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SOME INEQUALITIES FOR SEMI-INNER PRODUCTS ON COMPLEX BANACH SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. In this paper we establish some inequalities for two sub-semi-inner products $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2$ for which $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$ that are related to Schwarz's inequality. Some applications for particular semi-inner products generated by an element of norm one in the given Banach spaces as well as for some bounded operators that satisfy a Schwarz's type condition are given. Some norm and numerical radius inequalities for operators acting on smooth uniformly convex Banach spaces with more sugestive examples in the case of complex Hilbert spaces are also provided.

1. Introduction

In what follows, we assume that X is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [7] but the main properties of it were discovered by J. R. Giles [8], P. L. Papini [14], P. M. Miličić [10]–[12], I. Roşca [15], B. Nath [13] and others.

In this introductory section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 1. *The mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ will be called the semi-inner product in the sense of Lumer-Giles or L-G-s.i.p., for short, if the following properties are satisfied:*

(i) $[x+y,z] = [x,z] + [y,z]$ *for all* $x,\ y,\ z \in X$*;*
(ii) $[\lambda x,y] = \lambda [x,y]$ *for all* $x,\ y \in X$ *and* λ *a scalar in* $\mathbb{K}$*;*
(iii) $[x,x] \geq 0$ *for all* $x \in X$ *and* $[x,x] = 0$ *implies that* $x = 0$*;*
(iv) $|[x,y]|^2 \leq [x,x]\,[y,y]$ *for all* $x,\ y \in X$*;*
(v) $[x,\lambda y] = \bar{\lambda}\,[x,y]$ *for all* $x,\ y \in X$ *and* λ *a scalar in* $\mathbb{K}$*.*

The following results collects some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 1. *Let X be a linear space and $[\cdot,\cdot]$ a L-G-s.i.p on X. Then the following statements are true:*

(i) *The mapping* $X \ni x \xrightarrow{\|\cdot\|} [x,x]^{\frac{1}{2}} \in \mathbb{R}_+$ *is a norm on* X*;*
(ii) *For every* $y \in X$ *the functional* $X \ni x \xrightarrow{f_y} [x,y] \in \mathbb{K}$ *is a continuous linear functional on* X *endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality* $\|f_y\| = \|y\|$*.*

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Definition 2. *The mapping $J : X \to 2^{X^*}$, where X^* is the dual space of X, given by:*

$$J(x) := \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \ \ \|x^*\| = \|x\|\}, \ \ x \in X$$

will be called the normalised duality mapping of normed linear space $(X, \|\cdot\|)$.

Definition 3. *A mapping $\tilde{J} : X \to X^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in X.*

The following theorem due to I. Roşca [15] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for all } x, y \text{ in } X,$$

where $\tilde{J}$ is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *X is smooth;*
(ii) *There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.*

Before we can state a remarkable result due to J. R. Giles [8] that contains a classical characterization of smooth normed spaces, we need the following definition.

Definition 4. *A L-G-s.i.p. $[\cdot, \cdot]$ defined on the linear space X is said to be continuous* [8]*, if for every x, $y \in X$ one has the equality:*

$$\lim_{t \to 0} \operatorname{Re}[y, x + ty] = \operatorname{Re}[y, x]. \tag{1.1}$$

Now we can state the following well known result [8].

Theorem 2. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot, \cdot]$ a L-G-s.i.p. which generates the norm $\|\cdot\|$. Then $[\cdot, \cdot]$ is continuous if and only if the space X is smooth.*

As shown in [2, p. 22 - p. 23], the condition (1.1) can be relaxed to:

(ii') *The following limit exists:*

$$\lim_{t \to 0} \operatorname{Re}[y, x + ty]$$

for all x, $y \in X$.

Another result of this type is embodied in the following theorem [2, p. 23 - p. 24].

Theorem 3. *Let $(X, \|\cdot\|)$ be a normed space and $[\cdot, \cdot]$ a L-G-s.i.p. which generates the norm $\|\cdot\|$. Then the following statements are equivalent:*

(i) *X is smooth;*
(ii) *The following limit exists:*

$$\lim_{t \to 0} \frac{\operatorname{Re}[x, x + ty] - \|x\|^2}{t} \tag{1.2}$$

for all x, $y \in X$.

Moreover, if (i) or (ii) hold, then one has the equality

$$\lim_{t\to 0}\frac{\operatorname{Re}[x,x+ty]-\|x\|^2}{t}=\operatorname{Re}[y,x] \tag{1.3}$$

for all $x,\ y\in X$.

Further on, we will state a result due to Nath [13] containing a characterization of strictly convex spaces in terms of semi-inner product in Lumer-Giles' sense.

Theorem 4. *Let $(X,\|\cdot\|)$ be a normed linear space and $[\cdot,\cdot]$ a L-G-s.i.p. which generates its norm. Then the following statements are equivalent:*

- (i) *X is strictly convex;*
- (ii) *For every $x,\ y\in X$, $x,\ y\neq 0$ so that $[x,y]=\|x\|\,\|y\|$ there exists a positive number λ with $x=\lambda y$.*

A mapping $[\cdot,\cdot]:X\times X\to\mathbb{K}$ will be called a *sub-semi-inner product in the sense of Lumer-Giles* or s-L-G-*s.i.p.*, for short, if in the Definition 1 instead of condition (iii) we have the relaxed condition:

(iii') $[x,x]\geq 0$ *for all* $x\in X$.

We denote by $\mathcal{SS}(X)$ the class of all sub-semi-inner products defined on the linear space X.

