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SERIES REPRESENTATIONS OF THE REMAINDERS IN THE EXPANSIONS FOR CERTAIN TRIGONOMETRIC FUNCTIONS AND SOME RELATED INEQUALITIES, II

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ABSTRACT. We examine Wilker and Huygens-type inequalities involving trigonometric functions making use of results derived in Part I. The Papenfuss-Bach inequality representing upper and lower bounds for the function $x \sec^2 x - \tan x$ for $0 \le x < \pi/2$ is also investigated. An open problem posed by Sun and Zhu concerning this last inequality is established.

1. Introduction

In Part I [6] we derived series representations of the remainders in the expansions of certain trigonometric and hyperbolic functions and discussed some inequalities involving these functions. In Part II we apply some of the results obtained in [6] in the discussion of further trigonometric inequalities that are related to the Wilker, Huygens and Papenfuss-Bach inequalities.

In [24], Wilker proposed the following two open problems: Show that if $0 < x < \pi/2$, then

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} > 2 \tag{1.1}$$

and also find the largest constant c when the right-hand side of (1.1) is replaced by $2 + cx^3 \tan x$. This was proved in [22] and further extended by Chen and Cheung in [5], where it was shown that for $0 < x < \pi/2$,

$$2 + \frac{8}{45}x^4 + \frac{16}{315}x^5 \tan x < \left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} < 2 + \frac{8}{45}x^4 + \left(\frac{2}{\pi}\right)^6 x^5 \tan x,$$

with the constants $16/315$ and $(2/\pi)^6$ being the best possible. The Wilker-type inequality (1.1) has attracted much interest from many mathematicians and has motivated a large number of research papers involving different proofs, various generalizations and improvements (see [3–5, 8, 11, 14–18, 21, 22, 25–29, 31, 33–35] and the references cited therein).

A related inequality that is of interest to us is Huygens' inequality [12], which asserts that

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} > 3, \qquad 0 < |x| < \frac{\pi}{2}. \tag{1.2}$$

Chen and Cheung [5] showed that for $0 < x < \pi/2$,

$$3 + \frac{3}{20}x^3 \tan x < 2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} < 3 + \left(\frac{2}{\pi}\right)^4 x^3 \tan x, \tag{1.3}$$

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where the constants $3/20$ and $(2/\pi)^4$ are the best possible, and obtained a further improvement with the addition of terms involving $x^5\tan x$. These authors also posed three conjectures on Wilker and Huygens-type inequalities. As far as we know, these conjectures have not yet been proved.

Zhu [32] established some new inequalities of the Huygens-type for trigonometric and hyperbolic functions. Baricz and Sándor [3] pointed out that inequalities (1.1) and (1.2) are simple consequences of the arithmetic-geometric mean inequality, together with the well-known Lazarević-type inequality [13, p. 238] given by $(\cos x)^{1/3} < \sin x/x$ $(0 < |x| < \pi/2)$, or equivalently

$$\left(\frac{\sin x}{x}\right)^2 \frac{\tan x}{x} > 1, \qquad 0 < |x| < \frac{\pi}{2}. \tag{1.4}$$

Wu and Srivastava [27, Lemma 3] established another inequality, in which the trigonometric ratios in (1.1) are inverted, namely

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} > 2, \qquad 0 < |x| < \frac{\pi}{2} \tag{1.5}$$

and Chen and Sándor [8] established the following inequality chain:

$$\begin{aligned}\frac{(\sin x/x)^2 + \tan x/x}{2} &> \left(\frac{\sin x}{x}\right)^2 \left(\frac{\tan x}{x}\right) > \frac{2\,(\sin x/x) + \tan x/x}{3} \\ &> \left(\frac{\sin x}{x}\right)^{2/3} \left(\frac{\tan x}{x}\right)^{1/3} > \frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}\right] > \frac{2(x/\sin x) + x/\tan x}{3} > 1\end{aligned} \tag{1.6}$$

for $0 < |x| < \pi/2$. This can be seen to involve the Wilker and Huygens inequalities (1.1) and (1.2), together with their inverted forms and the inequality (1.4).

The final inequality we consider is the Papenfuss-Bach inequality. In [20], Papenfuss proposed the following problem to establish the inequality

$$x\sec^2 x - \tan x \le \frac{8\pi^2 x^3}{\left(\pi^2 - 4x^2\right)^2}, \qquad 0 \le x < \pi/2. \tag{1.7}$$

Bach [2] proved (1.7) and obtained a sharper upper bound in which the numerator $8\pi^2 x^3$ is replaced by $2\pi^4/3$. Ge [9, Theorem 1.3] presented a lower bound in (1.7) and proved that

$$\frac{64x^3}{\left(\pi^2 - 4x^2\right)^2} < x\sec^2 x - \tan x \le \frac{(2\pi^4/3)x^3}{\left(\pi^2 - 4x^2\right)^2}, \qquad 0 \le x < \pi/2, \tag{1.8}$$

where the constants 64 and $2\pi^4/3$ are the best possible.

Sun and Zhu [23, Theorem 1.5] obtained better lower and upper bounds for the Papenfuss-Bach inequality inequality in the form

$$\frac{\frac{2\pi^4}{3}x^3 + \left(\frac{8\pi^4}{15} - \frac{16\pi^2}{3}\right)x^5}{\left(\pi^2 - 4x^2\right)^2} < x\sec^2 x - \tan x < \frac{\frac{2\pi^4}{3}x^3 + \left(\frac{256}{\pi^2}\alpha - \frac{8\pi^2}{3}\right)x^5}{\left(\pi^2 - 4x^2\right)^2} \tag{1.9}$$

with $\alpha = 513/511$. They also posed the following open problem:

Open problem 1.1. *Let $0 < x < \pi/2$. Then* (1.9) *holds with $\alpha = 1$, where the constants $\frac{8\pi^4}{15} - \frac{16\pi^2}{3}$ and $\frac{256}{\pi^2} - \frac{8\pi^2}{3}$ are best possible.*

Our aim in Sections 2 and 3 is to develop the inverted forms of the Wilker and Huygens inequalities given in (1.5) and the last inequality in (1.6) to produce sharp inequalities. In Section 4, we present a series representation of the remainder in the expansion for $t\sec^2 t - \tan t$.

