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Some New Refinements of Jensen's Discrete Inequality

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Abstract.
Some new refinements of the discrete Jensen's inequality are given. New upper and lower bounds for the discrete Jensen gap functional are discussed. These bounds are of two types. The first type is a hybrid of bounds given by several authors in different works. The second type is appropriate for functions which are r-convex for some integer $r \geq 3$. A numerical example is presented. Some conjectures are made.

1. Introduction

The well-known discrete Jensen's inequlaity states that if $f : [a, b] \to \mathcal{R}$ is a convex function on $[a, b]$, $x_i \in [a, b]$, $p_i > 0$, $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} p_i = 1$, then

$$f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i f(x_i)\,. \tag{1}$$

In this paper, we shall be concerned with upper and lower bounds for the Jensen gap functional $J(f, n, \underline{p}, \underline{x})$ given by

$$J(f, n, \underline{p}, \underline{x}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right), \tag{2}$$

where $\underline{x} = (x_1, \ldots, x_n)$ and $\underline{p} = (p_1, \ldots, p_n)$. Note that Jensen's inequality states that $J(f, n, \underline{p}, \underline{x}) \geq 0$, if f is convex on $[a, b]$. Many classical inequalities in analysis are consequences of Jensen's inequality after suitably choosing f, $\underline{p}$ and $\underline{x}$, hence, the importance of this inequality.

In this paper, we are concerned with obtaining new upper and lower bounds for $J = J(f, n, \underline{p}, \underline{x})$ under various sets of assumptions on f. In some cases, we shall do so without requiring the convexity of f. However, in these cases, we shall require that the third and/or higher order derivatives of f exist on $[a, b]$.

The upper and lower bounds discussed below are one of two possible types. The first type is a 'hybrid' of bounds given by Dragomir in [1]–[3] and [7], and in From [8]. The second type is applicable to functions which have a certain type of higher convexity. In particular, f is r-convex for some positive integer $r \geq 3$ on $[a, b]$. We shall present a few numerical comparisons and make some conjectures based off of these comparisons. We assume $n \geq 2$ and $0 < p_i < 1$, $i = 1, 2, \ldots n$ throughout this paper, without loss of generality. Also, we assume $x_1 < x_2 < \cdots < x_n$, without loss of generality.

The research literature on Jensen's inequality and its refinements is vast, so no attempt will be made to cite many references. However, we discuss only the literature needed to discuss new upper and lower bounds for J, the Jensen gap.

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Next, we present some previously published bounds for J under various sets of assumptions on f. See references [1]–[7] and [9].

Theorem 1.1. *(Simic [9]). Suppose f is convex and differentiable on $[a,b]$. Then*

$$\begin{aligned} J(f,n,\underline{p},\underline{x}) &\leq f(a)+f(b)-2f\left(\frac{a+b}{2}\right) \\ &\equiv S_f(a,b)\,. \end{aligned} \tag{3}$$

Theorem 1.2. *(Theorem 1 of [3]). Suppose f is convex and differentiable on (a,b). Let $\overline{X}=\sum_{i=1}^{n} p_i x_i$. Suppose $x_i \in (a,b)$, $i=1,2,\ldots,n$. Then*

$$J(f,n,\underline{p},\underline{x}) \geq \left|\sum_{i=1}^{n} p_i |f(x_i)-f(\bar{x})| - \sum_{i=1}^{n} p_i |x_i-\bar{x}|\cdot|f'(\bar{x})|\right| \equiv L_{DS}\,. \tag{4}$$

Theorem 1.3. *(Theorem 1 of [2]). Suppose f is convex and differentiable on (a,b), $x_i \in (a,b)$, $i=1,2,\ldots,n$. Let $\bar{x}=\sum_{i=1}^{n} p_i x_i$. Then*

$$J(f,n,\underline{p},\underline{x}) \leq \sum_{i=1}^{n} p_i x_i f'(x_i) - \bar{x}\cdot\sum_{i=1}^{n} p_i f'(x_i) \equiv U_{D,1}\,. \tag{5}$$