We can introduce the following order relation amongst the elements of $\mathcal{SS}(X)$. For $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2\in\mathcal{SS}(X)$, we say that

$$[\cdot,\cdot]_2\succeq[\cdot,\cdot]_1 \ \text{ iff } \ [\cdot,\cdot]_2-[\cdot,\cdot]_1\in\mathcal{SS}(X). \tag{1.4}$$

In this paper we establish some inequalities for two sub-semi-inner products $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2$ for which $[\cdot,\cdot]_2\succeq[\cdot,\cdot]_1$ that are related to Schwarz's inequality. Some applications for particular semi-inner products generated by an element of norm one in the given Banach spaces as well as for some bounded operators that satisfy a Schwarz's type condition are given. Some norm and numerical radius inequalities for operators acting on smooth uniformly convex Banach spaces with more suggestive examples in the case of complex Hilbert spaces are also provided.

2. Inequalities for Semi-Inner Products

We have the following simple fact related to the structure of $\mathcal{SS}(X)$.

Proposition 3. *$\mathcal{SS}(X)$ is a cone in the linear space of all functions defined on $X\times X$.*

Proof. We observe that if $[\cdot,\cdot]\in\mathcal{SS}(X)$ and $\alpha\geq 0$, then obviously $\alpha[\cdot,\cdot]\in\mathcal{SS}(X)$.

Now, assume that $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2\in\mathcal{SS}(X)$. In order to prove that $[\cdot,\cdot]_1+[\cdot,\cdot]_2\in\mathcal{SS}(X)$, we must only prove that $[\cdot,\cdot]_1+[\cdot,\cdot]_2$ satisfies the Schwarz's inequality (iv) in the Definition 1.

Since $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2\in\mathcal{SS}(X)$, then for any $x,\ y\in X$ we have

$$\left|[x,y]_1\right|\leq[x,x]_1^{1/2}[y,y]_1^{1/2} \ \text{ and } \ \left|[x,y]_2\right|\leq[x,x]_2^{1/2}[y,y]_2^{1/2}.$$

If we add these inequalities we get

$$\left|[x,y]_1\right|+\left|[x,y]_2\right|\leq[x,x]_1^{1/2}[y,y]_1^{1/2}+[x,x]_2^{1/2}[y,y]_2^{1/2}$$

for any $x,\ y\in X$.

By the triangle inequality for the modulus, we have

$$\left|[x,y]_1+[x,y]_2\right|\leq\left|[x,y]_1\right|+\left|[x,y]_2\right|$$

for any $x,\ y\in X$.

By the elementary inequality

$$ab+cd\leq\left(a^2+c^2\right)^{1/2}\left(b^2+d^2\right)^{1/2}\text{ for }a,\ b,\ c,\ d\geq 0$$

we have

$$[x,x]_1^{1/2}[y,y]_1^{1/2}+[x,x]_2^{1/2}[y,y]_2^{1/2}\leq\left([x,x]_1+[x,x]_2\right)^{1/2}\left([y,y]_1+[y,y]_2\right)^{1/2}$$

for any $x,\ y\in X$.

Therefore

$$\left|[x,y]_1+[x,y]_2\right|\leq\left([x,x]_1+[x,x]_2\right)^{1/2}\left([y,y]_1+[y,y]_2\right)^{1/2}$$

for any $x,\ y\in X$, which proves that $[\cdot,\cdot]_1+[\cdot,\cdot]_2\in\mathcal{SS}(X)$. □

We can state the following result connected with the Schwarz's inequality:

Theorem 5. *Let $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2\in\mathcal{SS}(X)$ with $[\cdot,\cdot]_2\succeq[\cdot,\cdot]_1$. Then for any $x,\ y\in X$ we have*

$$\begin{aligned}(2.1)\quad &\|x\|_2^2\|y\|_2^2-\left|[x,y]_2\right|^2+\|x\|_1^2\|y\|_1^2-\left|[x,y]_1\right|^2-\left[\det\begin{pmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_2 & \|y\|_2\end{pmatrix}\right]^2\\ &\geq 2\left(\|x\|_1\|y\|_1\|x\|_2\|y\|_2-\left|[x,y]_1[x,y]_2\right|\right)\geq 0.\end{aligned}$$

In particular, we have

$$\begin{aligned}(2.2)\quad &\|x\|_2^2\|y\|_2^2-\left|[x,y]_2\right|^2+\|x\|_1^2\|y\|_1^2-\left|[x,y]_1\right|^2\\ &\geq\left[\det\begin{pmatrix}\|x\|_1 & \|y\|_1\\ \|x\|_2 & \|y\|_2\end{pmatrix}\right]^2\geq 0\end{aligned}$$

and

$$\begin{aligned}(2.3)\quad &\frac{1}{2}\left[\|x\|_2^2\|y\|_2^2-\left|[x,y]_2\right|^2+\|x\|_1^2\|y\|_1^2-\left|[x,y]_1\right|^2\right]\\ &\geq\|x\|_1\|y\|_1\|x\|_2\|y\|_2-\left|[x,y]_1[x,y]_2\right|\geq 0.\end{aligned}$$

Proof. Since $[\cdot,\cdot]_2-[\cdot,\cdot]_1\in\mathcal{SS}(X)$, then by Schwarz's inequality we have for any $x,\ y\in X$ that

$$(2.4)\quad\left(\|x\|_2^2-\|x\|_1^2\right)\left(\|y\|_2^2-\|y\|_1^2\right)\geq\left|[x,y]_2-[x,y]_1\right|^2.$$

