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HYPO-q-NORMS ON A CARTESIAN PRODUCT OF NORMED LINEAR SPACES

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Abstract. In this paper we introduce the hypo-q-norms on a Cartesian product of normed linear spaces. A representation of these norms in terms of bounded linear functionals of norm less than one, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}(\|\cdot\|_n) := \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n | \, \|\lambda\|_n \leq 1\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ \\ \left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [6], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B(\|\cdot\|_n)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.3}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

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It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
(ii) $\|\mathbf{x}+\mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$;
(iii) $\|\alpha\mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0)$, $(0, \lambda_2^0, \dots, 0)$, ..., $(0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$, $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right. \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\boldsymbol{\lambda} \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.6}$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the $n-$tuples of vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{1.7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{1.8}$$

The following result established in [6] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [6]). *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\| \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2, \tag{1.9}$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [6]). *For any* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \dots, x_n)$, *we have*

$$\|\mathbf{x}\|_e = \sup_{\|x\| \leq 1} \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}. \tag{1.10}$$

Motivated by the above results, in this paper we introduce the hypo-q-norms on a Cartesian product of normed linear spaces. A representation of these norms in terms of bounded linear functionals of norm less than one, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

2. General Results

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\| \leq 1} |f(x)| < \infty, \text{ where } f \in E^*.$$

We have the following representation result for the *hypo-q-norms* on E^n.

Theorem 3. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$. *For any* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \dots, x_n)$, *we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|f\| \leq 1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^q \right)^{1/q} \right\} \tag{2.1}$$

where $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|f\| \leq 1} \left\{ \sum_{j=1}^{n} |f(x_j)| \right\} \tag{2.2}$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1,\dots,n\}} \{\|x_j\|\}. \tag{2.3}$$

In particular,

$$\|\mathbf{x}\|_{h,e} = \sup_{\|f\| \leq 1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^2 \right)^{1/2} \right\}. \tag{2.4}$$

Proof. Using Hölder's discrete inequality for $p,\ q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_p \le 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \le \|\beta\|_q \tag{2.5}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j} \left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\begin{aligned}\left| \sum_{j=1}^{n} \alpha_j \beta_j \right| &= \left| \sum_{j=1}^{n} \frac{\overline{\beta_j} \left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}} \beta_j \right| = \frac{\sum_{j=1}^{n} \left|\beta_j\right|^q}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}} \\ &= \left(\sum_{j=1}^{n} \left|\beta_j\right|^q \right)^{1/q} = \|\beta\|_q\end{aligned}$$

and

$$\begin{aligned}\|\alpha\|_p^p &= \sum_{j=1}^{n} |\alpha_j|^p = \sum_{j=1}^{n} \frac{\left|\overline{\beta_j} \left|\beta_j\right|^{q-2}\right|^p}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)} = \sum_{j=1}^{n} \frac{\left(\left|\beta_j\right|^{q-1}\right)^p}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)} \\ &= \sum_{j=1}^{n} \frac{\left|\beta_j\right|^{qp-p}}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)} = \sum_{j=1}^{n} \frac{\left|\beta_j\right|^{q}}{\left(\sum_{k=1}^{n} |\beta_k|^q\right)} = 1.\end{aligned}$$

Therefore, by (2.5) we have the representation

$$\sup_{\|\alpha\|_p \le 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_q \tag{2.6}$$

for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$.

By Hahn-Banach theorem, we have for any $u \in E$, $u \neq 0$ that

$$\|u\| = \sup_{\|f\| \le 1} |f(u)|. \tag{2.7}$$

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ and $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then by (2.7) we have

$$\left\| \sum_{j=1}^{n} \alpha_j x_j \right\| = \sup_{\|f\| \le 1} \left| f\left(\sum_{j=1}^{n} \alpha_j x_j \right) \right| = \sup_{\|f\| \le 1} \left| \sum_{j=1}^{n} \alpha_j f(x_j) \right|. \tag{2.8}$$

By taking the supremum in this equality we have

$$\sup_{\|\alpha\|_p\le 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \sup_{\|\alpha\|_p\le 1}\left(\sup_{\|f\|\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right)$$
$$= \sup_{\|f\|\le 1}\left(\sup_{\|\alpha\|_p\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right) = \sup_{\|f\|\le 1}\left(\sum_{j=1}^{n}\left|f(x_j)\right|^q\right)^{1/2},$$

where for the last equality we used the representation (2.6).