Theorem 1.4. *(Dragomir [1]). Suppose f is convex on $[a,b]$. Suppose f is bounded on $[a,b]$. Let $d_f(x) = f(a)\left(\frac{b-x}{b-a}\right)+f(b)\left(\frac{x-a}{b-a}\right)$ be the equation of the chord connecting $(a,f(a))$ and $(b,f(b))$. Let*

$$\phi_f(x)=\left[\left(\frac{b-x}{b-a}\right)f(a)+\left(\frac{x-a}{b-a}\right)f(b)\right]-f(x)$$

be the error in approximating $f(x)$ by $\phi_f(x)$. Then

$$0 \leq \phi_f(x) \leq \frac{(b-x)(x-a)}{b-a}[f'_-(b)-f'_+(a)] \leq \left(\frac{b-a}{4}\right)(f'_-(b)-f'_+(a))\,, \tag{6}$$

where f'_- and f'_+ are the lateral derivatives of f.

Theorem 1.5. *(Theorem 2.3 of From [8]). Suppose f has a second derivative f'' continuous on $[a,b]$. Suppose $a \leq x_1 < x_2 < \cdots < x_n \leq b$. Let*

$$\begin{aligned} R_i &= p_i+p_{i+1}+\cdots+p_n=\sum_{j=i}^{n} p_j, \quad i=1,2,\ldots,n\,, \\ x_i^* &= \frac{p_{i+1}x_{i+1}+p_{i+2}x_{i+2}+\cdots+p_n x_n}{R_{i+1}}, \quad i=1,2,\ldots,n-1\,, \\ x^{(i)} &= \frac{p_i}{R_i}x_i+\left(1-\frac{p_i}{R_i}\right)x_i^*, \quad i=1,2,\ldots,n-1\,. \end{aligned}$$

Then there exists real numbers $\theta_1, \theta_2, \ldots, \theta_{n-1}$ with $x_i < \theta_i < x_i^*$, $i=1,2,\ldots,n-1$ with

$$\begin{aligned} J=J(f,n,\underline{p},\underline{x}) &= \sum_{i=1}^{n-1} \frac{f''(\theta_i)}{2} p_i \left(1-\frac{p_i}{R_i}\right)(x_i^*-x_i)^2 \\ &= \leq \sum_{i=1}^{n-1} \frac{1}{2} M_i p_i \left(1-\frac{p_i}{R_i}\right)(x_i^*-x_i)^2 \equiv U_J \end{aligned} \tag{7}$$

and

$$J = J(f, n, \underline{p}, \underline{x}) \geq \sum_{i=1}^{n-1} \frac{1}{2} m_i p_i \left(1 - \frac{p_i}{R_i}\right)(x_i^* - x_i)^2 \equiv L_J \,, \tag{8}$$

where

$$\begin{aligned} M_i &= \sup\{f''(t) : x_i \leq t \leq x_i^*\} \quad \textit{and} \\ m_i &= \inf\{f''(t) : x_i \leq t \leq x_i^*\}, \quad i = 1, 2, \ldots, n-1 \,. \end{aligned}$$

Theorem 1.6. *(Theorem 2.5 of From [8]). Suppose f has a continous third derivative $f^{(3)}$ on $[a, b]$.*

(a) If $f''(t) \geq 0$ and $f^{(3)}(t) \geq 0$ on $[a, b]$, then

$$h_1 \leq J(f, n, \underline{p}, \underline{x}) \leq h_2 \,, \tag{9}$$

where

$$h_1 = \sum_{j=1}^{n-1} W_j p_j \left(1 - \frac{p_j}{R_j}\right), \tag{10}$$

$$h_2 = \sum_{j=1}^{n-1} V_j p_j \left(1 - \frac{p_j}{R_j}\right), \tag{11}$$

$$V_j = (x_j^* - x_j) f'(x_j^*) - f(x_j^*) + f(x_j) \,, \tag{12}$$

$$W_j = f(x_j^*) - f(x_j) - (x_j^* - x_j) f'(x_j) \,, \tag{13}$$

and x_j^* and R_j are as given in Theorem 1.5, $j = 1, 2, \ldots, n-1$.