Observe that

$$\left|[x,y]_2-[x,y]_1\right|^2=\left|[x,y]_2\right|^2-2\operatorname{Re}\left[[x,y]_2\overline{[x,y]_1}\right]+\left|[x,y]_1\right|^2$$

and

$$\left(\|x\|_2^2-\|x\|_1^2\right)\left(\|y\|_2^2-\|y\|_1^2\right)=\|x\|_2^2\|y\|_2^2+\|x\|_1^2\|y\|_1^2-\|x\|_1^2\|y\|_2^2-\|x\|_2^2\|y\|_1^2$$

and by (2.4) we get

$$\begin{aligned}&\|x\|_2^2\|y\|_2^2+\|x\|_1^2\|y\|_1^2-\|x\|_1^2\|y\|_2^2-\|x\|_2^2\|y\|_1^2\\ &\geq\left|[x,y]_2\right|^2-2\operatorname{Re}\left[[x,y]_2\overline{[x,y]_1}\right]+\left|[x,y]_1\right|^2\end{aligned}$$

for any x, $y \in X$, which is equivalent to

$$\begin{aligned}
&\|x\|_2^2 \|y\|_2^2 - |[x,y]_2|^2 + \|x\|_1^2 \|y\|_1^2 - |[x,y]_1|^2 \\
&\geq \|x\|_1^2 \|y\|_2^2 + \|x\|_2^2 \|y\|_1^2 - 2 \operatorname{Re}\left[[x,y]_2 \overline{[x,y]_1}\right] \\
&= \|x\|_1^2 \|y\|_2^2 + \|x\|_2^2 \|y\|_1^2 - 2 \|x\|_1 \|y\|_2 \|x\|_2 \|y\|_1 \\
&+ 2 \|x\|_1 \|y\|_2 \|x\|_2 \|y\|_1 - 2 \operatorname{Re}\left[[x,y]_2 \overline{[x,y]_1}\right] \\
&= (\|x\|_1 \|y\|_2 - \|x\|_2 \|y\|_1)^2 \\
&+ 2\left(\|x\|_1 \|y\|_1 \|x\|_2 \|y\|_2 - \operatorname{Re}\left[[x,y]_2 \overline{[x,y]_1}\right]\right)
\end{aligned} \tag{2.5}$$

for any x, $y \in X$.

By the properties of the modulus we have

$$\operatorname{Re}\left[[x,y]_2 \overline{[x,y]_1}\right] \leq \left|\operatorname{Re}\left[[x,y]_2 \overline{[x,y]_1}\right]\right| \leq |[x,y]_1| \, |[x,y]_2|$$

and by (2.5) we get

$$\begin{aligned}
&\|x\|_2^2 \|y\|_2^2 - |[x,y]_2|^2 + \|x\|_1^2 \|y\|_1^2 - |[x,y]_1|^2 \\
&\geq (\|x\|_1 \|y\|_2 - \|x\|_2 \|y\|_1)^2 \\
&+ 2 (\|x\|_1 \|y\|_1 \|x\|_2 \|y\|_2 - |[x,y]_1| \, |[x,y]_2|)
\end{aligned}$$

for any x, $y \in X$.

Since by Schwarz's inequality for each of the sub-semi-inner products we have

$$\|x\|_1 \|y\|_1 \geq |[x,y]_1| \text{ and } \|x\|_2 \|y\|_2 \geq |[x,y]_2|,$$

then

$$\|x\|_1 \|y\|_1 \|x\|_2 \|y\|_2 - |[x,y]_1| \, |[x,y]_2| \geq 0,$$

which proves the last part of (2.1). □

For $[\cdot,\cdot] \in \mathcal{SS}(X)$ and $e \in X$, $e \neq 0$ we consider the functional $[\cdot,\cdot]_e : X \times X \to \mathbb{K}$ defined by

$$[x,y]_e = [x,e] \overline{[y,e]} \text{ for } x, \ y \in X.$$

We observe that $[\cdot,\cdot]_e$ is linear in the first variable, anti-homogeneous in the second variable and

$$[x,x]_e = |[x,e]|^2 \geq 0 \text{ for any } x \in X.$$

Also, we have

$$|[x,y]_e|^2 = \left|[x,e] \overline{[y,e]}\right|^2 = |[x,e]|^2 \, |[y,e]|^2 = [x,x]_e \, [y,y]_e \, ,$$

for $x, y \in X$, which shows that the Schwarz' s inequality (iv) is verified with equality. Therefore we conclude that $[\cdot,\cdot]_e \in \mathcal{SS}(X)$.

Corollary 1. *Let $[\cdot,\cdot] \in \mathcal{SS}(X)$ and $e \in X$ with the associated norm $\|e\| = 1$. Assume that $[\cdot,\cdot] \succeq [\cdot,\cdot]_e$, then for any x, $y \in X$ we have*

$$\begin{aligned}
&\left(\frac{1}{2}\left[\|x\| \|y\| + |[x,y]|\right] - |[x,e]\,[y,e]|\right)(\|x\| \|y\| - |[x,y]|) \\
&\geq \frac{1}{2}\left[\det\begin{pmatrix} \|x\| & \|y\| \\ |[x,e]| & |[y,e]| \end{pmatrix}\right]^2.
\end{aligned} \tag{2.6}$$

In particular, we have

$$\|x\|^2\|y\|^2 - |[x,y]|^2 \geq \left[\det\begin{pmatrix} \|x\| & \|y\| \\ |[x,e]| & |[y,e]| \end{pmatrix}\right]^2 \geq 0 \tag{2.7}$$

and

$$\frac{1}{2}\left[\|x\|\|y\| + |[x,y]|\right] \geq |[x,e][y,e]| \tag{2.8}$$

for any x, $y \in X$.

Proof. If we write the inequality (2.1) for $[\cdot,\cdot]_2 = [\cdot,\cdot]$ and $[\cdot,\cdot]_1 = [\cdot,\cdot]_e$, then we get

$$\|x\|^2\|y\|^2 - |[x,y]|^2 - (|[x,e]|\|y\| - \|x\||[y,e]|)^2$$

$$\geq 2|[x,e]||[y,e]|(\|x\|\|y\| - |[x,y]|) \geq 0,$$

for any x, $y \in X$.