Based on this representation, we establish new bounds for $x\sec^2 x - \tan x$ when $0 < x < \pi/2$. We also answer the open problem in (1.9).

2. A Wilker-type inequality

We first consider the Wilker-type inequality in (1.5). It is well known [10, p. 42] that

$$\cot x = \frac{1}{x} - \sum_{k=1}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!} x^{2k-1}, \qquad |x| < \pi. \tag{2.1}$$

Differentiating the expression in (2.1), we find

$$\left(\frac{x}{\sin x}\right)^2 = 1 + \sum_{k=1}^{\infty} \frac{2^{2k}(2k-1)|B_{2k}|}{(2k)!} x^{2k}, \qquad |x| < \pi. \tag{2.2}$$

From (2.1) and (2.2), we obtain that for $|x| < \pi$,

$$\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} = 2 + \sum_{k=1}^{\infty} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2}. \tag{2.3}$$

It follows from (2.3) that for every $N \in \mathbb{N}$,

$$\frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} x^{2N+2} < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - \left(2 + \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2}\right) \tag{2.4}$$

for $0 < |x| < \pi/2$.

In view of (2.4) it is natural to ask: What are the largest number λ_N and the smallest number μ_N such that the inequality

$$\lambda_N x^{2N+2} < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - \left(2 + \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2}\right) < \mu_N x^{2N+2}$$

holds for $x \in (0, \pi/2)$ and $N \in \mathbb{N}$? Theorem 2.1 answers this question. In what follows we shall require the summations

$$\sum_{k=1}^{\infty} \frac{1}{k^{2n}} = \frac{2^{2n-1}\pi^{2n}}{(2n)!}|B_{2n}|, \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^{2n}} = \frac{(2^{2n-1}-1)\pi^{2n}}{(2n)!}|B_{2n}|,$$
$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} = \frac{(2^{2n}-1)\pi^{2n}}{2(2n)!}|B_{2n}|, \tag{2.5}$$

where B_n denote the Bernoulli numbers, defined by the following generating functions:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi)$$

Theorem 2.1. *Let $N \geq 1$ be an integer. Then for $0 < t < \pi/2$,*

$$\lambda_N t^{2N+2} < \left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} - \left(2 + \sum_{j=1}^{N-1} \frac{j \cdot 2^{2j+3}|B_{2j+2}|}{(2j+2)!} t^{2j+2}\right) < \mu_N t^{2N+2} \tag{2.6}$$

with the best possible constants

$$\lambda_N = \frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} \tag{2.7}$$

and

$$\mu_N = \frac{64N}{\pi^{2N+2}} \sum_{k=1}^{\infty} \frac{1}{k^{2N-2}(4k^2-1)^2} - \frac{16(N-1)}{\pi^{2N+2}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}(4k^2-1)^2}. \tag{2.8}$$

Proof. It follows from [19, p. 118] that

$$\csc^2 t = \frac{1}{t^2} + 2\sum_{k=1}^{\infty} \frac{t^2+\pi^2k^2}{(t^2-\pi^2k^2)^2}, \quad \cot t = \frac{1}{t} + 2t\sum_{k=1}^{\infty} \frac{1}{t^2-\pi^2k^2}. \tag{2.9}$$

This then yields

$$\left(\frac{t}{\sin t}\right)^2 = 1 + \sum_{k=1}^{\infty} \frac{2t^4+2\pi^2k^2t^2}{(t^2-\pi^2k^2)^2}. \tag{2.10}$$

From (2.10) and the second expansion in (2.9), we obtain

$$\left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} = 2 + 4t^4 \sum_{k=1}^{\infty} \frac{1}{(\pi^2k^2-t^2)^2} = 2 + 4t^4 \sum_{k=1}^{\infty} \frac{1}{\pi^4k^4\left(1-(\frac{t}{\pi k})^2\right)^2}. \tag{2.11}$$

Using the following identity:

$$\frac{1}{(1-q)^2} = \sum_{j=1}^{N-1} jq^{j-1} + \frac{Nq^{N-1}}{1-q} + \frac{q^N}{(1-q)^2} \qquad (q \neq 1) \tag{2.12}$$

and the first sum in (2.5), we then have

$$\begin{aligned}
\left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} &= 2 + 4t^4 \sum_{k=1}^{\infty} \frac{1}{\pi^4k^4\left(1-(\frac{t}{\pi k})^2\right)^2} \\
&= 2 + 4t^4 \sum_{k=1}^{\infty} \frac{1}{\pi^4k^4} \left(\sum_{j=1}^{N-1} j \left(\frac{t}{\pi k}\right)^{2j-2} + \frac{N(\frac{t}{\pi k})^{2N-2}}{1-(\frac{t}{\pi k})^2} + \frac{(\frac{t}{\pi k})^{2N}}{(1-(\frac{t}{\pi k})^2)^2} \right) \\
&= 2 + \sum_{j=1}^{N-1} \frac{j \cdot 2^{2j+3}|B_{2j+2}|}{(2j+2)!} t^{2j+2} + \sum_{k=1}^{\infty} \frac{4Nt^{2N+2}}{\pi^{2N}k^{2N}(\pi^2k^2-t^2)} + \sum_{k=1}^{\infty} \frac{4t^{2N+4}}{\pi^{2N}k^{2N}(\pi^2k^2-t^2)^2} \\
&= 2 + \sum_{j=1}^{N-1} \frac{j \cdot 2^{2j+3}|B_{2j+2}|}{(2j+2)!} t^{2j+2} + \frac{4t^{2N+2}}{\pi^{2N}} V_N(t),
\end{aligned}$$

where

$$V_N(t) = \sum_{k=1}^{\infty} \frac{N\pi^2k^2-(N-1)t^2}{k^{2N}(\pi^2k^2-t^2)^2}.$$