(b) If $f''(t) \geq 0$ and $f^{(3)}(t) \leq 0$ on $[a, b]$, then

$$h_2 \leq J(f, n, \underline{p}, \underline{x}) \leq h_1$$

instead.

Theorem 1.7. *(Theorem 4 of Dragomir [6].) Let I be a closed subinterval of $\mathbb{R}$, let $a, b \in I$ with $a < b$ and let L be a nonnegative integer. If $f : I \to \mathbb{R}$ is such that the n^{th} derivative $f^{(n)}$ is of bounded variation on the interval $[a, b]$, then, for any x in $[a, b]$, we have the representation:*

$$\begin{aligned} f(x) = \left(\frac{b-x}{b-a}\right) f(a) \quad &+ \quad \left(\frac{x-a}{b-a}\right) f(b) \\ &+ \quad \frac{(b-x)(x-a)}{b-a} \sum_{k=1}^{L} \frac{1}{k!} \left\{(x-a)^{k-1} f^{(k)}(a) + (-1)^k (b-x)^{k-1} f^{(k)}(b)\right\} \\ &+ \quad \frac{1}{b-a} \int_a^b S_n(x, t) d(f^{(L)}(t)) \end{aligned} \tag{14}$$

where

$$S_n(x, t) = \frac{1}{n!} \times \begin{cases} (x-t)^L (b-x), & \textit{if} \quad a \leq t \leq x \,, \\ (-1)^{L+1} (t-x)^n (x-a), & \textit{if} \quad x < t \leq b \,, \end{cases} \tag{15}$$

and the sum in (14) is defined to be zero for $L = 0$.

Finally, we shall also need the famous Hermite-Hadamard inequality
Hermite-Hadamard inequality. Suppose f is convex on $[a,b]$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{\int_a^b f(t)dt}{b-a} \leq \frac{f(a)+f(b)}{2}\,.$$

2. New Hybrid Bounds

In this section, we shall present new bounds for the Jensen gap functional $J = J(f,n,\underline{p},\underline{x})$ under various sets of assumptions of f which can be considered bounds of hybrid type. First, we need the following lemma. The proof of Lemma 2.1 is part of the proof of Theorem 1.5 and is omitted.

Lemma 2.1. *For any real-valued function f on $[a,b]$, we have*

$$J = J(f,n,\underline{p},\underline{x}) = \sum_{j=1}^{n-1} R_j B_j\,,$$

where

$$B_j = \left[\frac{p_j}{R_j} f(x_j) + \left(1 - \frac{p_j}{R_j}\right) f(x_j^*)\right] - f(x^{(j)}) \tag{16}$$

$j = 1,2,\ldots,n-1$, where R_j, x_j^* and $x^{(j)}$ were given in Theorem 1.5.

Note that in Lemma 2.1, B_j represents the linear interpolation error when approximating $f(x)$ at $x = x^{(j)}$. An important observation is that B_j is a Jensen gap functional itself based on only two points (and not n points) restricted to the interval $[x_i, x_i^*]$, instead of the whole interval $[a,b]$. Thus, we may take any of the bounds given in Theorems 1.1–1.4 and apply them 'locally' on $[x_i, x_i^*]$ only; that is, we may bound B_j for each j individually using the $n = 2$ subcase versions of Theorems 1.1–1.4, multiply by R_j (per Lemma 2.1), and then sum over j to obtain new bounds for $J = J(f,n,\underline{p},\underline{x})$. We shall see that this can cometimes improve on the bounds of Theorems 1.1–1.4. Note that Lemma 2.1 is valid for any real-valued function f on $[a,b]$ and does not require convexity on any other restriction on f.