This is equivalent to

$$\begin{aligned}&\|x\|^2\|y\|^2 - |[x,y]|^2 - 2|[x,e]||[y,e]|(\|x\|\|y\| - |[x,y]|)\\ &\geq (|[x,e]|\|y\| - \|x\||[y,e]|)^2,\end{aligned}$$

which proves the desired result (2.6).

The inequality (2.7) follows by (2.2) for $[\cdot,\cdot]_2 = [\cdot,\cdot]$ and $[\cdot,\cdot]_1 = [\cdot,\cdot]_e$.

From (2.3) we get for $[\cdot,\cdot]_2 = [\cdot,\cdot]$ and $[\cdot,\cdot]_1 = [\cdot,\cdot]_e$ that

$$\frac{1}{2}\left[\|x\|^2\|y\|^2 - |[x,y]|^2\right] \geq |[x,e]||[y,e]|(\|x\|\|y\| - |[x,y]|) \geq 0,$$

namely

$$\begin{aligned}&\frac{1}{2}\left[\|x\|\|y\| + |[x,y]|\right](\|x\|\|y\| - |[x,y]|)\\ &\geq |[x,e]||[y,e]|(\|x\|\|y\| - |[x,y]|) \geq 0,\end{aligned}$$

and, since $\|x\|\|y\| - |[x,y]| \geq 0$, it implies that the inequality (2.8) holds true. $\square$

The following result also holds:

Theorem 6. *Let $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2 \in \mathcal{SS}(X)$ with $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$. Then for any x, $y \in X$ we have*

$$|[x,y]_2 - [x,y]_1| \leq \|x\|_2\|y\|_2 - \|x\|_1\|y\|_1. \tag{2.9}$$

We also have the refinement of Schwarz's inequality

$$\begin{aligned}|[x,y]_2| &\leq |[x,y]_2 - [x,y]_1| + |[x,y]_1|\\ &\leq |[x,y]_2 - [x,y]_1| + \|x\|_1\|y\|_1 \leq \|x\|_2\|y\|_2\end{aligned} \tag{2.10}$$

and the inequality

$$\begin{aligned}|[x,y]_1| &\leq \frac{1}{2}\left[|[x,y]_1 - [x,y]_2| + |[x,y]_1| + |[x,y]_2|\right]\\ &\leq \frac{1}{2}\left[|[x,y]_1 - [x,y]_2| + \|x\|_1\|y\|_1 + |[x,y]_2|\right]\\ &\leq \frac{1}{2}\left[\|x\|_2\|y\|_2 + |[x,y]_2|\right],\end{aligned} \tag{2.11}$$

for any x, $y \in X$.

Proof. We use the following elementary inequality

$$\left(a^2 - c^2\right)\left(b^2 - d^2\right) \le (ab - cd)^2, \tag{2.12}$$

that is equivalent to

$$a^2b^2 + c^2d^2 - c^2b^2 - a^2d^2 \le a^2b^2 - 2abcd + c^2d^2$$

or to $(cb - ad)^2 \ge 0$, which holds for any real numbers a, b, c and d.

If we use the inequality (2.12) then we have

$$\left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) \le \left(\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1\right)^2, \tag{2.13}$$

for any x, $y \in X$.

Also by (2.4) we get

$$\left|[x,y]_2 - [x,y]_1\right|^2 \le \left(\|x\|_2^2 - \|x\|_1^2\right)\left(\|y\|_2^2 - \|y\|_1^2\right) \tag{2.14}$$

for any x, $y \in X$.

From (2.13) and (2.14) we then get

$$\left|[x,y]_2 - [x,y]_1\right|^2 \le \left(\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1\right)^2 \tag{2.15}$$

for any x, $y \in X$.

Since $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$ it follows that $\|x\|_2 \ge \|x\|_1$ and $\|y\|_2 \ge \|y\|_1$ which imply that $\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1 \ge 0$. By taking the square root in (2.15) we get

$$\left|[x,y]_2 - [x,y]_1\right| \le \left|\|x\|_2\|y\|_2 - \|x\|_1\|y\|_1\right| = \|x\|_2\|y\|_2 - \|x\|_1\|y\|_1,$$

which proves the required result (2.9).

We observe that the first inequality (2.10) follows by the triangle inequality for modulus, the second inequality by Schwarz's inequality while the third one is equivalent to (2.9).

From the continuity of modulus we have

$$\left|[x,y]_1\right| - \left|[x,y]_2\right| \le \left|\left|[x,y]_2\right| - \left|[x,y]_1\right|\right| \le \left|[x,y]_2 - [x,y]_1\right|,$$

which implies that

$$\begin{aligned} 2\left|[x,y]_1\right| - \left|[x,y]_2\right| &\le \left|[x,y]_2 - [x,y]_1\right| + \left|[x,y]_1\right| \\ &\le \left|[x,y]_2 - [x,y]_1\right| + \|x\|_1\|y\|_1 \le \|x\|_2\|y\|_2 \end{aligned}$$

for any x, $y \in X$.

If we add in all terms of this inequality the same quantity $\left|[x,y]_2\right|$ then we get

$$\begin{aligned} 2\left|[x,y]_1\right| &\le \left|[x,y]_2 - [x,y]_1\right| + \left|[x,y]_1\right| + \left|[x,y]_2\right| \\ &\le \left|[x,y]_2 - [x,y]_1\right| + \|x\|_1\|y\|_1 + \left|[x,y]_2\right| \le \|x\|_2\|y\|_2 + \left|[x,y]_2\right|, \end{aligned}$$

which by division with 2 produces the desired result (2.11). □

Corollary 2. *Let* $[\cdot,\cdot] \in \mathcal{SS}(X)$ *and* $e \in X$ *with the associated norm* $\|e\| = 1$. *Assume that* $[\cdot,\cdot] \succeq [\cdot,\cdot]_e$, *then for any* x, $y \in X$ *we have*

$$\left|[x,y] - [x,e]\,\overline{[y,e]}\right| \le \|x\|\,\|y\| - \left|[x,e]\,[y,e]\right|, \tag{2.16}$$

$$\left|[x,y]\right| \le \left|[x,y] - [x,e]\,\overline{[y,e]}\right| + \left|[x,e]\,[y,e]\right| \le \|x\|\,\|y\| \tag{2.17}$$

and

$$\left|[x,e]\,[y,e]\right| \le \frac{1}{2}\left[\left|[x,y]-[x,e]\,\overline{[y,e]}\right| + \left|[x,e]\,[y,e]\right| + \left|[x,y]\right|\right] \le \frac{1}{2}\left[\|x\|\,\|y\| + \left|[x,y]\right|\right]. \tag{2.18}$$