Differentiation yields

$$V_N'(t) = \sum_{k=1}^{\infty} \frac{2t\Big((N+1)\pi^2k^2-(N-1)t^2\Big)}{k^{2N}(\pi^2k^2-t^2)^3} > 0.$$

Hence, $V_N(t)$ is strictly increasing for $t \in (0, \pi/2)$, and we have

$$\lambda_N t^{2N+2} < \left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} - \left(2 + \sum_{j=1}^{N-1} \frac{j \cdot 2^{2j+3}|B_{2j+2}|}{(2j+2)!} t^{2j+2}\right) < \mu_N t^{2N+2}$$

with

$$\lambda_N = \frac{4}{\pi^{2N}} V_N(0) \quad \text{and} \quad \mu_N = \frac{4}{\pi^{2N}} V_N\left(\frac{\pi}{2}\right).$$

Direct computations yield

$$V_N(0) = \frac{N}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^{2N+2}} = \frac{N \cdot 2^{2N+1} \pi^{2N} |B_{2N+2}|}{(2N+2)!}$$

and

$$V_N\left(\frac{\pi}{2}\right) = \frac{16N}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^{2N-2}(4k^2-1)^2} - \frac{4(N-1)}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^{2N}(4k^2-1)^2}.$$

Hence, the inequality (2.6) holds with the best possible constants given in (2.7) and (2.8). The proof of Theorem 2.1 is complete. □

Remark 2.1. *Direct computations yield*

$$\lambda_1 = \frac{2}{45}, \quad \mu_1 = \frac{4(\pi^2-8)}{\pi^4}$$

and

$$\lambda_2 = \frac{8}{945}, \quad \mu_2 = \frac{8(-720+90\pi^2-\pi^4)}{45\pi^6}.$$

We then obtain from (2.6) *that for* $0 < t < \pi/2$*,*

$$2 + \frac{2}{45} t^4 < \left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} < 2 + \frac{4(\pi^2-8)}{\pi^4} t^4, \tag{2.13}$$

where the constants $\frac{2}{45}$ *and* $4(\pi^2-8)/\pi^4$ *are the best possible, and*

$$2 + \frac{2}{45} t^4 + \frac{8}{945} t^6 < \left(\frac{t}{\sin t}\right)^2 + \frac{t}{\tan t} < 2 + \frac{2}{45} t^4 + \frac{8(-720+90\pi^2-\pi^4)}{45\pi^6} t^6, \tag{2.14}$$

where the constants $\frac{8}{945}$ *and* $8(-720+90\pi^2-\pi^4)/(45\pi^6)$ *are the best possible.*

The classical Euler gamma function is defined (for $x > 0$) by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \mathrm{d}t.$$

Its logarithmic derivative, denoted by $\psi(x) = \Gamma'(x)/\Gamma(x)$, is called the psi (or digamma) function, and $\psi^{(k)}(x)$ $(k \in \mathbb{N})$ are called the polygamma functions.

Theorem 2.2. *Let* $N \geq 0$ *be an integer. Then for* $0 < x < \pi/2$*,*

$$\alpha_N x^4 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - \left(2 + 4x^4 \sum_{k=1}^{N} \frac{1}{(\pi^2 k^2 - x^2)^2}\right) < \beta_N x^4 \tag{2.15}$$

with the best possible constants

$$\alpha_N = \frac{2\psi'''(N+1)}{3\pi^4} \quad \text{and} \quad \beta_N = \frac{8\Big((2N+1)^2 \psi'(N+\frac{1}{2}) - 4(N+1)\Big)}{(2N+1)^2 \pi^4}. \tag{2.16}$$

Proof. Write (2.11) as

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}=2+4x^4\sum_{k=1}^{N}\frac{1}{(\pi^2k^2-x^2)^2}+4x^4A_N(x),$$

where

$$A_N(x)=\sum_{k=N+1}^{\infty}\frac{1}{(\pi^2k^2-x^2)^2}.$$

Obviously, $A_N(x)$ is strictly increasing for $x\in(0,\pi/2)$. Hence, for $0<x<\pi/2$, we have

$$\alpha_N x^4<\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}-\left(2+4x^4\sum_{k=1}^{N}\frac{1}{(\pi^2k^2-x^2)^2}\right)<\beta_N x^4$$

with

$$\alpha_N=4A_N(0)=\frac{4}{\pi^4}\sum_{k=N+1}^{\infty}\frac{1}{k^4}\quad\text{and}\quad\beta_N=4A_N\left(\frac{\pi}{2}\right)=\frac{64}{\pi^4}\sum_{k=N+1}^{\infty}\frac{1}{\left(4k^2-1\right)^2}.$$

From the following formula (see [1, p. 260, Eq. (6.4.10)]):

$$\psi^{(n)}(z)=(-1)^{n+1}n!\sum_{k=0}^{\infty}\frac{1}{(z+k)^{n+1}},\qquad z\neq 0,-1,-2,\ldots,$$

we obtain

$$\sum_{k=N+1}^{\infty}\frac{1}{k^4}=\frac{\psi'''(N+1)}{6}.\tag{2.17}$$

We find[1]

$$\sum_{k=N+1}^{\infty}\frac{1}{\left(4k^2-1\right)^2}=\frac{1}{8}\psi'\left(N+\frac{1}{2}\right)-\frac{N+1}{2(2N+1)^2}.\tag{2.18}$$

Hence, the inequality (2.15) holds with the best possible constants given in (2.16). The proof of Theorem 2.2 is complete. □

Remark 2.2. *The choice $N=0$ in* (2.15) *yields* (2.13)*. The choice $N=1$ in* (2.15) *yields*

$$2+\frac{4x^4}{(\pi^2-x^2)^2}+\frac{2(\pi^4-90)}{45\pi^4}x^4<\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}<2+\frac{4x^4}{(\pi^2-x^2)^2}+\frac{4(9\pi^2-88)}{9\pi^4}x^4\tag{2.19}$$

for $0<x<\pi/2$, where the constants $2(\pi^4-90)/(45\pi^4)$ and $4(9\pi^2-88)/(9\pi^4)$ are the best possible.

