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ON MAPPINGS IN CONNECTION TO THE FEJÉR-HADAMARD INEQUALITY FOR COORDINATED CONVEX FUNCTIONS

G. FARID[1], M. MARWAN[2], AND ATIQ UR REHMAN[3]

Abstract. In this paper we give and investigate generalization of monotonic nondecreasing mapping in connection with the Hadamard inequality for convex functions on coordinates defined in [13]. We also give Lipschitzian mapping connected to the generalized Hadamard inequalities.

1. Introduction

A real valued function $f : I \to \mathbb{R}$, where I is an interval in $\mathbb{R}$, is called convex if

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y), \tag{1}$$

where $\alpha \in [0,1], x, y \in I$.
The most classical inequality for convex functions is stated in the following.

Theorem 1.1. *Let $f : I \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{2}$$

Inequality in (2) is known as the Hadamard inequality for convex functions. For generalizations, refinements, counterparts, and new Hadamard-type inequalities one can see for example [4, 5, 10, 11] and references there in.
In [6] Dragomir gave the Hadamard inequality on a rectangle in plane, by defining convex functions on coordinates.

Let $[a, b]$ and $[c, d]$ be two intervals in $\mathbb{R}$ and we define $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ which we will use in sequel throughout the paper.

Definition 1.2. A function $f : \Delta \to \mathbb{R}$ will be called convex on the coordinates if the partial mappings $f_y : [a, b] \to \mathbb{R}$, $f_y(u) := f(u, y)$, and $f_x : [a, b] \to \mathbb{R}$, $f_x(v) := f(x, v)$, are convex where defined for all $y \in [c, d]$ and $x \in [a, b]$.

Recall that a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x, y) + (1-\lambda)f(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$.
One can note that every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the coordinates but the converse is not true. For example, $f(x, y) = xy$ is convex on the coordinates in $\mathbb{R}^2$ but it is not convex.

In 1906, Fejér [12] (see also, [17, p. 138]) gave weighted version of (2) which appears as its generalization.

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Theorem 1.3. *Let $f : I \to \mathbb{R}$ be convex function on the interval I, with $a, b \in I$ with $a < b$ and $g : [a, b] \to \mathbb{R}^+$ is symmetric about $(a+b)/2$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{3}$$

In literature inequality (3) is famous as the Fejér–Hadamard inequality for convex functions. It is extensively studied by many researchers.
In the following we state the Fejér–Hadamard inequality [10] which is generalization of the Hadamard inequality for coordinated convex functions proved by Dragomir in 2001.

Theorem 1.4. *Let $f : \Delta \to \mathbb{R}$ be a convex functions on the coordinates in Δ. Also let $g_1 : [a, b] \to \mathbb{R}^+$ and $g_2 : [c, d] \to \mathbb{R}^+$ be two integrable and symmetric functions about $(a + b)/2$ and $(c + d)/2$ respectively. Then one has the following inequalities*

$$\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \frac{1}{2}\left[\frac{1}{G_1}\int_a^b f\left(x,\frac{c+d}{2}\right)g_1(x)dx+\frac{1}{G_2}\int_c^d f\left(\frac{a+b}{2},y\right)g_2(y)dy\right]\\
&\le \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx\\
&\le \frac{1}{4}\left[\frac{1}{G_1}\int_a^b g_1(x)f(x,c)dx+\frac{1}{G_1}\int_a^b g_1(x)f(x,d)dx\right.\\
&\left.+\frac{1}{G_2}\int_c^d g_2(y)f(a,y)dy+\frac{1}{G_2}\int_c^d g_2(y)f(b,y)dy\right]\\
&\le \frac{1}{4}\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right],
\end{aligned} \tag{4}$$

where

$$G_1 = \int_a^b g_1(x)dx \text{ and } G_2 = \int_c^d g_2(y)dy.$$

These inequalities are sharp.

In [13] authors have introduced some mappings in connection to the Hadamard inequality in two coordinates and discussed their interesting properties. In this paper we define and study properties of mapping in connection with the Fejér-Hadamard inequality for coordinated convex functions. Results of this paper are actually generalizations of results given in [13].

2. Main results

The following lemma is given in [13].

