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Magic Rectangles in Construction of Block-Wise Pan Magic Squares

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Abstract

The idea of magic rectangles is used to bring pan magic squares of orders 15, 21, 27 and 33, where 3×3 blocks are with equal sums entries and are semi-magic squares of order 3 (in rows and columns). The magic squares of order 9, 12, 18, 24, 30 and 36 are calculated, with the property that 3×3 blocks are magic squares of order 3 with different magic sums. All the magic squares constructed are pan diagonal except the orders 18 and 30. Exceptionally, the pan magic square of order 35 is of type 5×7 or 7×5. It is constructed in two different approaches. One with 25 blocks of equal sums magic squares of order 7 and second 49 blocks of equal sums magic squares of order 5.

Contents

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1 Introduction

Magic rectangles are very well known in the literature [1, 5, 7]. In this paper, we shall bring magic squares of type $3k$ by use of **magic rectangles** or **semi-magic rectangles**. The construction is such way that each 3×3 blocks are either magic squares of order 3 or sum of entries are of equal sums. In case of orders 12, 24 and 36 the 3×3 blocks are of magic squares of order 3. In case of orders 9, 15, 21, 27 and 33, the 3×3 blocks are semi-magic squares (in rows and columns) of same sums. All these magic squares are **pan magic squares** except the orders 18 and 30. In case of orders 15, 21, 27 and 33 the **magic triangles** are used. The case of order 35 is done separately in two different ways, one as 7×5 and another as 5×7. Even though it is not necessary, but in this case, we used the idea of magic rectangle of order $(5, 7)$. In this case also magic square is pan diagonal. Based on similar lines author recently [23, 24, 25] worked on block-wise construction of magic squares of orders $3k$ and $4k$. The idea of perfect square sum, and Pythagorean triples are developed in [21, 22].

1.1 Magic Rectangles

The idea of magic rectangles is not very much famous in the literature as of magic squares. In case of magic squares, we have sum of rows, columns and main diagonal of same sum. In case of magic rectangle the sum of rows and columns are same but of different values. Below are some basic examples of magic rectangles due to [5]:

Example 1.1. *Below is an example of a* ***magic rectangle of order (3,5)*** *using the numbers 1-15:*

(3,5)	C1	C2	C3	C4	C5	Total
R1	14	10	4	5	7	40
R2	1	3	8	13	15	40
R3	9	11	12	6	2	40
Total	24	24	24	24	24	

Example 1.2. *Below is an example of a* ***magic rectangle of order (3,7)*** *using the numbers 1-21:*

(3,7)	C1	C2	C3	C4	C5	C6	C7	Total
R1	1	12	13	6	17	20	8	77
R2	18	19	15	11	7	3	4	77
R3	14	2	5	16	9	10	21	77
Total	33	33	33	33	33	33	33	

Example 1.3. *Below is an example of a* ***magic rectangle of order (3,9)*** *using the numbers 1-27:*

(3,9)	C1	C2	C3	C4	C5	C6	C7	C8	C9	Total
R1	1	15	5	16	21	22	9	26	11	126
R2	24	25	18	20	14	8	10	3	4	126
R3	17	2	19	6	7	12	23	13	27	126
Total	42	42	42	42	42	42	42	42	42	

Example 1.4. *Below is an example of a* ***magic rectangle of order (3,11)*** *using the numbers 1-33:*

(3,11)	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	Total
R1	22	29	3	7	24	9	26	13	16	32	6	187
R2	1	20	30	23	19	17	15	11	4	14	33	187
R3	28	2	18	21	8	25	10	27	31	5	12	187
Total	51	51	51	51	51	51	51	51	51	51	51	

Example 1.5. *Below is an example of a* ***magic rectangle of order (3,13)*** *using the numbers 1-39:*

(3,13)	C1	C2	C3	C4	C5	C6	C7	C8	C9	C10	C11	C12	C13	Total
R1	34	2	21	26	8	22	10	31	16	33	37	5	15	260
R2	1	23	36	27	28	29	20	11	12	13	4	17	39	260
R3	25	35	3	7	24	9	30	18	32	14	19	38	6	260
Total	60	60	60	60	60	60	60	60	60	60	60	60	60	

Example 1.6. *Below is an example of a* ***magic rectangle of order (5,7)*** *using the numbers 1-35:*