Remark 2.3. *There is no strict comparison between the two lower bounds in* (2.14) *and* (2.19)*. Likewise, there is no strict comparison between the two upper bounds in* (2.14) *and* (2.19)*.*

Theorem 2.3 proves Conjecture 2 in [5].

[1] The formula (2.18) is established by induction on N in the appendix.

Theorem 2.3. *Let $N \geq 1$ be an integer. Then for $0 < x < \pi/2$, we have*

$$2 + \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2} + p_N x^{2N+1} \tan x < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x}$$
$$< 2 + \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2} + q_N x^{2N+1} \tan x \tag{2.20}$$

with the best possible constants

$$p_N = 0 \quad \textit{and} \quad q_N = \frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!}. \tag{2.21}$$

Proof. By (2.3), for $p_N = 0$, the first inequality in (2.20) holds. We now prove the second inequality in (2.20) with $q_N = N \cdot 2^{2N+3}|B_{2N+2}|/(2N+2)!$. Using (2.3) and the following expansion (see [10, p. 42]):

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1}, \qquad |x| < \frac{\pi}{2}, \tag{2.22}$$

we find

$$\begin{aligned}
&\frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} x^{2N+1} \tan x - \left(\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!} x^{2k+2} \right) \\
&= \frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} x^{2N+1} \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!} x^{2k-1} - \sum_{k=N+1}^{\infty} \frac{(k-1) \cdot 2^{2k+1}|B_{2k}|}{(2k)!} x^{2k} \\
&= \sum_{k=N+2}^{\infty} \left\{ \frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} \frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!} - \frac{(k-1) \cdot 2^{2k+1}|B_{2k}|}{(2k)!} \right\} x^{2k},
\end{aligned} \tag{2.23}$$

where we note that the term corresponding to $k = N+1$ vanishes.

We claim that for $k \geq N+2$,

$$\frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!} \frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!} > \frac{(k-1) \cdot 2^{2k+1}|B_{2k}|}{(2k)!}. \tag{2.24}$$

Using the inequality (see [1, p. 805])

$$\frac{2}{(2\pi)^{2n}(1-2^{1-2n})} > \frac{|B_{2n}|}{(2n)!} > \frac{2}{(2\pi)^{2n}}, \qquad n \geq 1, \tag{2.25}$$

it is sufficient to prove that for $k \geq N+2$,

$$\frac{N \cdot 2^{2N+3} \cdot 2}{(2\pi)^{2N+2}} \frac{2^{2k-2N}(2^{2k-2N}-1) \cdot 2}{(2\pi)^{2k-2N}} > \frac{2(k-1) \cdot 2^{2k+1}}{(2\pi)^{2k}(1-2^{1-2k})},$$

which can be rearranged as

$$N\left(\frac{2^{2k}}{2^{2N}} - 1\right) > \frac{\pi^2}{2}(k-1)\left(1 + \frac{2}{2^{2k}-2}\right), \qquad k \geq N+2.$$

Noting that $\pi^2/2 < 5$, it is enough to prove the following inequality:

$$N\left(\frac{2^{2k}}{2^{2N}} - 1\right) > 5(k-1)\left(1 + \frac{2}{2^{2k}-2}\right), \qquad k \geq N+2,$$

which can be rearranged as

$$\frac{N}{2^{2N}}2^{2k} - 5(k-1) > N + \frac{10(k-1)}{2^{2k}-2}, \qquad k \geq N+2.$$

Noting that the sequence

$$\frac{N}{2^{2N}}2^{2k} - 5(k-1)$$

is strictly increasing for $k \geq N+2$, and the sequence

$$\frac{10(k-1)}{2^{2k}-2}$$

is strictly decreasing for $k \geq 2$, it is enough to prove the following inequality:

$$\frac{N}{2^{2N}}2^{2(N+2)} - 5(N+1) > N + \frac{10(N+1)}{2^{2(N+2)}-2},$$

which can be rearranged as

$$(2N-1)2^{2N+3} > 3N, \qquad N \geq 1.$$

Obviously, the last inequality holds. This proves the claim (2.24). From (2.23), we obtain the second inequality in (2.20) with $q_N = N \cdot 2^{2N+3}|B_{2N+2}|/(2N+2)!$.

Write (2.20) as

$$p_N < \frac{\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!}x^{2k+2}}{x^{2N+1}\tan x} < q_N.$$

We find that

$$\lim_{x \to \frac{\pi}{2}} \frac{\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!}x^{2k+2}}{x^{2N+1}\tan x} = 0$$

and

$$\lim_{x \to 0} \frac{\left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} - 2 - \sum_{k=1}^{N-1} \frac{k \cdot 2^{2k+3}|B_{2k+2}|}{(2k+2)!}x^{2k+2}}{x^{2N+1}\tan x} = \frac{N \cdot 2^{2N+3}|B_{2N+2}|}{(2N+2)!}.$$

Hence, the inequality (2.20) holds with the best possible constants given in (2.21). The proof of Theorem 2.3 is complete. □

3. A Huygens-type inequality

We now turn our attention to the inverted form of the Huygens inequality in (1.2). Using (2.1) and the following expansion (see [10, p. 43]):

$$\csc x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2(2^{2k-1}-1)|B_{2k}|}{(2k)!}x^{2k-1}, \qquad |x| < \pi,$$

we find

$$2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} = 3 + \sum_{k=2}^{\infty} \frac{(2^{2k}-4)|B_{2k}|}{(2k)!}x^{2k}, \qquad |x| < \pi. \tag{3.1}$$

It follows from (3.1) that for every $N \in \mathbb{N}$,

$$\frac{(2^{2N+2}-4)|B_{2N+2}|}{(2N+2)!}x^{2N+2} < 2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} - \left(3 + \sum_{k=2}^{N} \frac{(2^{2k}-4)|B_{2k}|}{(2k)!}x^{2k}\right) \tag{3.2}$$

for $0 < |x| < \pi/2$.