This proves (2.1).

Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \max_{j\in\{1,\dots,n\}}|\alpha_j|\sum_{j=1}^{n}\left|\beta_j\right|$$

which implies that

$$\sup_{\|\alpha\|_\infty\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_1 \tag{2.9}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{\left|\beta_j\right|}\beta_j\right| = \sum_{j=1}^{n}\left|\beta_j\right| = \|\beta\|_1$$

and

$$\|\alpha\|_\infty = \max_{j\in\{1,\dots,n\}}|\alpha_j| = \max_{j\in\{1,\dots,n\}}\left|\frac{\overline{\beta_j}}{\left|\beta_j\right|}\right| = 1$$

and by (2.9) we get the representation

$$\sup_{\|\alpha\|_\infty\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_1 \tag{2.10}$$

for any $\beta=(\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) we have

$$\sup_{\|\alpha\|_\infty\le 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| = \sup_{\|\alpha\|_\infty\le 1}\left(\sup_{\|f\|\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right)$$
$$= \sup_{\|f\|\le 1}\left(\sup_{\|\alpha\|_\infty\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right) = \sup_{\|f\|\le 1}\left(\sum_{j=1}^{n}\left|f(x_j)\right|\right),$$

where for the last equality we used the equality (2.10), which proves the representation (2.2).

Finally, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \sum_{j=1}^{n}|\alpha_j| \max_{j\in\{1,\dots,n\}}\left|\beta_j\right|$$

which implies that

$$\sup_{\|\alpha\|_1\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_\infty \tag{2.11}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\ne 0$, let $j_0\in\{1,...,n\}$ such that $\|\beta\|_\infty = \max_{j\in\{1,\dots,n\}}\left|\beta_j\right| = \left|\beta_{j_0}\right|$. Consider $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}$ and $\alpha_j = 0$ for $j\ne j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j| = \frac{\left|\overline{\beta_{j_0}}\right|}{\left|\beta_{j_0}\right|} = 1 \text{ and } \left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}\beta_{j_0}\right| = \left|\beta_{j_0}\right| = \|\beta\|_\infty,$$

therefore by (2.11) we obtain the representation

$$\sup_{\|\alpha\|_1\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \|\beta\|_\infty \tag{2.12}$$

for any $\beta = (\beta_1,\dots,\beta_n)\in\mathbb{K}^n$.

By taking the supremum in the equality (2.8) and by using the equality (2.12), we have

$$\begin{aligned}
\sup_{\|\alpha\|_1\le 1}\left\|\sum_{j=1}^{n}\alpha_j x_j\right\| &= \sup_{\|\alpha\|_1\le 1}\left(\sup_{\|f\|\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right)\\
&= \sup_{\|f\|\le 1}\left(\sup_{\|\alpha\|_1\le 1}\left|\sum_{j=1}^{n}\alpha_j f(x_j)\right|\right) = \sup_{\|f\|\le 1}\left(\max_{j\in\{1,\dots,n\}}|f(x_j)|\right)\\
&= \max_{j\in\{1,\dots,n\}}\left(\sup_{\|f\|\le 1}|f(x_j)|\right) = \max_{j\in\{1,\dots,n\}}\{\|x_j\|\},
\end{aligned}$$

which proves (2.3). For the last equality we used the property (2.7). □

Corollary 1. *With the assumptions of Theorem 3 we have for $q\ge 1$ that*

$$\frac{1}{n^{1/q}}\|\mathbf{x}\|_{n,q} \le \|\mathbf{x}\|_{h,n,q} \le \|\mathbf{x}\|_{n,q} \tag{2.13}$$

for any any $\mathbf{x}\in E^n$.