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be convex function and let $a \le y_1 \le x_1 \le x_2 \le y_2 \le b$ and $x_1 + x_2 = y_1 + y_2$. Then*

$$f(x_1) + f(x_2) \le f(y_1) + f(y_2).$$

Theorem 2.2. *Suppose that $f:\Delta\subseteq\mathbb{R}^2\to\mathbb{R}$ is coordinated convex on Δ and the mapping $F_{g_1g_2}:[0,1]^2\to\mathbb{R}$ is defined by*

$$\begin{aligned}F_{g_1g_2}(t,s)=&\frac{1}{4G_1G_2}\int_a^b\int_c^d\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.\\&+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\\&+f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\\&\left.+f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]g_1(x)g_2(y)dydx.\end{aligned}$$

Then the followings are valid:
(i) The mapping $F_{g_1g_2}$ is coordinated convex on $[0,1]^2$.
(ii) The mapping $F_{g_1g_2}$ is coordinated monotonic nondecreasing on $[0,1]^2$.
(iii) $F_{g_1g_2}$ have bounds

$$\begin{aligned}&\inf_{(t,s)\in[0,1]^2}F_{g_1g_2}(t,s)\\&=\frac{1}{G_1G_2}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)g_1(2x-a)g_2(2y-c)dydx\right.\\&+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}f(x,y)g_1(2x-b)g_2(2y-c)dydx\\&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)g_1(2x-a)g_2(2y-d)dydx\\&\left.+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)g_1(2x-b)g_2(2y-d)dydx\right]\\&=F_{g_1g_2}(0,0),\end{aligned}$$

$$\sup_{(t,s)\in[0,1]^2}F_{g_1g_2}(t,s)=\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}=F_{g_1g_2}(1,1).$$

Proof. (i) If $s\in[0,1]$ is fixed, then for all $t_1,t_2\in[0,1]$ and $\alpha,\beta\geq0$ with $\alpha+\beta=1$, we have

$$\begin{aligned}&F_{g_1g_2}(\alpha t_1+\beta t_2,s)=\frac{1}{4G_1G_2}\int_a^b\int_c^d\\&\left[f\left(\left(\frac{1+(\alpha t_1+\beta t_2)}{2}\right)a+\left(\frac{1-(\alpha t_1+\beta t_2)}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)+\right.\\&f\left(\left(\frac{1+(\alpha t_1+\beta t_2)}{2}\right)b+\left(\frac{1-(\alpha t_1+\beta t_2)}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)+\\&f\left(\left(\frac{1+(\alpha t_1+\beta t_2)}{2}\right)a+\left(\frac{1-(\alpha t_1+\beta t_2)}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)+\\&\left.f\left(\left(\frac{1+(\alpha t_1+\beta t_2)}{2}\right)b+\left(\frac{1-(\alpha t_1+\beta t_2)}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]\times\\&g_1(x)g_2(y)dydx.\end{aligned}$$

Since we have for $z = a, b$

$$\left(\frac{1+(\alpha t_1+\beta t_2)}{2}\right) z + \left(\frac{1-(\alpha t_1+\beta t_2)}{2}\right) x =$$
$$\alpha\left(\frac{(1+t_1)z+(1-t_1)x}{2}\right) + \beta\left(\frac{(1+t_2)z+(1-t_2)x}{2}\right),$$

therefore

$$F_{g_1g_2}(\alpha t_1+\beta t_2, s) = \frac{1}{4G_1G_2}\int_a^b\int_c^d$$
$$\left[f\left(\alpha\left(\frac{(1+t_1)a+(1-t_1)x}{2}\right)+\beta\left(\frac{(1+t_2)a+(1-t_2)x}{2}\right), \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.$$
$$+ f\left(\alpha\left(\frac{(1+t_1)b+(1-t_1)x}{2}\right)+\beta\left(\frac{(1+t_2)b+(1-t_2)x}{2}\right), \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)$$
$$+ f\left(\alpha\left(\frac{(1+t_1)a+(1-t_1)x}{2}\right)+\beta\left(\frac{(1+t_2)a+(1-t_2)x}{2}\right), \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)$$
$$\left.+ f\left(\alpha\left(\frac{(1+t_1)b+(1-t_1)x}{2}\right)+\beta\left(\frac{(1+t_2)b+(1-t_2)x}{2}\right), \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]$$
$$\times g_1(x)g_2(y)dydx.$$