By choosing different bounds for B_j we can obtain new 'hybrid'-type bounds for J. These new bounds utilize the construction of Lemma 2.1 given in From [8] and locally bounds B_j using the bounds given in Theorems 1.1–1.4 for the $N = 2$ case applied locally.

First, let's consider the hybrid (H) version of Theorem 1.1.

Theorem 2.2. *Let f be convex and differentiable on $[a,b]$. Then*

$$J(f,n,\underline{p},\underline{x}) \leq \sum_{i=1}^{n-1} R_i \left[f(x_i) + f(x_i^*) - 2f\left(\frac{x_i + x_i^*)}{2}\right)\right] \equiv S_{f,H}(a,b)\,. \tag{17}$$

Proof. Apply Lemma 2.1. By Theorem 1.1, applied to f on $[x_i, x_i^*]$ instead of $[a,b]$ and with $n = 2$ instead, we obtain

$$B_i \leq f(x_i) + f(x_i^*) - 2f\left(\frac{x_i + x_i^*}{2}\right).$$

Multiplying this by R_i and summing over i, the result follows. □

The hybrid version of Theorem 1.2 follows below.

Theorem 2.3. *Suppose f is convex and differentiable on (a,b). Suppose $x_j \in (a,b)$, $j = 1,2,\dots,n$. Let*

$$\bar{x}_j = x^{(j)} = \frac{p_j}{R_j}x_j + \left(1 - \frac{p_j}{R_j}\right)x_j^*,\ j = 1,2,\dots,n-1\,.$$

$$\begin{aligned} C_j &= \tfrac{p_j}{R_j}|f(x_j) - f(\bar{x}_j)| + \left(1 - \tfrac{p_j}{R_j}\right)|f(x_j^*) - f(\bar{x}_j)| \\ D_j &= \tfrac{p_j}{R_j}|x_j - \bar{x}_j| \cdot |f'(\bar{x}_j)| + \left(1 - \tfrac{p_j}{R_j}\right) \cdot |f'(\bar{x}_j| \cdot |x_j^* - \bar{x}_j|\,. \end{aligned}$$

Then

$$J(f,n,\underline{p},\underline{x}) \geq \sum_{j=1}^{n-1} R_j \cdot |C_j - D_j| \equiv L_{DS,H}\,. \tag{18}$$

Proof. Theorem 1.2 gives $B_j \geq |C_j - D_j|$. Multiplying by R_j and summing over j, the result follows from Lemma 2.1, upon substituting $\frac{p_j}{R_j}$ for p_1 and $\left(1 - \frac{p_j}{R_j}\right)$ for p_2 in Theorem 1.2 for $n = 2$ instead on the interval $[x_i, x_i^*]$ instead of $[a,b]$. □

Theorem 2.4 below is the hybrid version of Theorem 1.3.

Theorem 2.4. *Suppose f is convex and differentiable on (a,b), $x_j \in (a,b)$, $j = 1,2,\dots,n-1$. Let*

$$\begin{aligned} E_j &= \tfrac{p_j}{R_j}x_j f'(x_j) + \left(1 - \tfrac{p_j}{R_j}\right)x_j^* f'(x_j^*) \\ F_j &= \bar{x}_j\left(\tfrac{p_j}{R_j}f'(x_j) + \left(1 - \tfrac{p_j}{R_j}\right)f'(x_j^*)\right), \end{aligned}$$

where $\bar{x}_j$ is given in Theorem 2.3. Then

$$J(f,n,\underline{p},\underline{x}) \leq \sum_{j=1}^{n-1} R_j(E_j - F_j) \equiv U_{D,1,H}\,. \tag{19}$$