3. Inequalities for Schwarz Type Operators

Following [5], on operator A on a complex Banach space $(X, \|\cdot\|)$ is said to be Hermitian if $[Ax, x]$ is real for any $x \in H$, where $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$.

Definition 5. *Let $(X, \|\cdot\|)$ be a complex Banach space and $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$. We say that the Hermitian operator $A : X \to X$ is of Schwarz type related to $[\cdot,\cdot]$ if $[Ax, x] \ge 0$ for any $x \in X$ and*

$$\left|[Ax,y]\right|^2 \le [Ax,x]\,[Ay,y] \tag{3.1}$$

for any x, $y \in X$. We write that $A \in \mathcal{S}_{[\cdot,\cdot]}(X)$.

We observe that the Hermitian operator $A : X \to X$ is of Schwarz type related to $[\cdot,\cdot]$ if and only if the functional $[\cdot,\cdot]_A : X \times X \to \mathbb{C}$, $[x,y]_A := [Ax, y]$ is a s-L-G-s.i.p. on X.

We observe that the identity operator I is of Schwarz type for any s-L-G-s.i.p. $[\cdot,\cdot]$ that generates the norm.

We observe that if A, $B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, then $A + B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ and $\alpha A \in \mathcal{S}_{[\cdot,\cdot]}(X)$ for any $\alpha \ge 0$. This shows that $\mathcal{S}_{[\cdot,\cdot]}(X)$ is a cone in the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators acting on X.

We can define on $\mathcal{S}_{[\cdot,\cdot]}(X)$ the following order relation $A \succeq B$ for A, $B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ if $A - B \in \mathcal{S}_{[\cdot,\cdot]}(X)$. We observe that for A, $B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, $A \succeq B$ iff $[\cdot,\cdot]_A \succeq [\cdot,\cdot]_B$ in the sense of the definition (1.4).

Theorem 7. *Let $(X, \|\cdot\|)$ be a complex Banach space and $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$. Assume that A, $B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ with $A \succeq B$. Then*

$$\begin{aligned}
&[Ax,x]\,[Ay,y] - \left|[Ax,y]\right|^2 + [Bx,x]\,[By,y] - \left|[Bx,y]\right|^2 \\
&\quad - \left[\det\begin{pmatrix} [Bx,x]^{1/2} & [By,y]^{1/2} \\ [Ax,x]^{1/2} & [Ay,y]^{1/2} \end{pmatrix}\right]^2 \\
&\ge 2\left([Bx,x]^{1/2}\,[By,y]^{1/2}\,[Ax,x]^{1/2}\,[Ay,y]^{1/2} - \left|[Ax,y]\,[Bx,y]\right|\right) \ge 0
\end{aligned} \tag{3.2}$$

for any x, $y \in X$.

In particular, we have

$$\begin{aligned}
&[Ax,x]\,[Ay,y] - \left|[Ax,y]\right|^2 + [Bx,x]\,[By,y] - \left|[Bx,y]\right|^2 \\
&\ge \left[\det\begin{pmatrix} [Bx,x]^{1/2} & [By,y]^{1/2} \\ [Ax,x]^{1/2} & [Ay,y]^{1/2} \end{pmatrix}\right]^2 \ge 0
\end{aligned} \tag{3.3}$$

and

$$\frac{1}{2}\left[[Ax,x]\,[Ay,y]-|[Ax,y]|^2+[Bx,x]\,[By,y]-|[Bx,y]|^2\right] \tag{3.4}$$
$$\geq [Bx,x]^{1/2}\,[By,y]^{1/2}\,[Ax,x]^{1/2}\,[Ay,y]^{1/2}-|[Ax,y]\,[Bx,y]|\geq 0.$$

for any $x,\ y\in X$.

We also have

$$|[Ax-Bx,y]|\leq [Ax,x]^{1/2}\,[Ay,y]^{1/2}-[Bx,x]^{1/2}\,[By,y]^{1/2} \tag{3.5}$$

for any $x,\ y\in X$.

Moreover, we have the refinement of Schwarz's inequality

$$|[Ax,y]|\leq |[Ax-Bx,y]|+|[Bx,y]| \tag{3.6}$$
$$\leq |[Ax-Bx,y]|+[Bx,x]^{1/2}\,[By,y]^{1/2}\leq [Ax,x]^{1/2}\,[Ay,y]^{1/2}$$

and the inequality

$$|[Bx,y]|\leq \frac{1}{2}\left[|[Ax-Bx,y]|+|[Bx,y]|+|[Ax,y]|\right] \tag{3.7}$$
$$\leq \frac{1}{2}\left[|[Ax-Bx,y]|+[Bx,x]^{1/2}\,[By,y]^{1/2}+|[Ax,y]|\right]$$
$$\leq \frac{1}{2}\left[[Ax,x]^{1/2}\,[Ay,y]^{1/2}+|[Ax,y]|\right],$$

for any $x,\ y\in X$.