In view of (3.2) it is natural to ask: What are the largest number a_N and the smallest number $b_N$such that the inequality

$$a_N x^{2N+2} < 2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} - \left(3 + \sum_{k=2}^{N} \frac{(2^{2k}-4)|B_{2k}|}{(2k)!} x^{2k}\right) < b_N x^{2N+2}$$

holds for $x \in (0, \pi/2)$ and $N \in \mathbb{N}$? Theorem 3.1 answers this question.

Theorem 3.1. *Let $N \geq 1$ be an integer. Then for $0 < |x| < \pi/2$,*

$$a_N x^{2N+2} < 2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} - \left(3 + \sum_{j=2}^{N} \frac{(2^{2j}-4)|B_{2j}|}{(2j)!} x^{2j}\right) < b_N x^{2N+2} \tag{3.3}$$

with the best possible constants

$$a_N = \frac{(2^{2N+2}-4)|B_{2N+2}|}{(2N+2)!} \tag{3.4}$$

and

$$b_N = \frac{8}{\pi^{2N+2}}\left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2N}(2k-1)} - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{2N}(2k+1)}\right) - \frac{4}{\pi^{2N+2}}\left(\sum_{k=1}^{\infty} \frac{1}{k^{2N}(2k-1)} - \sum_{k=1}^{\infty} \frac{1}{k^{2N}(2k+1)}\right). \tag{3.5}$$

Proof. By [6, Theorems 4, 5], we have

$$2\left(\frac{x}{\sin x}\right) = 2 + \sum_{j=1}^{N} \frac{(2^{2j+1}-4)|B_{2j}|}{(2j)!} x^{2j} + x^{2N+2} \sum_{k=1}^{\infty} \frac{4(-1)^{k+1}}{(k\pi)^{2N}\left((k\pi)^2 - x^2\right)} \tag{3.6}$$

and

$$\frac{x}{\tan x} = 1 - \sum_{j=1}^{N} \frac{2^{2j}|B_{2j}|}{(2j)!} x^{2j} - x^{2N+2} \sum_{k=1}^{\infty} \frac{2}{(k\pi)^{2N}\left((k\pi)^2 - x^2\right)}. \tag{3.7}$$

Adding these two expressions, we obtain

$$2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} = 3 + \sum_{j=2}^{N} \frac{(2^{2j}-4)|B_{2j}|}{(2j)!} x^{2j} + \frac{2x^{2N+2}}{\pi^{2N}} U_N(x),$$

where

$$U_N(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2 - (-1)^{k+1}}{k^{2N}\left((k\pi)^2 - x^2\right)}.$$

Differentiation yields

$$\frac{U_N'(x)}{2x} = \sum_{k=1}^{\infty} (-1)^{k+1} \alpha_k, \quad \alpha_k = \frac{2 - (-1)^{k+1}}{k^{2N}\left((k\pi)^2 - x^2\right)^2}. \tag{3.8}$$

Then it is easily seen that $\alpha_k > \alpha_{k+1}$ for $k \in \mathbb{N}$, $0 < x < \pi/2$ and $N \in \mathbb{N}$; thus for every $N \geq 1$, we have $U_N'(x) > 0$ for $0 < x < \pi/2$. Hence, for all $0 < x < \pi/2$ and $N \in \mathbb{N}$, we have

$$U_N(0) < U_N(x) < U_N\left(\frac{\pi}{2}\right).$$

Using (2.5), we find

$$a_N = \frac{2U_N(0)}{\pi^{2N}} = 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^{2N+2}} - 2\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{2N+2}}$$
$$= \frac{4(2^{2N+1}-1)}{(2N+2)!}|B_{2N+2}| - \frac{2\cdot 2^{2N+1}}{(2N+2)!}|B_{2N+2}| = \frac{(2^{2N+2}-4)|B_{2N+2}|}{(2N+2)!}$$

and

$$b_N = \frac{2U_N(\pi/2)}{\pi^{2N}} = 4\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^{2N}\big((k\pi)^2-(\pi/2)^2\big)} - 2\sum_{k=1}^{\infty}\frac{1}{(k\pi)^{2N}\big((k\pi)^2-(\pi/2)^2\big)}$$
$$= \frac{8}{\pi^{2N+2}}\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2N}(2k-1)} - \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^{2N}(2k+1)}\right)$$
$$- \frac{4}{\pi^{2N+2}}\left(\sum_{k=1}^{\infty}\frac{1}{k^{2N}(2k-1)} - \sum_{k=1}^{\infty}\frac{1}{k^{2N}(2k+1)}\right).$$

The proof of Theorem 3.1 is complete. □

Clearly,

$$a_1 = \frac{1}{60} \quad \text{and} \quad a_2 = \frac{1}{504}.$$

Direct computations yield

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2(2k-1)} = \pi - 2\ln 2 - \frac{\pi^2}{12}, \quad \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2(2k+1)} = 4 - 2\ln 2 - \pi + \frac{\pi^2}{12},$$

$$\sum_{k=1}^{\infty}\frac{1}{k^2(2k-1)} = -\frac{\pi^2}{6} + 4\ln 2, \quad \sum_{k=1}^{\infty}\frac{1}{k^2(2k+1)} = -4 + 4\ln 2 + \frac{\pi^2}{6},$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4(2k-1)} = 4\pi - 8\ln 2 - \frac{\pi^2}{3} - \frac{3}{2}\zeta(3) - \frac{7\pi^4}{720},$$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^4(2k+1)} = 16 - 4\pi - 8\ln 2 + \frac{\pi^2}{3} - \frac{3}{2}\zeta(3) + \frac{7\pi^4}{720},$$

$$\sum_{k=1}^{\infty}\frac{1}{k^4(2k-1)} = 16\ln 2 - \frac{2\pi^2}{3} - 2\zeta(3) - \frac{\pi^4}{90},$$

$$\sum_{k=1}^{\infty}\frac{1}{k^4(2k+1)} = -16 + 16\ln 2 + \frac{2\pi^2}{3} - 2\zeta(3) + \frac{\pi^4}{90},$$

where $\zeta(s)$ is the Riemann zeta function. Then, we obtain from (3.5)

$$b_1 = \frac{16(\pi-3)}{\pi^4} \quad \text{and} \quad b_2 = \frac{960\pi - \pi^4 - 2880}{15\pi^6}.$$