Proof. Apply Theorem 1.3 using $n = 2$ on $[x_j, x_j^*]$, replacing p_j by $\frac{p_j}{R_j}$ and p_2 by $\left(1 - \frac{p_j}{R_j}\right)$. Then $B_j \leq E_j - F_j$, $j = 1,2,\dots,n-1$ holds. Multiplying by R_j and summing over j, Lemma 2.1 gives the desired result. □

Remark 2.5. *It is interesting to note that if f is differentiable on $[a,b]$, then the hybird version of the first half of Theorem 1.4 provides an upper bound on $J(f,n,\underline{p},\underline{x})$ which coincides with $U_{D,1,H}$, the hybrid version of the upper bound U_D, given in Theorem 2.4. Thus, we do not obtain any new upper bounds in this case, and the hybrid versions of Theorems 1.3 and 1.4 coincide, when using the first half of (7) in Theorem 1.4, that is, using*

$$B_j \leq \frac{(x_j^* - x^{(j)})(x^{(j)} - x_j)}{x_j^* - x_j} \cdot (f'_-(x_j^*) - f'_+(x_j))\,. \tag{20}$$

However, the second half of (7) does provide a new hybrid upper bound, but it is clearly inferior to Theorem 2.4 upper bound. However, it does not require differentiability of f as is discussed below.

Theorem 2.6 is the hybrid version of Theorem 1.4.

Theorem 2.6. *Suppose f is convex and bounded on $[a,b]$. Let*

$$\begin{aligned} \alpha_j &= \tfrac{(x_j^* - x^{(j)}) \cdot (x^{(j)} - x_j)}{x_j^* - x_j}[f'_-(x_j^*) - f'_+(x_j)]\,, \\ \beta_j &= \tfrac{1}{4}(x_j^* - x_j) \cdot [f'_-(x_j^*) - f'_+(x_j)],\quad j = 1,2,\dots,n-1\,. \end{aligned}$$

Then

$$J(f,n,\underline{p},\underline{x}) \le \sum_{j=1}^{n-1} \alpha_j R_j \le \sum_{j=1}^{n-1} \beta_j R_j\,. \tag{21}$$

Proof. Apply Theorem 1.4 to f on $[x_j, x_j^*]$ using $n = 2$. Replacing a by x_j, b by x_j^* and x by $x^{(j)}$, we obtain $B_j \le \alpha_j \le \beta_j$, $j = 1, 2, \ldots, n-1$. Multiplying by R_j and summing over j, the result follows. □

We shall see later that the hybrid versions of the bounds given in Section 1 sometimes improve the original versions of these bounds.

3. Bounds for *r*-Convex Functions

In this section, we present new bounds for $J = J(f,n,\underline{p},\underline{x})$ in the case where f is r-convex for some choice of $r \ge 3$. We shall see that significant imrpovement on the bounds of Section 1 is possible.

Definition 3.1. *Let r be a positive integer. A function $f : [a,b] \to \mathcal{R}$ is r-convex on $[a,b]$, if the r^{th} derivative, $f^{(r)}(x) \ge 0$ for all $x \in [a,b]$.*

In this paper, we shall first consider the cases $r = 3$ and $r = 4$. First, we need the following lemma.

Lemma 3.2. *(Theorem 4.2 of From [8]). Suppose $f : [a,b] \to \mathcal{R}$ is 3-convex on $[a,b]$ and that $f^{(3)}$ is continuous on $[a,b]$. Then*

$$\text{(a)}\quad \frac{\int_a^b f(x)dx}{b-a} \le f\left(\frac{a+b}{2}\right) + \frac{b-a}{12}\cdot\left(f'(b) - f'\left(\frac{a+b}{2}\right)\right) \tag{22}$$

and

$$\text{(b)}\quad \frac{\int_a^b f(x)dx}{b-a} \ge f\left(\frac{a+b}{2}\right) + \frac{b-a}{12}\cdot\left(f'\left(\frac{a+b}{2}\right) - f'(a)\right). \tag{23}$$

First, we consider the case where f is 3-convex on $[a,b]$.