The proof follows by Theorems 5 and 6 for the s-L-G-s.i.p.s $[x,y]_A:=[Ax,y]$ and $[x,y]_B:=[Bx,y]$, $x,\ y\in X$ which satisfy the condition $[\cdot,\cdot]_A\succeq[\cdot,\cdot]_B$.

Corollary 3. *Let* $(X,\|\cdot\|)$ *be a complex Banach space and* $[\cdot,\cdot]$ *a s-L-G-s.i.p. that generates the norm* $\|\cdot\|$. *Assume that* $A\in\mathcal{S}_{[\cdot,\cdot]}(X)$ *and* $\alpha>0$ *with* $A\succeq\alpha I$. *Then*

$$[Ax,x]\,[Ay,y]-|[Ax,y]|^2+\alpha^2\left(\|x\|^2\,\|y\|^2-|[x,y]|^2\right) \tag{3.8}$$
$$-\alpha\left[\det\begin{pmatrix}\|x\| & \|y\|\\ [Ax,x]^{1/2} & [Ay,y]^{1/2}\end{pmatrix}\right]^2$$
$$\geq 2\alpha\left(\|x\|\,\|y\|\,[Ax,x]^{1/2}\,[Ay,y]^{1/2}-|[Ax,y]\,[x,y]|\right)\geq 0$$

for any $x,\ y\in X$.

In particular, we have

$$[Ax,x]\,[Ay,y]-|[Ax,y]|^2+\alpha^2\left(\|x\|^2\,\|y\|^2-|[x,y]|^2\right) \tag{3.9}$$
$$\geq \alpha\left[\det\begin{pmatrix}\|x\| & \|y\|\\ [Ax,x]^{1/2} & [Ay,y]^{1/2}\end{pmatrix}\right]^2\geq 0$$

and

$$\frac{1}{2}\left[[Ax,x]\,[Ay,y]-|[Ax,y]|^2+\alpha^2\left(\|x\|^2\,\|y\|^2-|[x,y]|^2\right)\right] \tag{3.10}$$
$$\geq \alpha\left(\|x\|\,\|y\|\,[Ax,x]^{1/2}\,[Ay,y]^{1/2}-|[Ax,y]\,[x,y]|\right)\geq 0.$$

for any $x,\ y\in X$.

We also have

$$|[Ax-\alpha x,y]| \le [Ax,x]^{1/2}[Ay,y]^{1/2} - \alpha\|x\|\|y\| \tag{3.11}$$

for any $x,\ y \in X$.

Moreover, we have the refinement of Schwarz's inequality

$$\begin{aligned} |[Ax,y]| &\le |[Ax-\alpha x,y]| + \alpha|[x,y]| \\ &\le |[Ax-\alpha x,y]| + \alpha\|x\|\|y\| \le [Ax,x]^{1/2}[Ay,y]^{1/2} \end{aligned} \tag{3.12}$$

and the inequality

$$\begin{aligned} \alpha|[x,y]| &\le \frac{1}{2}[|[Ax-\alpha x,y]| + \alpha|[x,y]| + |[Ax,y]|] \\ &\le \frac{1}{2}[|[Ax-\alpha x,y]| + \alpha\|x\|\|y\| + |[Ax,y]|] \\ &\le \frac{1}{2}\left[[Ax,x]^{1/2}[Ay,y]^{1/2} + |[Ax,y]|\right], \end{aligned} \tag{3.13}$$

for any $x,\ y \in X$.

Let X be a smooth uniformly convex Banach space and let $[\cdot,\cdot]$ be the unique semi-inner-product generating the norm $\|\cdot\|$ of X. If T is a bounded linear operator on X, namely $T \in \mathcal{B}(X)$, then the operator $T^\dagger$ mapping X to X is called the generalized adjoint of T if and only if $[Tx,y] = \left[x, T^\dagger y\right]$ for all x and y in X [6]. We shall call $T^\dagger$ the generalized adjoint of T. In the case X is a Hilbert space the generalized adjoint is the usual Hilbert space adjoint. In the above more general setting this operator is not usually linear.

Corollary 4. *Let X be a smooth uniformly convex Banach space and let $[\cdot,\cdot]$ be the unique semi-inner-product generating the norm $\|\cdot\|$ of X. Assume that $A,\ B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ with $A \succeq B$ and let T a bounded linear operator on X, then*

$$|[TBx,y]| \le \frac{1}{2}\left[[Ax,x]^{1/2}\left[TAT^\dagger y,y\right]^{1/2} + |[TAx,y]|\right], \tag{3.14}$$

for any $x,\ y \in X$.

The inequality (3.14) follows from (3.7) by taking $T^\dagger y$ instead of y.

A fundamental result due to Lumer [7], in the theory of complex Banach spaces, is that if $T \in \mathcal{B}(X)$, then

$$w(T) \le \|T\| \le 4w(T) \tag{3.15}$$

where $w(T) := \sup_{\|x\|=1}|[Tx,x]|$ is the numerical radius of the operator T and $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$. As shown by Glickfeld, the second inequality in (3.15) holds with $e = \exp(1)$ instead of 4 and e is the best possible constant.

Now if $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$, then

$$\sup_{\|y\|=1}|[x,y]| = \|x\| \text{ for any } x \in X. \tag{3.16}$$

If $x = 0$ the equality is obvious. If $x \neq 0$, then by Schwarz's inequality we have

$$|[x,y]| \le \|x\|\|y\| \text{ for any } y \in X.$$

By taking the supremum in this inequality we have

$$\sup_{\|y\|=1}|[x,y]| \le \|x\|.$$

On the other hand by taking $y_0 := \frac{x}{\|x\|}$ we have that $\|y_0\| = 1$ and since

$$\sup_{\|y\|=1} |[x,y]| \geq |[x,y_0]| = \left| \left[x, \frac{x}{\|x\|} \right] \right| = \frac{\|x\|^2}{\|x\|} = \|x\|$$

we get the desired equality (3.16).