From (3.3), we have, for $0 < |x| < \pi/2$,

$$3 + \frac{1}{60}x^4 < 2\left(\frac{x}{\sin x}\right) + \frac{x}{\tan x} < 3 + \frac{16(\pi-3)}{\pi^4}x^4, \tag{3.9}$$

where the constants $\frac{1}{60}$ and $16(\pi-3)/\pi^4$ are the best possible, and

$$3+\frac{1}{60}x^4+\frac{1}{504}x^6<2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}<3+\frac{1}{60}x^4+\frac{960\pi-\pi^4-2880}{15\pi^6}x^6, \tag{3.10}$$

where the constants $\frac{1}{504}$ and $(960\pi-\pi^4-2880)/(15\pi^6)$ are the best possible.

The formula (3.1) motivated us to establish Theorem 3.2.

Theorem 3.2. *Let $N\geq 1$ be an integer. Then for $0<x<\pi/2$, we have*

$$3+\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}+\rho_N x^{2N+1}\tan x<2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}$$
$$<3+\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}+\varrho_N x^{2N+1}\tan x \tag{3.11}$$

with the best possible constants

$$\rho_N=0 \quad \text{and} \quad \varrho_N=\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}. \tag{3.12}$$

Proof. By (3.1), for $\rho_N=0$, the first inequality in (3.11) holds. We now prove the second inequality in (3.11) with $\varrho_N=4(2^{2N}-1)|B_{2N+2}|/(2N+2)!$. Using the expansion [10, p. 44]

$$\tan t=\sum_{k=1}^{\infty}\frac{8t}{\pi^2(2k-1)^2-4t^2} \tag{3.13}$$

and (3.1), we find

$$\begin{aligned}
&\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}x^{2N+1}\tan x-\left(2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}-3-\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}\right)\\
&=\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}x^{2N+1}\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!}x^{2k-1}-\sum_{k=N+1}^{\infty}\frac{(2^{2k}-4)|B_{2k}|}{(2k)!}x^{2k}\\
&=\sum_{k=N+2}^{\infty}\left\{\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}\frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!}-\frac{(2^{2k}-4)|B_{2k}|}{(2k)!}\right\}x^{2k}.
\end{aligned} \tag{3.14}$$

We claim that for $k\geq N+2$,

$$\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}\frac{2^{2k-2N}(2^{2k-2N}-1)|B_{2k-2N}|}{(2k-2N)!}>\frac{(2^{2k}-4)|B_{2k}|}{(2k)!}. \tag{3.15}$$

Using the inequality (2.25), it is sufficient to prove that

$$\frac{4(2^{2N}-1)\cdot 2}{(2\pi)^{2N+2}}\frac{2^{2k-2N}(2^{2k-2N}-1)\cdot 2}{(2\pi)^{2k-2N}}>\frac{(2^{2k}-4)\cdot 2}{(2\pi)^{2k}\left(1-2^{1-2k}\right)}, \qquad k\geq N+2,$$

which can be rearranged as

$$\left(1-\frac{1}{2^{2N}}\right)\left(\frac{2^{2k}}{2^{2N}}-1\right)>\frac{\pi^2}{2}\left(1-\frac{2}{2^{2k}-2}\right), \qquad k\geq N+2.$$

Noting that $\pi^2/2<5$, it is enough to prove the following inequality:

$$\left(1-\frac{1}{2^{2N}}\right)\left(\frac{2^{2k}}{2^{2N}}-1\right)>5\left(1-\frac{2}{2^{2k}-2}\right), \qquad k\geq N+2,$$

which can be written as

$$\left(1-\frac{1}{2^{2N}}\right)\frac{2^{2k}}{2^{2N}}+\frac{1}{2^{2N}}+\frac{10}{2^{2k}-2}>6, \qquad k\geq N+2.$$

It is enough to prove the following inequality:

$$\left(1-\frac{1}{2^{2N}}\right)\frac{2^{2k}}{2^{2N}}+\frac{1}{2^{2N}}>6, \qquad k\geq N+2. \tag{3.16}$$

Clearly,

$$\left(1-\frac{1}{2^{2N}}\right)\frac{2^{2k}}{2^{2N}}+\frac{1}{2^{2N}}\geq\left(1-\frac{1}{2^{2N}}\right)\frac{2^{2N+4}}{2^{2N}}+\frac{1}{2^{2N}}=16-\frac{15}{2^{2N}}, \qquad k\geq N+2.$$

In order to prove (3.16), it suffices to show that

$$16-\frac{15}{2^{2N}}>6, \qquad N\geq 1,$$

that is,

$$2^{2N+1}>3, \qquad N\geq 1.$$

Obviously, the last inequality holds. This proves the claim (3.15). From (3.14), we obtain the second inequality in (3.11) with $\varrho_N=4(2^{2N}-1)|B_{2N+2}|/(2N+2)!$.

Write (3.11) as

$$\rho_N<\frac{2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}-3-\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}}{x^{2N+1}\tan x}<\varrho_N.$$

We find

$$\lim_{x\to\frac{\pi}{2}}\frac{2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}-3-\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}}{x^{2N+1}\tan x}=0$$

and

$$\lim_{x\to 0}\frac{2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}-3-\sum_{j=2}^{N}\frac{(2^{2j}-4)|B_{2j}|}{(2j)!}x^{2j}}{x^{2N+1}\tan x}=\frac{4(2^{2N}-1)|B_{2N+2}|}{(2N+2)!}.$$

Hence, the inequality (3.11) holds with the best possible constants given in (3.12). The proof of Theorem 3.2 is complete. $\square$

Remark 3.1. *For $0<|x|<\pi/2$, we have*

$$3+ax^3\tan x<2\left(\frac{x}{\sin x}\right)+\frac{x}{\tan x}<3+bx^3\tan x \tag{3.17}$$

with the best possible constants

$$a=0 \quad \text{and} \quad b=\frac{1}{60}. \tag{3.18}$$

There is no strict comparison between the two upper bounds in (3.9) *and* (3.17).