Theorem 3.3. *Suppose f is 3-convex on $[a,b]$ and $f^{(3)}$ is continuous on $[a,b]$. Then*

$$J(f,n,\underline{p},\underline{x}) \ge \sum_{j=1}^{n-1} R_j \cdot \alpha_{1,j} \equiv L_1(3) \tag{24}$$

and

$$J(f,n,\underline{p},\underline{x}) \le \sum_{j=1}^{n-1} R_j B_{1,j} \equiv U_1(3)\,, \tag{25}$$

where

$$\begin{aligned}
\alpha_{1,j} &= \xi_j \cdot \left[f'\left(\frac{x_j^* + x^{(j)}}{2}\right) - \left(\frac{f'(x_j^*) + f'(x^{(j)})}{2}\right)\right] \\
\beta_{1,j} &= \xi_j \cdot \left[\left(\frac{f'(x_j^*) + f'(x^{(j)})}{2}\right) - f'\left(\frac{x_j + x^{(j)}}{2}\right)\right], \\
\xi_j &= \frac{(x^{(j)} - x_j)\cdot(x_j^* - x^{(j)})}{(x_j^* - x_j)}, \quad j = 1, 2, \ldots, n-1\,,
\end{aligned} \tag{26}$$

and where x_j^* and $x^{(j)}$ are given in Theorem 1.5.

Proof. Let B_j be as given in (16) of Lemma 2.1. By Theorem 1.7, with $L = 0$ using $a = x_j$, $b = x_j^*$, and $x = x^{(j)}$, we obtain

$$B_j = \xi_j \left[\int_{x^{(j)}}^{x_j^*} \frac{f'(t)}{x_j^* - x^{(j)}} dt - \int_{x_j}^{x^{(j)}} \frac{f'(t)}{x^{(j)} - x_j} dt \right]. \tag{27}$$

Since f is 3-convex, f' is convex. By the Hermite-Hadamard inequality applied to both integrals, we obtain

$$B_j \geq \xi_j \left[f'\left(\frac{x_j^* + x^{(j)}}{2} \right) - \left(\frac{f'(x_j^*) + f'(x^{(j)})}{2} \right) \right] = \alpha_{1,j} \tag{28}$$

and

$$B_j \leq \xi_j \left[\left(\frac{f'(x_j^*) + f'(x^{(j)})}{2} \right) - f'\left(\frac{x_j + x^{(j)}}{2} \right) \right] = \beta_{1,j}\,. \tag{29}$$

Multiplying both sides of (28)–(29) by $R_j \geq 0$ and summing over j produces the lower bound $L_1(3)$ and upper bound $U_1(3)$ for a 3-convex function f on $[a, b]$. This completes the proof. □

Next, we apply Lemma 3.2 to the case in which f is 4-convex on $[a, b]$.

Theorem 3.4. *Suppose f is 4-convex on $[a, b]$ and that $f^{(4)}$ is continuous on $[a, b]$. Then*

$$\text{(a)} \quad J(f, n, \underline{p}, \underline{x}) \geq \sum_{j=1}^{n-1} R_j \cdot \alpha_{2,j} \equiv L_2(4)\,, \tag{30}$$

and

$$\text{(b)} \quad J(f, n, \underline{p}, \underline{x}) \leq \sum_{j=1}^{n-1} R_j \cdot \beta_{2,j} \equiv U_2(4)\,, \tag{31}$$

where

$$\begin{aligned} \alpha_{2,j} \;=\; & (-\xi_j) \cdot \left[\left(f'\left(\frac{x^{(j)} + x_j}{2} \right) + \frac{1}{12} \cdot (x^{(j)} - x_j) \cdot \left(f''(x^{(j)}) - f''\left(\frac{x^{(j)} + x_j}{2} \right) \right) \right) \right. \\ & \left. - \left(f'\left(\frac{x^{(j)} + x_j^*}{2} \right) \right) + \frac{1}{12} \cdot (x_j^* - x^{(j)}) \cdot \left(f''\left(\frac{x_j^* + x^{(j)}}{2} \right) - f''(x^{(j)}) \right) \right], \end{aligned} \tag{32}$$

$$\begin{aligned} \beta_{2,j} \;=\; & (-\xi_j) \cdot \left[\left(f'\left(\frac{x_j + x^{(j)}}{2} \right) + \frac{1}{12} (x^{(j)} - x_j) \cdot \left(f''\left(\frac{x_j + x^{(j)}}{2} \right) - f''(x_j) \right) \right) \right. \\ & \left. - \left(f'\left(\frac{x_j^* + x^{(j)}}{2} \right) \right) + \frac{1}{12} (x_j^* - x^{(j)}) \cdot \left(f''(x_j^*) - f''\left(\frac{x_j^* + x^{(j)}}{2} \right) \right) \right], \end{aligned} \tag{33}$$

ξ_j is given by (26) in Theorem 3.3, and x_j^* and $x^{(j)}$ are given in Theorem 1.5. Let B_j be given by (16) in Lemma 2.1.

Proof. Proceeding as in the proof of Theorem 3.3, from (27) in the proof of Theorem 3.3, we have

$$B_j = \xi_j \left[\frac{\int_{x^{(j)}}^{x_j^*} f'(t) dt}{x_j^* - x^{(j)}} - \frac{\int_{x_j}^{x^{(j)}} f'(t) dt}{x^{(j)} - x_j} \right]. \tag{34}$$

Since f is 4-convex, f' is 3-convex. Now apply part (b) of Lemma 3.2 to the first integral in (34) and apply part (a) of Lemma 3.2 to the second integral in (34) after replacing f by f'. We then obtain

$$B_j \geq \alpha_{2,j} \quad \text{and} \quad B_j \leq \beta_{2,j}, \quad j = 1, 2, \ldots, n-1\,.$$

Multiplying by R_j and summing over j proves the theorem, after application of Lemma 2.1. □

Now let's consider the case where f is r-convex for some even positive integer r. The next theorem states that upper bounds for the Jensen gap functional can be obtained in this case.

Theorem 3.5. *Suppose that $f : [a, b] \to \mathbb{R}$ is r-convex on $[a, b]$ for some even integer $r \geq 2$, that is, $f^{(r)}(x) \geq 0$, $a \leq x \leq b$, and that f satisfies the conditions of Theorem 1.7 with $L = r - 1$. Then $J(f, n, p, x) \leq \sum_{j=1}^{n-1} G_j R_j \equiv U(r)$, where*

$$G_j = \frac{(x_j^* - x^{(j)})(x_j - x^{(j)})}{(x_j^* - x_j)} \cdot \sum_{k=1}^{r-1} \frac{1}{k!} \left\{ x^{(j)} - x_j)^{k-1} f^{(k)}(x_j) + (-1)^k \cdot (x_j^* - x^{(j)})^{k-1} f^{(k)}(x_j^*) \right\}, \tag{35}$$

$j = 1, 2, \ldots, n-1$ and R_j is given in Theorem 1.5.

Proof. Apply Lemma 2.1. By Theorem 1.7, with $L = r - 1$,

$$B_j = \frac{p_j}{R_j} f(x_j) + \left(1 - \frac{p_j}{R_j}\right) f(x_j^*) - f(x^{(j)}) \leq G_j,\, j = 1, 2, \ldots, n-1\,. \tag{36}$$

Multiplying (36) by R_j and summing over j, we obtain the desired result, since $S_n(x^{(j)}, t) \geq 0 \forall_j$, $a \leq t \leq b$, if $L = r - 1$ is odd. □

Next, we present a numerical example and make some conjectures.