By using the definition of coordinated convexity on first coordinate,

$$\begin{aligned}
&F_{g_1g_2}(\alpha t_1+\beta t_2, s) \le \frac{1}{4G_1G_2}\\
&\times \alpha\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_1}{2}\right)a+\left(\frac{1-t_1}{2}\right)x, \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.\\
&+ f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)x, \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\\
&+ f\left(\left(\frac{1+t_1}{2}\right)a+\left(\frac{1-t_1}{2}\right)x, \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\\
&\left.+ f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)x, \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right] g_1(x)g_2(y)dydx\\
&+ \frac{\beta}{4G_1G_2}\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_2}{2}\right)a+\left(\frac{1-t_2}{2}\right)x, \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.\\
&+ f\left(\left(\frac{1+t_2}{2}\right)b+\left(\frac{1-t_2}{2}\right)x, \left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\\
&+ f\left(\left(\frac{1+t_2}{2}\right)a+\left(\frac{1-t_2}{2}\right)x, \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\\
&\left.+ f\left(\left(\frac{1+t_2}{2}\right)b+\left(\frac{1-t_2}{2}\right)x, \left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right] g_1(x)g_2(y)dydx\\
&= \alpha F_{g_1g_2}(t_1, s) + \beta F_{g_1g_2}(t_2, s).
\end{aligned}$$

If $t \in [0,1]$ is fixed, then for all $s_1, s_2 \in [0,1]$ and $\alpha, \beta \ge 0$ with $\alpha+\beta = 1$, we also have

$$F_{g_1g_2}(t, \alpha s_1+\beta s_2) \le \alpha F_{g_1g_2}(t, s_1) + \beta F_{g_1g_2}(t, s_2).$$

This proves that $F_{g_1 g_2}$ is coordinated convex on $[0,1]^2$.

(ii) Now to prove $F_{g_1 g_2}$ is coordinated monotonic nondecreasing on $[0,1]^2$ we fix $s \in [0,1]$ and let $0 \le t_1 \le t_2 \le 1$, $a \le x \le b$. We have

$$
\begin{aligned}
&F_{g_1 g_2}(t_1, s) \\
&= \frac{1}{4G_1G_2} \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s}{2}\right) c + \left(\frac{1-s}{2}\right) y \right) \right. \\
&+ f\left(\left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s}{2}\right) c + \left(\frac{1-s}{2}\right) y \right) \\
&+ f\left(\left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s}{2}\right) d + \left(\frac{1-s}{2}\right) y \right) \\
&+ \left. f\left(\left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s}{2}\right) d + \left(\frac{1-s}{2}\right) y \right) \right] g_1(x) g_2(y) dy dx.
\end{aligned}
$$

Since f is coordinated convex on Δ and

$$
\begin{aligned}
&\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x \le \left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x \\
&\le \left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) (b+a-x) \le \left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) (b+a-x)
\end{aligned}
$$

with

$$
\begin{aligned}
&\left[\left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x \right] + \left[\left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) (b+a-x) \right] \\
&= \left[\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x \right] + \left[\left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) (b+a-x) \right].
\end{aligned}
$$

Then by using Lemma 2.1, we have

(5)

$$
\begin{aligned}
&F_{g_1 g_2}(t_1, s) \le \frac{1}{4G_1G_2} \times \\
&\int_a^b \int_c^d \left[f\left(\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s}{2}\right) c + \left(\frac{1-s}{2}\right) y \right) \right] g_1(x) g_2(y) dy dx \\
&+ \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) (b+a-x), \left(\frac{1+s}{2}\right) c + \left(\frac{1-s}{2}\right) y \right) \right] \times \\
&g_1(b+a-x) g_2(y) dy dx \\
&+ \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s}{2}\right) d + \left(\frac{1-s}{2}\right) y \right) \right] g_1(x) g_2(y) dy dx \\
&+ \int_a^b \int_c^d \left[f\left(\left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) (b+a-x), \left(\frac{1+s}{2}\right) d + \left(\frac{1-s}{2}\right) y \right) \right] \times \\
&g_1(b+a-x) g_2(y) dy dx,
\end{aligned}
$$

here we use the following equality
(6)

$$\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right]g_1(x)g_2(y)dydx$$
$$=\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_1}{2}\right)b+\left(\frac{1-t_1}{2}\right)(b+a-x),\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right]\times$$
$$g_1(b+a-x)g_2(y)dydx.$$