4. The Papenfuss-Bach inequality

In this section we consider refinements to the Papenfuss-Bach inequality stated in (1.7) and answer the open problem in (1.9).

It follows from [10, p. 44] that

$$\sec^2\frac{\pi x}{2}=\frac{4}{\pi^2}\sum_{k=1}^{\infty}\left\{\frac{1}{(2k-1-x)^2}+\frac{1}{(2k-1+x)^2}\right\}.$$

Replacement of x by $2t/\pi$ yields

$$\sec^2 t=\frac{4}{\pi^2}\sum_{k=1}^{\infty}\left\{\frac{1}{(2k-1-\frac{2t}{\pi})^2}+\frac{1}{(2k-1+\frac{2t}{\pi})^2}\right\}. \tag{4.1}$$

From (3.13) and (4.1), we have

$$t\sec^2 t-\tan t=\frac{64t^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4\left(1-\left(\frac{2t}{\pi(2k-1)}\right)^2\right)^2}. \tag{4.2}$$

Using (2.12) and the third summation in (2.5), we obtain from (4.2) the series representation of the remainder in the expansion for $\sec^2 t-\tan t/t$:

$$\begin{aligned}&t\sec^2 t-\tan t\\&=\frac{64t^3}{\pi^4}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^4}\left(\sum_{j=1}^{N-1}j\left(\frac{2t}{\pi(2k-1)}\right)^{2j-2}+\frac{N\left(\frac{2t}{\pi(2k-1)}\right)^{2N-2}}{1-\left(\frac{2t}{\pi(2k-1)}\right)^2}+\frac{\left(\frac{2t}{\pi(2k-1)}\right)^{2N}}{\left(1-\left(\frac{2t}{\pi(2k-1)}\right)^2\right)^2}\right)\\&=\sum_{j=1}^{N-1}\frac{2j\cdot 2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!}t^{2j+1}+\kappa_N(t),\end{aligned} \tag{4.3}$$

where

$$\begin{aligned}\kappa_N(t)=&\frac{N\cdot 2^{2N+4}t^{2N+1}}{\pi^{2N}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)}\\&+\frac{2^{2N+6}t^{2N+3}}{\pi^{2N}}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^{2N}\left(\pi^2(2k-1)^2-4t^2\right)^2}.\end{aligned} \tag{4.4}$$

Theorem 4.1. *Let $N\geq 1$ be an integer. Then for $0<t<\pi/2$, we have*

$$\begin{aligned}L_N(t)<t\sec^2 t-\tan t&-\sum_{j=1}^{N-1}\frac{2j\cdot 2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!}t^{2j+1}\\&-\frac{N\cdot 2^{2N+4}t^{2N+1}}{\pi^{2N}(\pi^2-4t^2)}-\frac{2^{2N+6}t^{2N+3}}{\pi^{2N}(\pi^2-4t^2)^2}<M_N(t),\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}L_N(t)=&\frac{N\cdot 2^{2N+4}t^{2N+1}}{\pi^{2N+2}}\left\{\frac{(2^{2N+2}-1)\pi^{2N+2}|B_{2N+2}|}{2\cdot(2N+2)!}-1\right\}\\&+\frac{2^{2N+6}t^{2N+3}}{\pi^{2N+4}}\left\{\frac{(2^{2N+4}-1)\pi^{2N+4}|B_{2N+4}|}{2\cdot(2N+4)!}-1\right\}\end{aligned}$$

and

$$M_N(t) = \frac{N \cdot 2^{2N+2} t^{2N+1}}{\pi^{2N+2}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N} k(k-1)} + \frac{2^{2N+2} t^{2N+3}}{\pi^{2N+4}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N} k^2 (k-1)^2}.$$

Proof. Write (4.3) as

$$\begin{aligned} t \sec^2 t - \tan t = {} & \sum_{j=1}^{N-1} \frac{2j \cdot 2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!} t^{2j+1} \\ & + \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}(\pi^2 - 4t^2)} + \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}} I_N(t) \\ & + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}(\pi^2 - 4t^2)^2} + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}} J_N(t), \end{aligned} \tag{4.6}$$

where

$$I_N(t) = \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N} \left(\pi^2 (2k-1)^2 - 4t^2\right)}$$

and

$$J_N(t) = \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N} \left(\pi^2 (2k-1)^2 - 4t^2\right)^2}.$$

Obviously, $I_N(t)$ and $J_N(t)$ are both strictly increasing for $t \in (0, \pi/2)$. We then obtain from (4.6) that

$$\begin{aligned} & \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}} I_N(0) + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}} J_N(0) \\ & \quad < t \sec^2 t - \tan t - \sum_{j=1}^{N-1} \frac{2j \cdot 2^{2j+2}(2^{2j+2}-1)|B_{2j+2}|}{(2j+2)!} t^{2j+1} \\ & \qquad - \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}(\pi^2 - 4t^2)} - \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}(\pi^2 - 4t^2)^2} \\ & \quad < \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}} I_N\left(\frac{\pi}{2}\right) + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}} J_N\left(\frac{\pi}{2}\right). \end{aligned}$$