Example 3.6. *Let $f(x) = e^x$, $n = 5$, $x_i = i$, $p_i = \frac{1}{5}$, $i = 1, 2, 3, 4, 5$. Then we obtain $J = J(f, n, \underline{p}, \underline{x}) = 26.56$. The values of the bounds discussed in Sections 1 and 2 are as follows:*
Lower bounds for J: $L_J = 9.32$, $h_1 = 17.87$, $L_{D,5} = 14.48$, $L_{DS,H} = 5.53$, $L_1(3) = 23.32$, $L_2(4) = 25.88$. We see that $L_{DS,H}$, the hybrid version of L_{DS} is worse than L_{DS}, but the reverse is true for the upper bounds.
Upper bounds for J: $U_J = 53.86$, $h_2 = 31.82$, $U_{D,1} = 67.72$, $U_{0,1,H} = 49.69$, $S_f(a, b) = S_f(1.0, 5.0) = 110.96$, $S_{f,H}(a, b) = S_{f,H}(1.0, 5.0) = 58.54$, $U_1(3) = 28.90$, and $U_2(4) = 27.04$. Here, $U_{D,1,H}$, the hybrid version of $U_{D,1}$, improves on $U_{D,1}$ upper bound. Also, $S_{f,H}(a, b)$, the hybrid version of $S_f(a, b)$, improves on $S_f(a, b)$. In many examples, it has been noticed that $U_{D,1,H}$ and $S_{f,H}(a, b)$ improve on $U_{D,1}$ and $S_f(a, b)$, respectively, but no proof has been found, in the case where f is 3-convex on $[a, b]$. We make these conjectures here. We see that $L_1(3)$, $U_1(3)$, $L_2(4)$ and $U_2(4)$ are very good bounds and often improves on h_1, h_2 bounds for smaller n. But h_2 is 'asymptotically' the best upper bound as $n \to \infty$ (see Theorem 4.3b of From [8]) and often beats $U_1(3)$ and $U_2(4)$ for very large n.

The table below presents the values of $U(r)$ upper bounds for various even values of r (odd values of $L = r - 1$):

r	:	2	4	6	8	10
$U(r)$	:	49.69	28.10	26.66	26.561	26.555515

r	:	12	14	16
$U(r)$	:	26.555306	26.555300	26.555 299876

These bounds are very good for $r \geq 4$, as the exact value of $J = J(f, n, p, x)$ is 26.555299874. We see that very good upper bounds are obtained. For $r = 2$, the $U(2)$ upper bound is slightly better than $U_j = 53.86$ and both bounds use only second derivative information. Both $U_2(4)$ and $U(4)$ assume f is 4-convex. In this example, $U_2(4)$ is a slightly better upper bound, since $U_2(4) = 27.04 < U(4) = 28.10$.

Remark 3.7. *Bounds for the case of r odd ($L - 1$ even) can also be obtained by applying Theorem 1.5 to the integral in (14) involving $S_n(x, t)$. These formulas will be more complicated. It is much easier to obtain better bounds by increasing r to the next higher even value.*

4. Concluding Remarks

In this paper, we have presented new bounds for the discrete Jensen gap of two types. The first type is of hybrid type and in some cases improves upon the original bounds. A conjecture was made. The second type of bound assumes f is r-convex on $[a, b]$. These bounds are very good, but require more differentiability assumptions. Some new inequalities of Hermite-Hadamard type are currently being investigated based off of the results in this paper.

Finally, as an application area, we mention that the bounds given in Section 3 produce very good refinements of the classical arithmetic mean-geometric mean inequality when letting $f(x) = Lnx$, the natural log of x. This has been verified for many choices of n, $\underline{p}$ and $\underline{x}$. These refinements compare very favorably to many previously published refinements and has been verified in many numerical comparisons done.

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