Proposition 4. *Let $(X, \|\cdot\|)$ be a complex Banach space and $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$.*

(i) *If $T \in \mathcal{B}(X)$, then*

$$\|T\| = \sup_{\|x\|=\|y\|=1} |[Tx,y]| ; \tag{3.17}$$

(ii) *If $A \in \mathcal{S}_{[\cdot,\cdot]}(X)$, then*

$$\|A\| = w(A) . \tag{3.18}$$

Proof. (i) We have by (3.16) that

$$\sup_{\|x\|=\|y\|=1} |[Tx,y]| = \sup_{\|x\|=1} \left(\sup_{\|y\|=1} |[Tx,y]| \right) = \sup_{\|x\|=1} \|Tx\| = \|T\| .$$

(ii) By (3.1) we have

$$\begin{aligned} \|A\|^2 &= \sup_{\|x\|=\|y\|=1} |[Ax,y]|^2 \leq \sup_{\|x\|=\|y\|=1} ([Ax,x]\,[Ay,y]) \\ &= \sup_{\|x\|=1} ([Ax,x]) \sup_{\|y\|=1} [Ay,y] = w^2(A) , \end{aligned}$$

namely $\|A\| \leq w(A)$. Since the reverse inequality holds for any operator, we obtain the desired result (3.18). □

Corollary 5. *Let X be a smooth uniformly convex Banach space and let $[\cdot,\cdot]$ be the unique semi-inner-product generating the norm $\|\cdot\|$ of X. Assume that $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ with $A \succeq B$ and let T a bounded linear operator on X, then*

$$\|TBx\| \leq \frac{1}{2} \left[[Ax,x]^{1/2} w^{1/2} \left(TAT^{\dagger} \right) + \|TAx\| \right] , \tag{3.19}$$

for any $x \in X$, where $w\left(TAT^{\dagger}\right) := \sup_{\|y\|=1} \left| \left[TAT^{\dagger} y, y \right] \right|$ as in the case of linear operators.

We also have

$$\|TB\| \leq \frac{1}{2} \left[\|A\|^{1/2} w^{1/2} \left(TAT^{\dagger} \right) + \|TA\| \right] \tag{3.20}$$

and

$$w(TB) \leq \frac{1}{2} \left[\|A\|^{1/2} w^{1/2} \left(TAT^{\dagger} \right) + w(TA) \right] . \tag{3.21}$$

Proof. If we take the supremum in the inequality (3.14) then we get by (3.16) that

$$\begin{aligned}\|TBx\| &= \sup_{\|y\|=1} |[TBx,y]| \\ &\le \frac{1}{2}\sup_{\|y\|=1}\left[[Ax,x]^{1/2}\left[TAT^{\dagger}y,y\right]^{1/2} + |[TAx,y]|\right] \\ &\le \frac{1}{2}\sup_{\|y\|=1}\left[[Ax,x]^{1/2}\sup_{\|y\|=1}\left[TAT^{\dagger}y,y\right]^{1/2} + \sup_{\|y\|=1}|[TAx,y]|\right] \\ &= \frac{1}{2}\left[[Ax,x]^{1/2} w^{1/2}\left(TAT^{\dagger}\right) + \|TAx\|\right]\end{aligned}$$

for any $x \in X$, which proves (3.19).

Now, if we take the supremum in (3.19), then we get

$$\begin{aligned}\|TB\| &= \sup_{\|x\|=1}\|TBx\| \\ &\le \frac{1}{2}\sup_{\|x\|=1}\left[[Ax,x]^{1/2} w^{1/2}\left(TAT^{\dagger}\right) + \|TAx\|\right] \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}[Ax,x]^{1/2} w^{1/2}\left(TAT^{\dagger}\right) + \sup_{\|x\|=1}\|TAx\|\right] \\ &= \frac{1}{2}\left[\|A\|^{1/2} w^{1/2}\left(TAT^{\dagger}\right) + \|TA\|\right]\end{aligned}$$

since, by (3.18) $\sup_{\|x\|=1}[Ax,x]^{1/2} = w(A) = \|A\|$.

From (3.14) we have for $y = x$ that

$$|[TBx,x]| \le \frac{1}{2}\left[[Ax,x]^{1/2}\left[TAT^{\dagger}x,x\right]^{1/2} + |[TAx,x]|\right], \tag{3.22}$$

for any $x \in X$.

Taking the supremum in (3.22) we get

$$\begin{aligned}w(TB) &= \sup_{\|x\|=1}|[TBx,x]| \\ &\le \frac{1}{2}\sup_{\|x\|=1}\left[[Ax,x]^{1/2}\left[TAT^{\dagger}x,x\right]^{1/2} + |[TAx,x]|\right] \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}\left([Ax,x]^{1/2}\left[TAT^{\dagger}x,x\right]^{1/2}\right) + \sup_{\|x\|=1}|[TAx,x]|\right] \\ &\le \frac{1}{2}\left[\sup_{\|x\|=1}[Ax,x]^{1/2}\sup_{\|x\|=1}\left[TAT^{\dagger}x,x\right]^{1/2} + \sup_{\|x\|=1}|[TAx,x]|\right] \\ &= \frac{1}{2}\left[\|A\|^{1/2} w^{1/2}\left(TAT^{\dagger}\right) + w(TA)\right],\end{aligned}$$

which proves (3.21). □

4. The Case of Hilbert Spaces

Let $(H,\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $e \in H$ with $\|e\| = 1$. If we take $[\cdot,\cdot] = \langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]_e = \langle\cdot,e\rangle\langle e,\cdot\rangle$ then we observe that

$$(x,y)_e := [x,y] - [x,y]_e = \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle,\ x,\ y \in H$$

is linear in the first variable and anti-linear in the second and, by Schwarz's inequality in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$,

$$(x,x)_e = [x,x] - [x,x]_e = \|x\|^2 - |\langle x,e\rangle|^2 \geq 0 \text{ for any } x \in H.$$