Direct computations yield

$$\begin{aligned} L_N(t) &= \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N}} I_N(0) + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N}} J_N(0) \\ &= \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N+2}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N+2}} + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N+4}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N+4}} \\ &= \frac{N \cdot 2^{2N+4} t^{2N+1}}{\pi^{2N+2}} \left\{ \frac{(2^{2N+2}-1)\pi^{2N+2}|B_{2N+2}|}{2 \cdot (2N+2)!} - 1 \right\} \\ & \quad + \frac{2^{2N+6} t^{2N+3}}{\pi^{2N+4}} \left\{ \frac{(2^{2N+4}-1)\pi^{2N+4}|B_{2N+4}|}{2 \cdot (2N+4)!} - 1 \right\} \end{aligned}$$

and

$$\begin{aligned} M_N(t) &= \frac{N\cdot 2^{2N+4}t^{2N+1}}{\pi^{2N}} I_N\left(\frac{\pi}{2}\right) + \frac{2^{2N+6}t^{2N+3}}{\pi^{2N}} J_N\left(\frac{\pi}{2}\right) \\ &= \frac{N\cdot 2^{2N+2}t^{2N+1}}{\pi^{2N+2}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N}k(k-1)} \\ &\quad + \frac{2^{2N+2}t^{2N+3}}{\pi^{2N+4}} \sum_{k=2}^{\infty} \frac{1}{(2k-1)^{2N}k^2(k-1)^2}. \end{aligned}$$

The proof of Theorem 4.1 is complete. □

With the evaluations

$$\sum_{k=2}^{\infty} \frac{1}{(2k-1)^4 k(k-1)} = 9 - \frac{\pi^4}{24} - \frac{\pi^2}{2}$$

and

$$\sum_{k=2}^{\infty} \frac{1}{(2k-1)^4 k^2(k-1)^2} = -59 + \frac{13\pi^2}{3} + \frac{\pi^4}{6},$$

the choice $N=2$ in (4.5) yields

$$\frac{P(x)}{\left(\pi^2-4x^2\right)^2} < x\sec^2 x - \tan x < \frac{Q(x)}{\left(\pi^2-4x^2\right)^2}, \quad 0<x<\frac{\pi}{2}, \tag{4.7}$$

where

$$\begin{aligned} P(x) &= \frac{2\pi^4}{3}x^3 + \frac{8\pi^2(\pi^2-10)}{15}x^5 + \frac{2(322560+1680\pi^4-672\pi^6+17\pi^8)}{315\pi^4}x^7 \\ &\quad + \frac{16(168-17\pi^2)}{315}x^9 + \frac{32(17\pi^8-161280)}{315\pi^8}x^{11} \end{aligned}$$

and

$$\begin{aligned} Q(x) &= \frac{2\pi^4}{3}x^3 + \frac{32(156-6\pi^2-\pi^4)}{3\pi^2}x^5 + \frac{64(-657+37\pi^2+3\pi^4)}{3\pi^4}x^7 \\ &\quad + \frac{512(285-19\pi^2-\pi^4)}{3\pi^6}x^9 + \frac{512(-354+26\pi^2+\pi^4)}{3\pi^8}x^{11}. \end{aligned}$$

The inequality (4.7) is an improvement on the inequality (1.9).

Remark 4.1. *In fact, the lower bound in* (4.7) *is larger than the one in* (1.9)*, and the upper bound in* (4.7) *is smaller than the one in* (1.9)*. Hence, the inequality* (1.9) *holds true. If we write* (1.9) *as*

$$\frac{8\pi^4}{15} - \frac{16\pi^2}{3} < \frac{(x\sec^2 x - \tan x)(\pi^2-4x^2)^2 - \frac{2\pi^4}{3}x^3}{x^5} < \frac{256}{\pi^2} - \frac{8\pi^2}{3},$$

we find that

$$\lim_{x\to 0} \frac{(x\sec^2 x - \tan x)(\pi^2-4x^2)^2 - \frac{2\pi^4}{3}x^3}{x^5} = \frac{8\pi^4}{15} - \frac{16\pi^2}{3}$$

and

$$\lim_{x\to \pi/2} \frac{(x\sec^2 x - \tan x)(\pi^2-4x^2)^2 - \frac{2\pi^4}{3}x^3}{x^5} = \frac{256}{\pi^2} - \frac{8\pi^2}{3}.$$

Hence, the inequality (1.9) *holds for* $0 < x < \pi/2$, *and the constants* $\frac{8\pi^4}{15} - \frac{16\pi^2}{3}$ *and* $\frac{256}{\pi^2} - \frac{8\pi^2}{3}$ *are the best possible.*

Appendix: A proof of (2.18)

For $N = 0$ in (2.18), we find that

$$\sum_{k=1}^{\infty} \frac{1}{\big(4k^2-1\big)^2} = \frac{\pi^2-8}{16} \quad \text{and} \quad \frac{1}{8}\psi'\left(\frac{1}{2}\right) - \frac{1}{2} = \frac{\pi^2-8}{16}.$$

This shows that the formula (2.18) holds for $N = 0$.

Now we assume that the formula (2.18) holds for some $N \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Then, for $N \mapsto N+1$ in (2.18), by using the induction hypothesis and the following relation:

$$\psi'(z+1) = \psi'(z) - \frac{1}{z^2},$$

we have

$$\begin{aligned}
\sum_{k=N+2}^{\infty} \frac{1}{\big(4k^2-1\big)^2} &= \sum_{k=N+1}^{\infty} \frac{1}{\big(4k^2-1\big)^2} - \frac{1}{\big(4(N+1)^2-1\big)^2} \\
&= \frac{1}{8}\psi'\left(N+\frac{1}{2}\right) - \frac{N+1}{2(2N+1)^2} - \frac{1}{\big(4(N+1)^2-1\big)^2} \\
&= \frac{1}{8}\psi'\left(N+\frac{1}{2}\right) - \frac{1}{8(N+\frac{1}{2})^2} - \frac{N+2}{2\big(2N+3\big)^2} \\
&= \frac{1}{8}\psi'\left(N+\frac{3}{2}\right) - \frac{N+2}{2\big(2N+3\big)^2}.
\end{aligned}$$

Thus, by the principle of mathematical induction, the formula (2.18) holds for all $N \in \mathbb{N}_0$.

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