From (5) and (6) we get

$$\begin{aligned}
&F_{g_1g_2}(t_1,s)\leq\frac{1}{4G_1G_2}\times\\
&\int_a^b\int_c^d\left[f\left(\left(\frac{1+t_2}{2}\right)a+\left(\frac{1-t_2}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\right.\\
&+f\left(\left(\frac{1+t_2}{2}\right)b+\left(\frac{1-t_2}{2}\right)x,\left(\frac{1+s}{2}\right)c+\left(\frac{1-s}{2}\right)y\right)\\
&+f\left(\left(\frac{1+t_2}{2}\right)a+\left(\frac{1-t_2}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\\
&\left.+f\left(\left(\frac{1+t_2}{2}\right)b+\left(\frac{1-t_2}{2}\right)x,\left(\frac{1+s}{2}\right)d+\left(\frac{1-s}{2}\right)y\right)\right]g_1(x)g_2(y)dydx\\
&=F_{g_1g_2}(t_2,s).
\end{aligned}$$

This shows that $F_{g_1g_2}(t,s)$ is coordinated nondecreasing for all $t\in[0,1]$.
If $t\in[0,1]$ is fixed, then for all $s\in[0,1]$, we also have $F_{g_1g_2}(t,s)$ is coordinated nondecreasing for all $s\in[0,1]$. Thus the mapping $F_{g_1g_2}$ is coordinated monotonic nondecreasing on $[0,1]^2$.

(iii) It follows from (ii) that, for all $(t,s)\in[0,1]^2$,

$$\begin{aligned}
F_{g_1g_2}(t,s)&\geq F_{g_1g_2}(0,s)\geq F_{g_1g_2}(0,0)\\
&=\frac{1}{4G_1G_2}\int_a^b\int_c^d\left[f\left(\frac{a+x}{2},\frac{c+y}{2}\right)+f\left(\frac{a+x}{2},\frac{d+y}{2}\right)\right.\\
&\left.+f\left(\frac{b+x}{2},\frac{c+y}{2}\right)+f\left(\frac{b+x}{2},\frac{d+y}{2}\right)\right]g_1(x)g_2(y)dydx\\
&=\frac{1}{G_1G_2}\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}f(x,y)g_1(2x-a)g_2(2y-c)dydx\right.\\
&+\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}f(x,y)g_1(2x-b)g_2(2y-c)dydx\\
&+\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d f(x,y)g_1(2x-a)g_2(2y-d)dydx\\
&\left.+\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d f(x,y)g_1(2x-b)g_2(2y-d)dydx\right]
\end{aligned}$$

and

$$\begin{aligned} F_{g_1g_2}(t,s) &\leq F_{g_1g_2}(t,1) \leq F_{g_1g_2}(1,1) \\ &= \frac{1}{4G_1G_2}\int_a^b\int_c^d [f(a,c)+f(a,d)+f(b,c)+f(b,d)]g_1(x)g_2(y)dydx \\ &= \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}. \end{aligned}$$

This completes the proof. □

Remark 2.3. If we put $g_1 \equiv 1$ and $g_2 \equiv 1$ in Theorem 2.2, then we get Theorem 1 of [13].

Now we are interested to give some results related to Lipshitzian mapping. The following definition is given in [15, p. 305].

Definition 2.4. Consider a function $f : A \to \mathbb{R}$ defined on a subset A of $\mathbb{R}^m, m \in N$. Let $L = (L_1, L_2, \dots, L_m)$ where $L_i \geq 0, i = 1, 2, \dots, m$. We say that f is L-Lipchitian function if

$$|f(x) - f(y)| \leq \sum_{i=1}^{m} L_i|x_i - y_i| \tag{7}$$

for all $x, y \in A$.

For desired results we need the following lemma, which is due to Levin and Stečkin [17, p. 200].

Lemma 2.5. *Let f is convex on $[a,b]$ and g is symmetric about $(a+b)/2$ and non-decreasing on $[a,(a+b)/2]$. Then*

$$\int_a^b f(x)g(x)dx \leq \frac{1}{b-a}\int_a^b f(x)dx\int_a^b g(x)dx. \tag{8}$$

For the functions $F_{g_1g_2}$ defined in Theorem 2.2, and $\widehat{H}$ defined as follows

$$\widehat{H}(t,s) = \frac{1}{G_1G_2}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2}, sy+(1-s)\frac{c+d}{2}\right)g_1(x)g_2(y)dydx,$$

we have the following result.

Theorem 2.6. *Let $f : \Delta = [a,b]\times[c,d] \to \mathbb{R}$ satisfy Lipschitzian conditions. That is, for (t_1,s_1) and (t_2,s_2) belong to Δ, we have*

$$|f(t_1,s_1) - f(t_2,s_2)| \leq L_1|t_1-t_2| + L_2|s_1-s_2|$$

where L_1 and L_2 are positive constants. Then

$$|F_{g_1g_2}(t_1,s_1) - F_{g_1g_2}(t_2,s_2)| \leq \frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4} \tag{9}$$

and

$$|\widehat{H}(t_1,s_1) - \widehat{H}(t_2,s_2)| \leq \frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4}. \tag{10}$$

Proof. For (t_1, s_1) and (t_2, s_2) belong to Δ, we have

$$
\begin{aligned}
&|F_{g_1g_2}(t_1,s_1) - F_{g_1g_2}(t_2,s_2)| \leq \frac{1}{4G_1G_2} \\
&\times \int_a^b \int_c^d \left[\left| f\left(\left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s_1}{2}\right) c + \left(\frac{1-s_1}{2}\right) y \right) \right. \right. \\
&- \left. f\left(\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s_2}{2}\right) c + \left(\frac{1-s_2}{2}\right) y \right) \right| \\
&+ \left| f\left(\left(\frac{1+t_1}{2}\right) a + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s_1}{2}\right) d + \left(\frac{1-s_1}{2}\right) y \right) \right. \\
&- \left. f\left(\left(\frac{1+t_2}{2}\right) a + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s_2}{2}\right) d + \left(\frac{1-s_2}{2}\right) y \right) \right| \\
&+ \left| f\left(\left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s_1}{2}\right) c + \left(\frac{1-s_1}{2}\right) y \right) \right. \\
&- \left. f\left(\left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s_2}{2}\right) c + \left(\frac{1-s_2}{2}\right) y \right) \right| \\
&+ \left| f\left(\left(\frac{1+t_1}{2}\right) b + \left(\frac{1-t_1}{2}\right) x, \left(\frac{1+s_1}{2}\right) d + \left(\frac{1-s_1}{2}\right) y \right) \right. \\
&- \left. \left. f\left(\left(\frac{1+t_2}{2}\right) b + \left(\frac{1-t_2}{2}\right) x, \left(\frac{1+s_2}{2}\right) d + \left(\frac{1-s_2}{2}\right) y \right) \right| \right] \\
&\times g_1(x)g_2(y)dydx.
\end{aligned}
$$

Since f satisfies lipshitzian condition we have

$$
\begin{aligned}
&|F_{g_1g_2}(t_1,s_1) - F_{g_1g_2}(t_2,s_2)| \leq \frac{1}{4G_1G_2} \int_a^b \int_c^d \\
&\left[L_1 \left| \left(\frac{t_1-t_2}{2}\right) a + \left(\frac{t_2-t_1}{2}\right) x \right| + L_2 \left| \left(\frac{s_1-s_2}{2}\right) c + \left(\frac{s_2-s_1}{2}\right) y \right| \right. \\
&+ L_1 \left| \left(\frac{t_1-t_2}{2}\right) a + \left(\frac{t_2-t_1}{2}\right) x \right| + L_2 \left| \left(\frac{s_1-s_2}{2}\right) d + \left(\frac{s_2-s_1}{2}\right) y \right| \\
&+ L_1 \left| \left(\frac{t_1-t_2}{2}\right) b + \left(\frac{t_2-t_1}{2}\right) x \right| + L_2 \left| \left(\frac{s_1-s_2}{2}\right) c + \left(\frac{s_2-s_1}{2}\right) y \right| \\
&+ \left. L_1 \left| \left(\frac{t_1-t_2}{2}\right) b + \left(\frac{t_2-t_1}{2}\right) x \right| + L_2 \left| \left(\frac{s_1-s_2}{2}\right) d + \left(\frac{s_2-s_1}{2}\right) y \right| \right] g_1(x)g_2(y)dydx \\
&= \frac{L_1}{2G_1} \int_a^b \left| \left(\frac{t_1-t_2}{2}\right) a + \left(\frac{t_2-t_1}{2}\right) x \right| + \left| \left(\frac{t_1-t_2}{2}\right) b + \left(\frac{t_2-t_1}{2}\right) x \right| g_1(x)dx \\
&+ \frac{L_2}{2G_2} \int_c^d \left| \left(\frac{s_1-s_2}{2}\right) c + \left(\frac{s_2-s_1}{2}\right) y \right| + \left| \left(\frac{s_1-s_2}{2}\right) d + \left(\frac{s_2-s_1}{2}\right) y \right| g_2(y)dy,
\end{aligned}
$$

using Lemma 2.5 we get

$$
|F_{g_1g_2}(t_1,s_1) - F_{g_1g_2}(t_2,s_2)| \leq \frac{L_1|t_1-t_2|(b-a) + L_2|s_1-s_2|(d-c)}{4}.
$$

Now

$$
\begin{aligned}
&|\widehat{H}(t_1,s_1)-\widehat{H}(t_2,s_2)| \\
&\leq \frac{1}{G_1G_2}\int_a^b\int_c^d \left| f\left(t_1x+(1-t_1)\frac{a+b}{2}, s_1y+(1-s_1)\frac{c+d}{2}\right)\right. \\
&\left. - f\left(t_2x+(1-t_2)\frac{a+b}{2}, s_2y+(1-s_2)\frac{c+d}{2}\right)\right| g_1(x)g_2(y)dydx \\
&\leq \frac{1}{G_1G_2}\int_a^b\int_c^d \left[L_1\left|(t_1-t_2)(x-\frac{a+b}{2})\right| + L_2\left|(s_1-s_2)(y-\frac{c+d}{2})\right|\right] \times \\
&g_1(x)g_2(y)dydx \\
&= \frac{L_1\left|(t_1-t_2)\right|}{G_1}\int_a^b\left|x-\frac{a+b}{2}\right|g_1(x)dx + \frac{L_2\left|(s_1-s_2)\right|}{G_2}\int_c^d\left|y-\frac{c+d}{2}\right|g_2(y)dy.
\end{aligned}
$$

Again using Lemma 2.5 we get

$$|H_{g_1g_2}(t_1,s_1)-H_{g_1g_2}(t_2,s_2)| \leq \frac{L_1|t_1-t_2|(b-a)+L_2|s_1-s_2|(d-c)}{4}.$$

This completes the proof. □

Remark 2.7. If we put $g_1 \equiv 1$ and $g_2 \equiv 1$ in Theorem 2.6, then we get Theorem 2 of [13].

Remark 2.8. If we take $t_1 = 0, t_2 = 1, s_1 = 0$, and $s_2 = 1$ in Theorem 2.6, then inequalities (9) and (10) reduce to

$$
\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4} - \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx\right| \\
&\leq \frac{L_1(b-a)+L_2(d-c)}{4}
\end{aligned}
\tag{11}
$$

and

$$\left|f\left(\frac{a+b}{2},\frac{c+d}{2}\right) - \frac{1}{G_1G_2}\int_a^b\int_c^d f(x,y)g_1(x)g_2(y)dydx\right| \leq \frac{L_1(b-a)+L_2(d-c)}{4}. \tag{12}$$

2.9. **Concluding remarks.** It is a beauty of mathematics that a topic in Mathematics always have further connections which cannot be end. We have generalized the Hadamard inequality for convex functions on coordinates and here we find its applications as extension of known results.

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1 -COMSATS Institute of Information Technology, Attock Campus, Pakistan.
E-mail address: `faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk`

2 -COMSATS Institute of Information Technology, Attock Campus, Pakistan.
E-mail address: `marwan78642@gmail.com`

3 -COMSATS Institute of Information Technology, Attock Campus, Pakistan.
E-mail address: `atiq@mathcity.org`