In particular, we have

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2 \le \|\mathbf{x}\|_{h,e} \le \|\mathbf{x}\|_2 \tag{2.14}$$

for any any $\mathbf{x}\in E^n$.

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $f \in E^*$ with $\|f\| \le 1$, then for $q \ge 1$

$$\left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \le \left(\sum_{j=1}^{n} (\|f\| |x_i|)^q\right)^{1/q} = \|f\| \left(\sum_{j=1}^{n} |x_i|^q\right)^{1/q} = \|f\| \|\mathbf{x}\|_{n,q}$$

and by taking the supremum over $\|f\| \le 1$, we get the second inequality in (2.13).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\ldots,n\}} \{|f(x_j)|\} \le \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q}$$

and by taking the supremum over $\|f\| \le 1$, we get

$$\sup_{\|f\|\le 1} \left(\max_{j\in\{1,\ldots,n\}} \{|f(x_j)|\}\right) \le \sup_{\|f\|\le 1} \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \tag{2.15}$$

and since

$$\begin{aligned}\sup_{\|f\|\le 1} \left(\max_{j\in\{1,\ldots,n\}} \{|f(x_j)|\}\right) &= \max_{j\in\{1,\ldots,n\}} \left\{\sup_{\|f\|\le 1} |f(x_j)|\right\} \\ &= \max_{j\in\{1,\ldots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty},\end{aligned}$$

then by (2.15) we get

$$\|\mathbf{x}\|_{n,\infty} \le \|\mathbf{x}\|_{h,n,q} \text{ for any } \mathbf{x} \in E^n. \tag{2.16}$$

Since

$$\left(\sum_{j=1}^{n} \|x_j\|^q\right)^{1/q} \le \left(n \|\mathbf{x}\|_{n,\infty}^q\right)^{1/q} = n^{1/q} \|\mathbf{x}\|_{n,\infty}$$

then also

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \le \|\mathbf{x}\|_{n,\infty} \text{ for any } \mathbf{x} \in E^n. \tag{2.17}$$

By utilising the inequalities (2.16) and (2.17) we obtain the first inequality in (2.13). □

Remark 1. *In the case of inner product spaces the inequality (2.14) has been obtained in a different and more difficult way* [6] *by employing the rotation-invariant normalised positive Borel measure on the unit sphere.*

Corollary 2. *With the assumptions of Theorem 3 we have for $r \ge q \ge 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \le \|\mathbf{x}\|_{h,n,q} \le n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r} \tag{2.18}$$

for any any $\mathbf{x} \in E^n$.

In particular, for $q \ge 2$ we have

$$\|\mathbf{x}\|_{h,n,q} \le \|\mathbf{x}\|_{h,e} \le n^{\frac{q-2}{2q}} \|\mathbf{x}\|_{h,n,q} \tag{2.19}$$

and for $1 \le q \le 2$ we have

$$\|\mathbf{x}\|_{h,e} \le \|\mathbf{x}\|_{h,n,q} \le n^{\frac{2-q}{2q}} \|\mathbf{x}\|_{h,e} \tag{2.20}$$

for any any $\mathbf{x} \in E^n$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1, ..., n$ and $r \geq q > 0$ (see for instance [8])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{2.21}$$

Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and $f \in E^*$ with $\|f\| \leq 1$, then for $r \geq q \geq 1$ we have

$$\left(\sum_{j=1}^{n} |f(x_j)|^r\right)^{1/r} \leq \left(\sum_{j=1}^{n} |f(x_j)|^q\right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^{n} |f(x_j)|^r\right)^{1/r}. \tag{2.22}$$

By taking the supremum over $f \in E^*$ with $\|f\| \leq 1$ and using Theorem 3, we get (2.18). □

Remark 2. *If we take $q = 1$ in (2.18), then we get*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|\mathbf{x}\|_{h,n,r} \tag{2.23}$$

for any any $\mathbf{x} \in E^n$.

In particular, for $r = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,1} \leq \sqrt{n} \|\mathbf{x}\|_{h,e} \tag{2.24}$$

for any any $\mathbf{x} \in E^n$.

3. Some Reverse Inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [4] (see also [5, Theorem 5. 14])

Lemma 1. *Let a, $A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \quad \text{for each} \quad j \in \{1, \ldots, n\}. \tag{3.1}$$

Then for any $\mathbf{w} = (w_1, \ldots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j\right)^2 \leq \frac{1}{4}(A-a)^2 \left(\sum_{j=1}^{n} w_j y_j^2\right)^2. \tag{3.2}$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [9]) the following counterparts of (CBS)- inequality (see also [5, Theorem 5.20 & 5.21])

Lemma 2. *Assume that $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ are such that there exists a, A, b, B with the property that:*

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \ldots, n\} \tag{3.3}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}}\right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{3.4}$$

and

Lemma 3. *Assume that* $\mathbf{a}$, $\mathbf{b}$ *are nonnegative sequences and there exists* γ, Γ *with the property that*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any} \quad j \in \{1, \ldots, n\}. \tag{3.5}$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{3.6}$$

We have the following result:

Theorem 4. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$. *Then we have*

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2, \tag{3.7}$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty} \tag{3.8}$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|\mathbf{x}\|_{n,\infty}. \tag{3.9}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and put $R = \max_{j \in \{1,\ldots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $f \in E^*$ with $\|f\| \leq 1$ then $|f(x_j)| \leq \|f\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

If we write the inequality (3.2) for $z_j = |f(x_j)|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^{n} |f(x_j)|^2 - \left(\sum_{j=1}^{n} |f(x_j)| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $f \in E^*$ with $\|f\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |f(x_j)|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |f(x_j)| \right)^2 + \frac{1}{4} n R^2 \tag{3.10}$$

for any $f \in E^*$ with $\|f\| \leq 1$.

By taking the supremum in (3.10) over $f \in E^*$ with $\|f\| \leq 1$ we get (3.7).

If we write the inequality (3.4) for $a_j = |f(x_j)|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^{n} |f(x_j)|^2 - \left(\sum_{j=1}^{n} |f(x_j)| \right)^2 \leq n R \sum_{j=1}^{n} |f(x_j)|,$$

for any $f \in E^*$ with $\|f\| \leq 1$.

This implies that

$$\sum_{j=1}^{n} |f(x_j)|^2 \leq \frac{1}{n} \left(\sum_{j=1}^{n} |f(x_j)| \right)^2 + R \sum_{j=1}^{n} |f(x_j)|, \tag{3.11}$$

for any $f \in E^*$ with $\|f\| \leq 1$.

By taking the supremum in (3.11) over $f \in E^*$ with $\|f\| \leq 1$ we get (3.8).

Finally, if we write the inequality (3.6) for $a_j = |f(x_j)|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \leq \left(n \sum_{j=1}^{n} |f(x_j)|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} |f(x_j)| \leq \frac{1}{4} nR,$$

for any $f \in E^*$ with $\|f\| \leq 1$.

This implies that

$$\left(\sum_{j=1}^{n} |f(x_j)|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |f(x_j)| + \frac{1}{4} \sqrt{n} R, \tag{3.12}$$

for any $f \in E^*$ with $\|f\| \leq 1$.

By taking the supremum in (3.12) over $f \in E^*$ with $\|f\| \leq 1$ we get (3.9). □

Further, we recall the *Čebyšev's inequality* for *synchronous* n-tuples of vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, namely if $(a_j - a_k)(b_j - b_k) \geq 0$ for any j, $k \in \{1, ..., n\}$, then

$$\frac{1}{n} \sum_{j=1}^{n} a_j b_j \geq \frac{1}{n} \sum_{j=1}^{n} a_j \frac{1}{n} \sum_{j=1}^{n} b_j. \tag{3.13}$$

In 1950, Biernacki et al. obtained the following discrete version of Grüss' inequality

Lemma 4. *Assume that* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *are such that there exists real numbers* a, A, b, B *with the property that:*

$$a \leq a_j \leq A \quad \text{and} \quad b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\}. \tag{3.14}$$

Then

$$\begin{aligned} &\left| \frac{1}{n} \sum_{j=1}^{n} a_j b_j - \frac{1}{n} \sum_{j=1}^{n} a_j \frac{1}{n} \sum_{j=1}^{n} b_j \right| \\ &\leq \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \left(1 - \frac{1}{n} \left\lceil \frac{n}{2} \right\rceil \right) (A - a)(B - b) \\ &= \frac{1}{n^2} \left\lceil \frac{n^2}{4} \right\rceil (A - a)(B - a) \leq \frac{1}{4} (A - a)(B - b), \end{aligned} \tag{3.15}$$

where $\lceil x \rceil$ *gives the largest integer less than or equal to* x.

The following result also holds:

Theorem 5. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \dots, x_n)$. *Then for* q, $r \geq 1$ *we have*

$$\begin{aligned} \|\mathbf{x}\|_{h,n,q+r}^{q+r} &\leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n,\infty}^{q+r} \\ &\leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} + \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^{q+r}. \end{aligned} \tag{3.16}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j \in \{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $f \in E^*$ with $\|f\| \le 1$ then $|f(x_j)| \le \|f\| \|x_j\| \le R$ for any $j \in \{1,...,n\}$.

If we take into the inequality (3.15) $a_j = |f(x_j)|^q$, $b_j = |f(x_j)|^r$, $a = 0$, $A = R^q$, $b = 0$ and $B = R^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^{q+r} - \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^q \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^r \right| \le \frac{1}{n^2} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r}. \tag{3.17}$$

On the other hand, since the sequences $\{a_j\}_{j=1,\dots,n}$, $\{b_j\}_{j=1,\dots,n}$ are synchronous, then by (3.13) we have

$$0 \le \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^{q+r} - \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^q \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^r.$$

Using (3.17) we then get

$$\sum_{j=1}^{n} |f(x_j)|^{q+r} \le \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^q \sum_{j=1}^{n} |f(x_j)|^r + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \tag{3.18}$$

for any $f \in E^*$ with $\|f\| \le 1$.

By taking the supremum in (3.18), we get

$$\begin{aligned} &\sup_{\|f\| \le 1} \left\{ \sum_{j=1}^{n} |f(x_j)|^{q+r} \right\} \\ &\le \frac{1}{n} \sup_{\|f\| \le 1} \left\{ \sum_{j=1}^{n} |f(x_j)|^q \sum_{j=1}^{n} |f(x_j)|^r \right\} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \\ &\le \frac{1}{n} \sup_{\|f\| \le 1} \left\{ \sum_{j=1}^{n} |f(x_j)|^q \right\} \sup_{\|f\| \le 1} \left\{ \sum_{j=1}^{n} |f(x_j)|^r \right\} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r}, \end{aligned}$$

which proves the first inequality in (3.16).

The second part of (3.16) is obvious. □

Corollary 3. *With the assumptions of Theorem 5 and if* $r \ge 1$, *then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \le \frac{1}{n} \|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n,\infty}^{2r} \le \frac{1}{n} \|\mathbf{x}\|_{h,n,r}^{2r} + \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^{2r}. \tag{3.19}$$

In particular, for $r = 1$ *we get*

$$\|\mathbf{x}\|_{h,e}^{2} \le \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n,\infty}^{2} \le \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^{2} + \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^{2}. \tag{3.20}$$

The first inequality in (3.20) is better than the second inequality in (3.7).

4. Reverse Inequalities Via Forward Difference

For an n-tuple of complex numbers $\mathbf{a} = (a_1, \dots, a_n)$ with $n \ge 2$ consider the $(n-1)$-tuple built by the aid of forward differences $\Delta\mathbf{a} = (\Delta a_1, ..., \Delta a_{n-1})$ where $\Delta a_k := a_{k+1} - a_k$ where $k \in \{1, ..., n-1\}$. Similarly, if $\mathbf{x} = (x_1, \dots, x_n) \in E^n$ is an n-tuple of vectors we also can consider in a similar way the $(n-1)$-tuple $\Delta\mathbf{x} = (\Delta x_1, ..., \Delta x_{n-1})$.

We obtained the following Grüss' type inequalities in terms of forward differences:

Lemma 5. *Assume that* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{b} = (b_1, \dots, b_n)$ *are* n*-tuples of complex numbers. Then*

$$\left| \frac{1}{n} \sum_{j=1}^{n} a_j b_j - \frac{1}{n} \sum_{j=1}^{n} a_j \frac{1}{n} \sum_{j=1}^{n} b_j \right| \tag{4.1}$$
$$\leq \begin{cases} \frac{1}{12}\left(n^2-1\right) \|\Delta \mathbf{a}\|_{n-1,\infty} \|\Delta \mathbf{b}\|_{n-1,\infty}, & [7], \\ \frac{1}{6}\frac{n^2-1}{n} \|\Delta \mathbf{a}\|_{n-1,\alpha} \|\Delta \mathbf{b}\|_{n-1,\beta} \text{ where } \alpha,\ \beta > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, & [2], \\ \frac{1}{2}\left(1-\frac{1}{n}\right) \|\Delta \mathbf{a}\|_{n-1,1} \|\Delta \mathbf{b}\|_{n-1,1}, & [3]. \end{cases}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ *and* $\frac{1}{2}$ *are best possible in (4.1).*

The following result also holds:

Theorem 6. *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$ *and* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \dots, x_n)$. *Then for* q, $r \geq 1$ *we have*

$$\|\mathbf{x}\|_{h,n,q+r}^{q+r} \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^{q} \|\mathbf{x}\|_{h,n,r}^{r} \tag{4.2}$$
$$+ \begin{cases} \frac{1}{12} qr\left(n^2-1\right) n \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{n-1,\infty}^{2}, \\ \frac{1}{6}\left(n^2-1\right) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h,n-1,\alpha} \|\Delta \mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \frac{1}{2}(n-1) qr \|\mathbf{x}\|_{n,\infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h,n-1,1}^{2}. \end{cases}$$

Proof. Let $\mathbf{x} \in E^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $f \in E^*$ with $\|f\| \leq 1$. If we take into the inequality (4.1) $a_j = |f(x_j)|^q$, $b_j = |f(x_j)|^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^{q+r} - \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^q \frac{1}{n} \sum_{j=1}^{n} |f(x_j)|^r \right| \tag{4.3}$$
$$\leq \begin{cases} \frac{1}{12}\left(n^2-1\right) \max_{j=1,\dots,n-1} \left|\Delta |f(x_j)|^q\right| \max_{j=1,\dots,n-1} \left|\Delta |f(x_j)|^r\right|, \\ \frac{1}{6}\frac{n^2-1}{n} \left(\sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^q\right|^{\alpha}\right)^{1/\alpha} \left(\sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^r\right|^{\beta}\right)^{1/\beta} \\ \text{where } \alpha,\ \beta > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \frac{1}{2}\left(1-\frac{1}{n}\right) \sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^q\right| \sum_{j=1}^{n-1} \left|\Delta |f(x_j)|^r\right|. \end{cases}$$

We use the following elementary inequality for powers $p \geq 1$

$$|a^p - b^p| \leq pR^{p-1} |a - b|$$

where $a, b \in [0, R]$.

Put $R = \max_{j \in \{1,\dots,n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. Then for any $f \in E^*$ with $\|f\| \leq 1$ we have $|f(x_j)| \leq \|f\| \|x_j\| \leq R$ for any $j \in \{1, ..., n\}$.

Therefore

$$\left|\Delta |f(x_j)|^q\right| = \left||f(x_{j+1})|^q - |f(x_j)|^q\right| \leq qR^{q-1} \left||f(x_{j+1})| - |f(x_j)|\right| \tag{4.4}$$
$$\leq qR^{q-1} |f(x_{j+1}) - f(x_j)| = qR^{q-1} |f(\Delta x_j)|$$

for any $j=1,...,n-1$, where $\Delta x_j = x_{j+1}-x_j$ is the forward difference.

On the other hand, since the sequences $\{a_j\}_{j=1,\dots,n}$, $\{b_j\}_{j=1,\dots,n}$ are synchronous, then we have

$$0 \le \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q+r} - \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\,\frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{r} \tag{4.5}$$

and by the first inequality in (4.3) we get

$$\begin{aligned}
&\sum_{j=1}^{n}|f(x_j)|^{q+r} \\
&\le \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r} \\
&+ \frac{1}{12}\left(n^2-1\right) nqR^{q-1}\max_{j=1,\dots,n-1}|f(\Delta x_j)|\, rR^{r-1}\max_{j=1,\dots,n-1}|f(\Delta x_j)| \\
&= \frac{1}{n}\sum_{j=1}^{n}|f(x_j)|^{q}\sum_{j=1}^{n}|f(x_j)|^{r} \\
&+ \frac{1}{12}\left(n^2-1\right) nqrR^{q+r-2}\left(\max_{j=1,\dots,n-1}|f(\Delta x_j)|\right)^2
\end{aligned} \tag{4.6}$$

for any $f\in E^*$ with $\|f\|\le 1$.

Taking the supremum over $f\in E^*$ with $\|f\|\le 1$ in (4.6) we get the first branch in the inequality (4.2).

We also have, by (4.4), that

$$\begin{aligned}
\left(\sum_{j=1}^{n-1}\left|\Delta|f(x_j)|^q\right|^{\alpha}\right)^{1/\alpha} &\le \left[\left(qR^{q-1}\right)^{\alpha}\sum_{j=1}^{n-1}|f(\Delta x_j)|^{\alpha}\right]^{1/\alpha} \\
&= qR^{q-1}\left(\sum_{j=1}^{n-1}|f(\Delta x_j)|^{\alpha}\right)^{1/\alpha}
\end{aligned}$$

and, similarly,

$$\left(\sum_{j=1}^{n-1}\left|\Delta|f(x_j)|^r\right|^{\beta}\right)^{1/\beta} \le rR^{r-1}\left(\sum_{j=1}^{n-1}|f(\Delta x_j)|^{\beta}\right)^{1/\beta}$$

where $\alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1$.

By the second inequality in (4.3) and by (4.5) we have

$$\begin{aligned}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q+r} & \\ \leq \frac{1}{n}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{r} & \\ +\frac{1}{6}\left(n^{2}-1\right)\left(\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{q}\right|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{r}\right|^{\beta}\right)^{1/\beta} & \\ \leq \frac{1}{n}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{r} & \\ +\frac{1}{6}\left(n^{2}-1\right)qrR^{q+r-2}\left(\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|^{\alpha}\right)^{1/\alpha}\left(\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|^{\beta}\right)^{1/\beta} &\end{aligned}\tag{4.7}$$

for any $f\in E^{*}$ with $\|f\|\leq 1$, where α, $\beta>1$, $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

Taking the supremum over $f\in E^{*}$ with $\|f\|\leq 1$ in (4.7) we get the second branch in the inequality (4.2).

We also have, by (4.4), that

$$\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{q}\right|\leq qR^{q-1}\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|$$

and

$$\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{r}\right|\leq rR^{r-1}\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|.$$

By the third inequality in (4.3) and by (4.5) we have

$$\begin{aligned}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q+r} & \leq \frac{1}{n}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{r} \\ & +\frac{1}{2}\left(n-1\right)\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{q}\right|\sum_{j=1}^{n-1}\left|\Delta\left|f\left(x_{j}\right)\right|^{r}\right| \\ & \leq \frac{1}{n}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{q}\sum_{j=1}^{n}\left|f\left(x_{j}\right)\right|^{r} \\ & +\frac{1}{2}\left(n-1\right)qrR^{q+r-2}\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|\sum_{j=1}^{n-1}\left|f\left(\Delta x_{j}\right)\right|\end{aligned}\tag{4.8}$$

for any $f\in E^{*}$ with $\|f\|\leq 1$.

Taking the supremum over $f\in E^{*}$ with $\|f\|\leq 1$ in (4.8) we get the third branch in the inequality (4.2). □

Corollary 4. *With the assumptions of Theorem 6 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h,n,2r}^{2r} \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,r}^{2r} + \begin{cases} \frac{1}{12}r^2\left(n^2-1\right)n\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6}r^2\left(n^2-1\right)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}r^2(n-1)\|\mathbf{x}\|_{n,\infty}^{2r-2}\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{4.9}$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^2 \leq \frac{1}{n}\|\mathbf{x}\|_{h,n,1}^2 + \begin{cases} \frac{1}{12}\left(n^2-1\right)n\|\Delta\mathbf{x}\|_{n-1,\infty}^2, \\ \\ \frac{1}{6}\left(n^2-1\right)\|\Delta\mathbf{x}\|_{h,n-1,\alpha}\|\Delta\mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha,\ \beta>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1, \\ \\ \frac{1}{2}(n-1)\|\Delta\mathbf{x}\|_{h,n-1,1}^2. \end{cases} \tag{4.10}$$

References

[1] M. Biernacki, H. Pidek and C. Ryll-Nardzewski, Sur une inégalité entre des intégrales définies. (French) *Ann. Univ. Mariae Curie-Skłodowska. Sect. A.* **4**, (1950). 1–4.

[2] S. S. Dragomir, Another Grüss type inequality for sequences of vectors in normed linear spaces and applications, *J. Comp. Analysis & Appl.,* **4**(2) (2002), 157-172.

[3] S. S. Dragomir, A Grüss type inequality for sequences of vectors in normed linear spaces, *Tamsui Oxf. J. Math. Sci.*, **20**(2) (2004), 143-159.

[4] S. S. Dragomir, A counterpart of Schwarz's inequality in inner product spaces, *East Asian Math. J.*, **20** (1) (2004), 1-10. Preprint *RGMIA Res. Rep. Coll.* **6** (2003), Supplement, Art. 18. [Online http://rgmia.org/papers/v6e/CSIIPS.pdf].

[5] S. S. Dragomir, A survey on Cauchy-Bunyakovsky-Schwarz type discrete inequalities. *J. Inequal. Pure Appl. Math.* **4** (2003), no. 3, Article 63, 142 pp. [Online https://www.emis.de/journals/JIPAM/article301.html?sid=301].

[6] S. S. Dragomir, The hypo-Euclidean norm of an n-tuple of vectors in inner product spaces and applications. *J. Inequal. Pure Appl. Math.* 8 (2007), No. 2, Article 52, 22 pp. [Online https://www.emis.de/journals/JIPAM/article854.html?sid=854].

[7] S. S. Dragomir and G. L. Booth, On a Grüss-Lupaş type inequality and its applications for the estimation of p-moments of guessing mappings, *Math. Comm.,* **5** (2000), 117-126.

[8] A. Sheikhhosseini, M. S. Moslehian and K. Shebrawi, Inequalities for generalized Euclidean operator radius via Young's inequality. *J. Math. Anal. Appl.* **445** (2017), no. 2, 1516–1529.

[9] O. Shisha and B. Mond, Bounds on differences of means, Inequalities, Academic Press Inc., New York, 1967, 293-308.

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