Therefore $(\cdot,\cdot)_e$ is a nonnegative Hermitian from on the complex linear space H and thus satisfy the Schwarz inequality

$$|(x,y)_e|^2 \leq (x,x)_e\,(y,y)_e\,.$$

Using the terminology introduced above, we then have $[\cdot,\cdot] \succeq [\cdot,\cdot]_e$ and by Corollary 1 we have for any x, y, $e \in H$ with $\|e\| = 1$ that

$$\left(\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] - |\langle x,e\rangle\,\langle y,e\rangle|\right)\left(\|x\|\,\|y\| - |\langle x,y\rangle|\right) \geq \frac{1}{2}\left[\det\begin{pmatrix} \|x\| & \|y\| \\ |\langle x,e\rangle| & |\langle y,e\rangle| \end{pmatrix}\right]^2. \tag{4.1}$$

In particular, we have

$$\|x\|^2\,\|y\|^2 - |\langle x,y\rangle|^2 \geq \left[\det\begin{pmatrix} \|x\| & \|y\| \\ |\langle x,e\rangle| & |\langle y,e\rangle| \end{pmatrix}\right]^2 \geq 0 \tag{4.2}$$

and the *Buzano's* well known inequality

$$\frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right] \geq |\langle x,e\rangle\,\langle y,e\rangle| \tag{4.3}$$

for any x, y, $e \in H$ with $\|e\| = 1$.

The inequality (4.2) has been obtained in [3].

From Corollary 2 we also have for any x, y, $e \in H$ with $\|e\| = 1$ that

$$|\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle| \leq \|x\|\,\|y\| - |\langle x,e\rangle\,\langle e,y\rangle|\,, \tag{4.4}$$

$$|\langle x,y\rangle| \leq |\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle| + |\langle x,e\rangle\,\langle e,y\rangle| \leq \|x\|\,\|y\|\,, \tag{4.5}$$

and

$$\begin{aligned} |\langle x,e\rangle\,\langle e,y\rangle| &\leq \frac{1}{2}\left[|\langle x,y\rangle - \langle x,e\rangle\,\langle e,y\rangle| + |\langle x,e\rangle\,\langle e,y\rangle| + |\langle x,y\rangle|\right] \\ &\leq \frac{1}{2}\left[\|x\|\,\|y\| + |\langle x,y\rangle|\right]. \end{aligned} \tag{4.6}$$

The refinement of Schwarz's inequality (4.5) was obtained in 1985 by the author, see [1] or [4].

We recall the selfadjoint operator $P : H \to H$ is called nonnegative if $\langle Px,x\rangle \geq 0$ for any $x \in H$. If A, B are nonnegative operators with $A \geq B$, namely $A - B \geq 0$, then all inequalities in Theorem 7 hold with the inner product $\langle\cdot,\cdot\rangle$ instead of the $[\cdot,\cdot]$ the s-L-G-s.i.p. that generates the norm $\|\cdot\|$.

Proposition 5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $A \geq B \geq 0$. Then for any T, $V \in \mathcal{B}(H)$, we have*

$$|\langle TBV^*x,y\rangle| \leq \frac{1}{2}\left[\langle VAV^*x,x\rangle^{1/2}\,\langle TAT^*y,y\rangle^{1/2} + |\langle TAV^*x,y\rangle|\right], \tag{4.7}$$

for any x, $y \in H$.

We have the vector norm inequality

$$\|TBV^*x\| \leq \frac{1}{2}\left[\langle VAV^*x, x\rangle^{1/2}\|TAT^*\|^{1/2} + \|TAV^*x\|\right], \tag{4.8}$$

for any $x \in H$, the operator norm inequality

$$\|TBV^*\| \leq \frac{1}{2}\left[\|VAV^*\|^{1/2}\|TAT^*\|^{1/2} + \|TAV^*\|\right] \tag{4.9}$$

and the numerical radius inequality

$$w(TBV^*) \leq \frac{1}{2}\left[\|VAV^*\|^{1/2}\|TAT^*\|^{1/2} + w(TAV^*)\right]. \tag{4.10}$$

Proof. The inequality (4.7) follows by (3.14) by replacing x with V^*x and using the properties of the adjoint operator in Hilbert spaces.

The rest follows by (4.7) by a similar approach outlined in the proof of Corollary 5. $\square$

If we take $V = T^*$ then we get by (4.9) and (4.10) that

$$\|TBT\| \leq \frac{1}{2}\left[\|T^*AT\|^{1/2}\|TAT^*\|^{1/2} + \|TAT\|\right] \tag{4.11}$$

and

$$w(TBT) \leq \frac{1}{2}\left[\|T^*AT\|^{1/2}\|TAT^*\|^{1/2} + w(TAT)\right]. \tag{4.12}$$

Moreover, if we take $A = I$ and $B = C$ with $0 \leq C \leq I$, then by (4.9) and (4.10) we get

$$\|TCV^*\| \leq \frac{1}{2}\left[\|V\|\|T\| + \|TV^*\|\right] \tag{4.13}$$

and the numerical radius inequality

$$w(TCV^*) \leq \frac{1}{2}\left[\|V\|\|T\| + w(TV^*)\right], \tag{4.14}$$

for any $T, V \in \mathcal{B}(H)$.

From (4.11) and (4.12) we get

$$\|TCT\| \leq \frac{1}{2}\left(\|T\|^2 + \left\|T^2\right\|\right) \tag{4.15}$$

and

$$w(TCT) \leq \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right], \tag{4.16}$$

for any $T \in \mathcal{B}(H)$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa