmatrix}\frac{1}{m}\sum_{k=1}^{m}x_k & \frac{1}{n-m}\sum_{k=m+1}^{n}x_k\\ \frac{1}{m}\sum_{k=1}^{m}y_k & \frac{1}{n-m}\sum_{k=m+1}^{n}y_k\end{bmatrix}\right|^2$$
$$\geq\frac{m(n-m)}{n^2}\left(\det\begin{bmatrix}\left|\frac{1}{m}\sum_{k=1}^{m}x_k\right| & \left|\frac{1}{n-m}\sum_{k=m+1}^{n}x_k\right|\\ \left|\frac{1}{m}\sum_{k=1}^{m}y_k\right| & \left|\frac{1}{n-m}\sum_{k=m+1}^{n}y_k\right|\end{bmatrix}\right)^2$$

for any n-tuples of complex numbers $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$.

If we take $m=n-1$ in (3.17), then we get

$$\frac{1}{n}\sum_{k=1}^{n}|x_k|^2\frac{1}{n}\sum_{k=1}^{n}|y_k|^2-\left|\frac{1}{n}\sum_{k=1}^{n}x_k\overline{y_k}\right|^2 \tag{3.18}$$
$$\geq\frac{n-1}{n^2}\left|\det\begin{bmatrix}\frac{1}{n-1}\sum_{k=1}^{n-1}x_k & x_n\\ \frac{1}{n-1}\sum_{k=1}^{n-1}y_k & y_n\end{bmatrix}\right|^2$$
$$\geq\frac{n-1}{n^2}\left(\det\begin{bmatrix}\left|\frac{1}{n-1}\sum_{k=1}^{n-1}x_k\right| & |x_n|\\ \left|\frac{1}{n-1}\sum_{k=1}^{n-1}y_k\right| & |y_n|\end{bmatrix}\right)^2,$$

for any n-tuples of complex numbers $x=(x_1,...,x_n)$, $y=(y_1,...,y_n)$.

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