(5,7)	C1	C2	C3	C4	C5	C6	C7	Total
R1	26	19	8	31	25	13	4	126
R2	20	6	34	24	14	1	27	126
R3	3	7	15	18	21	29	33	126
R4	9	35	22	12	2	30	16	126
R5	32	23	11	5	28	17	10	126
Total	90	90	90	90	90	90	90	

Example 1.7. *Below is an example of a* ***magic rectangle of order (4,6)*** *using the numbers 1-24:*

(4,6)	C1	C2	C3	C4	C5	C6	Total
R1	1	2	3	22	23	24	75
R2	19	20	21	4	5	6	75
R3	18	17	16	9	8	7	75
R4	12	11	10	15	14	13	75
Total	50	50	50	50	50	50	

Above examples can be seen in sites of Aale [1] and Nakamura [5]. According to Nakamura [5], we can easily construct magic rectangles of orders (odd, odd) and (even, even), but of order (odd, even) don't exists. They are generally applied to construct magic cubes, etc. Here, our aim is to construct pan magic square of odd orders, such as of order 15, 21, 27 and 33 by use of above magic rectangles. The work is given for the order 9 to 36. In some case, idea of sem-magic rectangle is used to construct magic squares, such as of order 12, 24 and 36. The orders 18 and 30 are just magic squares multiple of 3k. Even though a pan magic square of order 35 can be constricted by without use of magic rectangle of of order (5,7), but still use have used to construct pan magic square of order 35. The pan magic square of order 24 can be constructed as each block of order 4 with equal sums. Here we don't require to use the magic rectangle of order (4,6). We have used it construct with each block of order 6.

2 Magic Squares of Orders $3k$

In this section, we shall give magic squares of order $3k$. From order 9 to 36. All of them are either sub-blocks of magic squares of order 3 or sum of entries of each block of 3 with equal sums. Except the orders of 18 and 30, all others are pan magic squares. In case of orders 15, 21, 27 and 33 the idea magic rectangles are used.

2.1 Pan Magic Square of Order 9

In order to construct pan magic squares of order 9 we shall use the idea of Latin squares decomposition of magic square of order 3.

Example 2.1. *Let's consider Latin squares decomposition of magic square of order 3 given by*

A			6
2	3	1	6
1	2	3	6
3	1	2	6
6	6	6	6

B			6
1	3	2	6
3	2	1	6
2	1	3	6
6	6	6	6

AB			15
4	9	2	15
3	5	7	15
8	1	6	15
15	15	15	15

The magic square AB is obtained by using the operation

$$AB := 10 \times (A - 1) + B.$$

In order to construct pan magic square of order 9, let us consider a magic square of order 3 given in Example 2.1. For simplicity, let's rewrite it:

Example 2.2. *The magic square of order 3 is given by*

(3,3)	1	2	3	Total
R1	4	9	2	15
R2	3	5	7	15
R3	8	1	6	15
Total	15	15	15	

Using columns combinations given in Example 2.2 let's construct 9 blocks of order 3 with the operation $AB := 9 \times (A - 1) + B$, and put them according to following structure lead us to a magic square of order 9:

Structure 2.1. *Let's consider 9 blocks of order 3 given as below:*

11	12	13
21	22	23
31	32	33

Example 2.3. *The* ***pan magic square*** *of order 9 constructed according to Structure 2.1 is given by*

		369	369	369	369	369	369	369	369	369
	22	71	30	27	64	32	20	69	34	369
369	35	21	67	28	23	72	33	25	65	369
369	66	31	26	68	36	19	70	29	24	369
369	40	8	75	45	1	77	38	6	79	369
369	80	39	4	73	41	9	78	43	2	369
369	3	76	44	5	81	37	7	74	42	369
369	58	53	12	63	46	14	56	51	16	369
369	17	57	49	10	59	54	15	61	47	369
369	48	13	62	50	18	55	52	11	60	369
	369	369	369	369	369	369	369	369	369	369

In this case, the magic sum is $S_{9\times 9} = 369$. The sum of all the entries of each 3×3 blocks are the same sums as of magic square, i.e., $S_9 = 369$. The middle block of order 3 is a magic square, and other 8 sub-blocks are semi-magic squares of order 3, i.e, only in rows and columns.

The Latin squares arising due to above construction are given in example below.

Example 2.4. *The Latin squares distributions of pan magic square of order 9 of Example 2.3 are given by*

A		45	45	45	45	45	45	45	45	45
	3	8	4	3	8	4	3	8	4	45
45	4	3	8	4	3	8	4	3	8	45
45	8	4	3	8	4	3	8	4	3	45
45	5	1	9	5	1	9	5	1	9	45
45	9	5	1	9	5	1	9	5	1	45
45	1	9	5	1	9	5	1	9	5	45
45	7	6	2	7	6	2	7	6	2	45
45	2	7	6	2	7	6	2	7	6	45
45	6	2	7	6	2	7	6	2	7	45
	45	45	45	45	45	45	45	45	45	45

B		45	45	45	45	45	45	45	45	45
	4	8	3	9	1	5	2	6	7	45
45	8	3	4	1	5	9	6	7	2	45
45	3	4	8	5	9	1	7	2	6	45
45	4	8	3	9	1	5	2	6	7	45
45	8	3	4	1	5	9	6	7	2	45
45	3	4	8	5	9	1	7	2	6	45
45	4	8	3	9	1	5	2	6	7	45
45	8	3	4	1	5	9	6	7	2	45
45	3	4	8	5	9	1	7	2	6	45
	45	45	45	45	45	45	45	45	45	45

The magic square of Example 2.3 is obtained by the application of the operation $9\times(A-1)+B$. The Latin squares A and B are not diagonalize.

2.2 Pan Magic Square of Order 12

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In [23], the author constricted block-wise pan magic square of order 12, where each block of order 4 is a pan magic square of order 4 with same sum. Here, the ais is to construct a block-wise pan magic square of order 12, where each block of order 3 a magic square with different magic sums. It is well known that a magic square of order 12 with 144 numbers 1-144 has a magic sum $S_{12\times 12} := 870$. If we divide 870 by 4 we get a fraction vaule, i.e., $\frac{870}{4} := 217.5$. It implies that we unable to construct block-wise magic square of order 12 with each block of order 3 with sums. In this case, we shall construct block-wise magic square of order 12 with each block of order 3 with different magic sums. This shall be done by use of magic square of order 3 given in Example 2.1 with composite magic square of order 4.

Example 2.5. *The pan diagonal magic square of order 4 is given by*

A		10	10	10	10
	2	3	1	4	10
10	1	4	2	3	10
10	4	1	3	2	10
10	3	2	4	1	10
	10	10	10	10	10

B		10	10	10	10
	3	4	1	2	10
10	2	1	4	3	10
10	4	3	2	1	10
10	1	2	3	4	10
	10	10	10	10	10

AB		34	34	34	34
	7	12	1	14	34
34	2	13	8	11	34
34	16	3	10	5	34
34	9	6	15	4	34
	34	34	34	34	34

The magic squares AB is obtained using the following operation

$$AB := 4 \times (A-1) + B.$$

Example 2.6. *The composite pan magic square of order 4 arising due to A and B, with* $C := 10 \times A + B$ *is given by*

C		110	110	110	110
	23	34	11	42	110
110	12	41	24	33	110
110	44	13	32	21	110
110	31	22	43	14	110
	110	110	110	110	110

Since it is impossible to make magic rectangle of order $(3,4)$, let us consider following **semi-magic rectangle** order $(3,4)$, i.e., equality only in rows:

Example 2.7. *Let's consider the following* ***semi-magic rectangle*** *of order (3,4):*

(3,4)	1	2	3	4	Total
R1	1	6	7	12	26
R2	2	5	8	11	26
R3	3	4	9	10	26
Total	6	15	24	33	

Note 2.1. *If we add the numbers from 1 to 12, we have total sum as 78. It is impossible to divided 78 in four equal parts, i.e.,* $\frac{78}{4} = 19.5$. *This is the reason, why we are unable to make magic rectangle of order (3,4).*

Applying the columns values given in Example 2.7 over the Example 2.1, with the operation $AB := 12 \times (A - 1) + B$, we get 16 blocks of magic squares of order 3 with different magic sums. Below are few examples:

• Block 12

1			6
2	3	1	6
1	2	3	6
3	1	2	6
6	6	6	6

2			15
6	4	5	15
4	5	6	15
5	6	4	15
15	15	15	15

12			51
18	28	5	51
4	17	30	51
29	6	16	51
51	51	51	51

- **Block 34**

3			*24*
7	9	8	*24*
9	8	7	*24*
8	7	9	*24*
24	*24*	*24*	*24*

4			*33*
12	10	11	*33*
10	11	12	*33*
11	12	10	*33*
33	*33*	*33*	*33*

34			*285*
96	106	83	*285*
82	95	108	*285*
107	84	94	*285*
285	*285*	*285*	*285*

The 16 blocks lead us to two pan magic squares of orders 12 and 4. The order 4 is due to the sums of each magic square order 3 and order 12 is with total values. See below these two magic squares.

Example 2.8. *The 16 blocks of magic squares constructed above and keeping according to composite magic square of order 4 given in Example 2.6, we get the following* ***pan magic square*** *of order 12.*

		870	870	870	870	870	870	870	870	870	870	870	870
	55	45	68	96	106	83	13	27	2	126	112	137	870
870	69	56	43	82	95	108	3	14	25	136	125	114	870
870	44	67	57	107	84	94	26	1	15	113	138	124	870
870	18	28	5	121	111	134	60	46	71	91	105	80	870
870	4	17	30	135	122	109	70	59	48	81	92	103	870
870	29	6	16	110	133	123	47	72	58	104	79	93	870
870	132	118	143	19	33	8	90	100	77	49	39	62	870
870	142	131	120	9	20	31	76	89	102	63	50	37	870
870	119	144	130	32	7	21	101	78	88	38	61	51	870
870	85	99	74	54	40	65	127	117	140	24	34	11	870
870	75	86	97	64	53	42	141	128	115	10	23	36	870
870	98	73	87	41	66	52	116	139	129	35	12	22	870
	870	870	870	870	870	870	870	870	870	870	870	870	870

In this case, the magic sum is $S_{12\times 12} := 870$. Each 3×3 block is a magic square of order 3 with different magic sums giving again a pan magic square of order 4 given in example below.

Example 2.9. *The magic sums of 16 blocks of magic squares of order 3 constructed above give us again a pan magic square of order 4 given by*

		870	870	870	870
	168	285	42	375	870
870	51	366	177	276	870
870	393	60	267	150	870
870	258	159	384	69	870
	870	870	870	870	870

2.3 Pan Magic Square of Order 15

In previous work [22], the author calculated pan magic square of order 15, where each 5×5 blocks are pan diagonal magic squares of order 5 with equal magic sums. Here our aim is to construct pan magic square of order 15 with each 3×3 blocks are of equal sums of entries. We shall use the idea of magic-triangle of order (3,5) to construct this magic square. Let's rewrite the magic-triangle of order (3,5) given in Example 1.1 as below:

Example 2.10. *The magic rectangle of order (3,5) is given by*

(3,5)	1	2	3	4	5	Total
R1	14	10	4	5	7	40
R2	1	3	8	13	15	40
R3	9	11	12	6	2	40
Total	24	24	24	24	24	

Let's construct 25 blocks of order 3 using the columns of above magic-rectangle 2.10 in a magic square of order 3 given in Example 2.1 by applying the operation $AB := 15 \times (A-1) + B$. Let's put these 25 blocks according to following structure:

Structure 2.2. *Let's consider 25 blocks of order 3 given as below:*

11	12	13	14	15
21	22	23	24	25
31	32	33	34	35
41	42	43	44	45
51	52	53	54	55

Below are few examples of semi-magic squares of order 3 constructed by applying the columns values given in Example 2.10 over the Example 2.1 by using the operation $AB := 15 \times (A-1) + B$:

• Block 13

1			24
13	5	6	24
6	13	5	24
5	6	13	24
24	24	24	39

3			3
8	15	1	24
15	1	8	24
1	8	15	24
24	24	24	24

13			318
188	75	76	339
90	181	68	339
61	83	195	339
339	339	339	564

• Block 32

3			24
1	15	8	24
8	1	15	24
15	8	1	24
24	24	24	3

2			9
7	14	3	24
14	3	7	24
3	7	14	24
24	24	24	24

32			324
7	224	108	339
119	3	217	339
213	112	14	339
339	339	339	24

• Block 54

5			24
12	2	10	24
10	12	2	24
2	10	12	24
24	24	24	36

4			33
9	4	11	24
4	11	9	24
11	9	4	24
24	24	24	24

54			348
174	19	146	339
139	176	24	339
26	144	169	339
339	339	339	519

According to above Structure 2.2 we have a **pan magic square** of order 15 given in the example below.

Example 2.11. *According to distribution given in Example 1.1, Structure 2.2 and 25 blocks of order 3, we have a **pan magic square** of order 15 given by*

		1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695
	186	65	88	187	74	78	188	75	76	189	64	86	190	62	87	1695
1695	80	193	66	89	183	67	90	181	68	79	191	69	77	192	70	1695
1695	73	81	185	63	82	194	61	83	195	71	84	184	72	85	182	1695
1695	36	200	103	37	209	93	38	210	91	39	199	101	40	197	102	1695
1695	95	43	201	104	33	202	105	31	203	94	41	204	92	42	205	1695
1695	208	96	35	198	97	44	196	98	45	206	99	34	207	100	32	1695
1695	6	215	118	7	224	108	8	225	106	9	214	116	10	212	117	1695
1695	110	13	216	119	3	217	120	1	218	109	11	219	107	12	220	1695
1695	223	111	5	213	112	14	211	113	15	221	114	4	222	115	2	1695
1695	156	50	133	157	59	123	158	60	121	159	49	131	160	47	132	1695
1695	125	163	51	134	153	52	135	151	53	124	161	54	122	162	55	1695
1695	58	126	155	48	127	164	46	128	165	56	129	154	57	130	152	1695
1695	171	20	148	172	29	138	173	30	136	174	19	146	175	17	147	1695
1695	140	178	21	149	168	22	150	166	23	139	176	24	137	177	25	1695
1695	28	141	170	18	142	179	16	143	180	26	144	169	27	145	167	1695
	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695	1695

In this case, the magic sum is $S_{15\times15} = 1695$. Each 3×3 blocks are **semi-magic squares** with equal semi-magic sums, i.e., $S_{3\times3} = 339$ (in rows and columns).

2.4 Magic Square of Order 18

In the previous work [24], the author constructed magic square of order 18 with sub-blocks of order 6. This have been done in two different ways. One when sum of entries of each block are equal, and secondly, when there are 9 magic squares of order 6 with different magic sums. Here the aim is to construct magic square of order 18 with each 3×3 sub-blocks as magic squares of order 3 with different magic sums. Since we know that the sum of all the numbers from 1 to 18 is 171. It is impossible to divided it in 6 equal parts, i.e., $\frac{171}{6} := 28.5$. This is the reason, we don't have magic rectangle of order (3,6) for sequential numbers from 1 to 18. Due to this we shall make magic squares of order 3 with different magic sums to complete a magic square of order 18. The final construction is based on the magic square of order 6 and **semi-magic rectangle** of order (3,6). Both are given in examples below.

Example 2.12. *Let us consider a magic square of order 6.*

AB						111
1	23	28	34	17	8	111
29	7	35	14	21	5	111
12	6	13	27	31	22	111
32	16	4	24	10	25	111
19	33	11	3	30	15	111
18	26	20	9	2	36	111
111	111	111	111	111	111	111

Example 2.13. *The Latin squares distributions of magic square of order 6 given in Example 2.12 is given by*

A						21
1	4	5	6	3	2	21
5	2	6	3	4	1	21
2	1	3	5	6	4	21
6	3	1	4	2	5	21
4	6	2	1	5	3	21
3	5	4	2	1	6	21
21	21	21	21	21	21	21

B						21
1	5	4	4	5	2	21
5	1	5	2	3	5	21
6	6	1	3	1	4	21
2	4	4	6	4	1	21
1	3	5	3	6	3	21
6	2	2	3	2	6	21
21	21	21	21	21	21	21

The **magic square** of order 6 given in Example 2.12 is obtained as

$$AB := 6 \times (A - 1) + B$$

where A is a diagonal Latin square of order 6, and B just a simple distribution of numbers.

Example 2.14. *The composite magic square of order 6 arising due to A and B, with $C := 10 \times A + B$ is given by*

C						231
11	45	54	64	35	22	231
55	21	65	32	43	15	231
26	16	31	53	61	44	231
62	34	14	46	24	51	231
41	63	25	13	56	33	231
36	52	42	23	12	66	231
231	231	231	231	231	231	231

Since it is impossible to make magic rectangle of order (3,6), let us consider following **semi-magic rectangle** order (3,6), i.e., equalities only in rows:

Example 2.15. *Let's consider the following **semi-magic rectangle** of order (3,4):*

(3,6)	1	2	3	4	5	6	Total
R1	1	6	7	12	13	18	57
R2	2	5	8	11	14	17	57
R3	3	4	9	10	15	16	57
Total	6	15	24	33	42	51	

Applying the columns values given in Example 2.15 over the Example 2.1, we get 36 blocks of magic squares of order 3, where the operation used is $AB := 18 \times (A - 1) + B$. Below are few examples.

• Block 13

1			6
2	3	1	6
1	2	3	6
3	1	2	6
6	6	6	6

3			24
7	9	8	24
9	8	7	24
8	7	9	24
24	24	24	24

13			78
25	45	8	78
9	26	43	78
44	7	27	78
78	78	78	78

- **Block 42**

4			*33*
11	10	12	*33*
12	11	10	*33*
10	12	11	*33*
33	*33*	*33*	*33*

2			*15*
6	4	5	*15*
4	5	6	*15*
5	6	4	*15*
15	*15*	*15*	*15*

42			*555*
186	166	203	*555*
202	185	168	*555*
167	204	184	*555*
555	*555*	*555*	*555*

- **Block 65**

6			*51*
17	16	18	*51*
18	17	16	*51*
16	18	17	*51*
51	*51*	*51*	*51*

5			*42*
13	15	14	*42*
15	14	13	*42*
14	13	15	*42*
42	*42*	*42*	*42*

65			*906*
301	285	320	*906*
321	302	283	*906*
284	319	303	*906*
906	*906*	*906*	*906*

Let us put these 36 blocks according to composite magic square of order 6 given in Example 2.12 we get magic square of order 18 given in example below.

Example 2.16. *According to the values given in equation, the magic square of order 18 is given by*

																		2925
19	*39*	*2*	*193*	*177*	*212*	*246*	*262*	*227*	*300*	*280*	*317*	*139*	*159*	*122*	*78*	*58*	*95*	*2925*
3	*20*	*37*	*213*	*194*	*175*	*226*	*245*	*264*	*316*	*299*	*282*	*123*	*140*	*157*	*94*	*77*	*60*	*2925*
38	*1*	*21*	*176*	*211*	*195*	*263*	*228*	*244*	*281*	*318*	*298*	*158*	*121*	*141*	*59*	*96*	*76*	*2925*
247	*267*	*230*	*73*	*57*	*92*	*301*	*285*	*320*	*132*	*148*	*113*	*187*	*171*	*206*	*31*	*51*	*14*	*2925*
231	*248*	*265*	*93*	*74*	*55*	*321*	*302*	*283*	*112*	*131*	*150*	*207*	*188*	*169*	*15*	*32*	*49*	*2925*
266	*229*	*249*	*56*	*91*	*75*	*284*	*319*	*303*	*149*	*114*	*130*	*170*	*205*	*189*	*50*	*13*	*33*	*2925*
90	*70*	*107*	*36*	*52*	*17*	*127*	*147*	*110*	*241*	*261*	*224*	*289*	*273*	*308*	*192*	*172*	*209*	*2925*
106	*89*	*72*	*16*	*35*	*54*	*111*	*128*	*145*	*225*	*242*	*259*	*309*	*290*	*271*	*208*	*191*	*174*	*2925*
71	*108*	*88*	*53*	*18*	*34*	*146*	*109*	*129*	*260*	*223*	*243*	*272*	*307*	*291*	*173*	*210*	*190*	*2925*
294	*274*	*311*	*138*	*154*	*119*	*30*	*46*	*11*	*198*	*178*	*215*	*84*	*64*	*101*	*235*	*255*	*218*	*2925*
310	*293*	*276*	*118*	*137*	*156*	*10*	*29*	*48*	*214*	*197*	*180*	*100*	*83*	*66*	*219*	*236*	*253*	*2925*
275	*312*	*292*	*155*	*120*	*136*	*47*	*12*	*28*	*179*	*216*	*196*	*65*	*102*	*82*	*254*	*217*	*237*	*2925*
181	*165*	*200*	*295*	*279*	*314*	*85*	*69*	*104*	*25*	*45*	*8*	*252*	*268*	*233*	*133*	*153*	*116*	*2925*
201	*182*	*163*	*315*	*296*	*277*	*105*	*86*	*67*	*9*	*26*	*43*	*232*	*251*	*270*	*117*	*134*	*151*	*2925*
164	*199*	*183*	*278*	*313*	*297*	*68*	*103*	*87*	*44*	*7*	*27*	*269*	*234*	*250*	*152*	*115*	*135*	*2925*
144	*160*	*125*	*240*	*256*	*221*	*186*	*166*	*203*	*79*	*63*	*98*	*24*	*40*	*5*	*306*	*286*	*323*	*2925*
124	*143*	*162*	*220*	*239*	*258*	*202*	*185*	*168*	*99*	*80*	*61*	*4*	*23*	*42*	*322*	*305*	*288*	*2925*
161	*126*	*142*	*257*	*222*	*238*	*167*	*204*	*184*	*62*	*97*	*81*	*41*	*6*	*22*	*287*	*324*	*304*	*2925*
2925	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*	*2925*

The above magic square is with magic sum $S_{18\times18} = 2925$. Each 3×3 block a magic square of order 3 with different magic sums giving again a magic square of order 6 given in example below.

Example 2.17. *The sum of magic squares of order 3 of Example 2.16 give a magic square of order 6 given by*

						2925
60	582	735	897	420	231	2925
744	222	906	393	564	96	2925
267	105	384	726	870	573	2925
879	411	87	591	249	708	2925
546	888	258	78	753	402	2925
429	717	555	240	69	915	2925
2925	2925	2925	2925	2925	2925	2925

2.5 Pan Magic Square of Order 21

In previous work [22], the author calculated pan magic square of order 21, where each 7×7 blocks are pan diagonal magic squares of order 7 with equal magic sums. Here our aim is to construct pan magic square of order 21 with each 3×3 blocks are of equal sums entries. We shall use the idea of magic-rectangle of order (3,7) to construct this magic square. For simiplicity, let's rewrite the magic rectangle of order (3,7) given in Example 1.2 as below:

Example 2.18. *The **magic rectangle** of order (3,7) is given by*

(3,7)	1	2	3	4	5	6	7	Total
R1	1	12	13	6	17	20	8	77
R2	18	19	15	11	7	3	4	77
R3	14	2	5	16	9	10	21	77
Total	33	33	33	33	33	33	33	

Let's construct 49 blocks of order 3 using the columns of above magic rectangle 2.18 in a magic square of order 3 given in Example 2.1 by applying the operation $AB := 21 \times (A-1) + B$. Below are few examples.

- **Block 17**

1			33
12	6	15	33
15	12	6	33
6	15	12	33
33	33	33	24

7			30
16	7	10	33
7	10	16	33
10	16	7	33
33	33	33	33

17			660
247	112	304	663
301	241	121	663
115	310	238	663
663	663	663	726

- **Block 36**

3			33
9	5	19	33
19	9	5	33
5	19	9	33
33	33	33	27

6			39
3	17	13	33
17	13	3	33
13	3	17	33
33	33	33	33

36			669
171	101	391	663
395	181	87	663
97	381	185	663
663	663	663	537

• Block 57

⑤			*33*
8	4	21	*33*
21	8	4	*33*
4	21	8	*33*
33	*33*	*33*	*24*

⑦			*30*
16	7	10	*33*
7	10	16	*33*
10	16	7	*33*
33	*33*	*33*	*33*

(57)			*660*
163	70	430	*663*
427	157	79	*663*
73	436	154	*663*
663	*663*	*663*	*474*

Based on same procedure, we can construct total 49 blocks of sem-magic squares of order 3.

Structure 2.3. *Let's put these 49 blocks according to following structure:*

11	12	13	14	15	16	17
21	22	23	24	25	26	27
31	32	33	34	35	36	37
41	42	43	44	45	46	47
51	52	53	54	55	56	57
61	62	63	64	65	66	67
71	72	73	74	75	76	77

According to above Structure refs7a a **pan magic square** of order 21 is given in the example below.

Example 2.19. *According to distribution given in Example 2.18, Structure 2.3 and 49* ***semi-magic squares*** *of order 3, we have a* ***pan magic square*** *of order 21 given by*

		4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641
	246	111	306	232	123	308	250	110	303	233	125	305	252	109	302	234	122	307	247	112	304	4641
4641	300	243	120	312	245	106	299	240	124	314	242	107	298	239	126	311	244	108	301	241	121	4641
4641	117	309	237	119	295	249	114	313	236	116	296	251	113	315	235	118	297	248	115	310	238	4641
4641	288	363	12	274	375	14	292	362	9	275	377	11	294	361	8	276	374	13	289	364	10	4641
4641	6	285	372	18	287	358	5	282	376	20	284	359	4	281	378	17	286	360	7	283	373	4641
4641	369	15	279	371	1	291	366	19	278	368	2	293	365	21	277	370	3	290	367	16	280	4641
4641	183	90	390	169	102	392	187	89	387	170	104	389	189	88	386	171	101	391	184	91	388	4641
4641	384	180	99	396	182	85	383	177	103	398	179	86	382	176	105	395	181	87	385	178	100	4641
4641	96	393	174	98	379	186	93	397	173	95	380	188	92	399	172	97	381	185	94	394	175	4641
4641	225	405	33	211	417	35	229	404	30	212	419	32	231	403	29	213	416	34	226	406	31	4641
4641	27	222	414	39	224	400	26	219	418	41	221	401	25	218	420	38	223	402	28	220	415	4641
4641	411	36	216	413	22	228	408	40	215	410	23	230	407	42	214	412	24	227	409	37	217	4641
4641	162	69	432	148	81	434	166	68	429	149	83	431	168	67	428	150	80	433	163	70	430	4641
4641	426	159	78	438	161	64	425	156	82	440	158	65	424	155	84	437	160	66	427	157	79	4641
4641	75	435	153	77	421	165	72	439	152	74	422	167	71	441	151	76	423	164	73	436	154	4641
4641	267	342	54	253	354	56	271	341	51	254	356	53	273	340	50	255	353	55	268	343	52	4641
4641	48	264	351	60	266	337	47	261	355	62	263	338	46	260	357	59	265	339	49	262	352	4641
4641	348	57	258	350	43	270	345	61	257	347	44	272	344	63	256	349	45	269	346	58	259	4641
4641	204	132	327	190	144	329	208	131	324	191	146	326	210	130	323	192	143	328	205	133	325	4641
4641	321	201	141	333	203	127	320	198	145	335	200	128	319	197	147	332	202	129	322	199	142	4641
4641	138	330	195	140	316	207	135	334	194	137	317	209	134	336	193	139	318	206	136	331	196	4641
	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641	4641

In this case, the magic sum is $S_{21\times21} = 4641$. The 3×3 blocks are **semi-magic squares** with semi-magic sums $S_{3\times3} = 663$ (in rows and columns).

2.6 Pan Magic Square of Order 24

In [23], the author constricted block-wise pan magic square of order 24, where each block of order 4 is a pan magic square of order 4 with equal magic sums. Here, the aim is to construct a **block-wise pan magic square** of order 24, where each block of order 3 a magic square with different magic sums. It is well known that a magic square of order 24 with 576 numbers 1-576 has a magic sum $S_{24\times24} := 16400$. If we divide 16400 by 8 we get a fraction vaule, i.e., $\frac{6924}{8} := 865.5$. It implies that we are unable to construct block-wise magic square of order 24 with equal sum blocks of order 3. In this case, we shall construct block-wise magic square of order 24 with each block of order 3 having different magic square sums. This shall be done by use of magic square of order 3 given in Example 2.1 with composite **pan magic square** of order 8. In this case, we have considered a **pan magic square** of order 8 with bimagic property. See the example below.

Example 2.20. ***Pan magic square*** *of order 8 is given by*

		260	260	260	260	260	260	260	260
	29	40	1	60	30	39	2	59	260
260	4	57	32	37	3	58	31	38	260
260	64	5	36	25	63	6	35	26	260
260	33	28	61	8	34	27	62	7	260
260	21	48	9	52	22	47	10	51	260
260	12	49	24	45	11	50	23	46	260
260	56	13	44	17	55	14	43	18	260
260	41	20	53	16	42	19	54	15	260
	260	260	260	260	260	260	260	260	260

In this case, the magic sum is $S_{8\times8} = 260$. Each 4×4 block is a pan magic square of order 4 with equal magic sums given as $S_{4\times4} := 130$. Also sum of all four members of each 2×2 blocks are the same as of magic square of order 4, i.e., 130.

Example 2.21. *The Latin square decompositions of magic square of order 8 given in Example 2.20 is given by*

A		36	36	36	36	36	36	36	36
	4	5	1	8	4	5	1	8	36
36	1	8	4	5	1	8	4	5	36
36	8	1	5	4	8	1	5	4	36
36	5	4	8	1	5	4	8	1	36
36	3	6	2	7	3	6	2	7	36
36	2	7	3	6	2	7	3	6	36
36	7	2	6	3	7	2	6	3	36
36	6	3	7	2	6	3	7	2	36
	36	36	36	36	36	36	36	36	36

B		36	36	36	36	36	36	36	36
	5	8	1	4	6	7	2	3	36
36	4	1	8	5	3	2	7	6	36
36	8	5	4	1	7	6	3	2	36
36	1	4	5	8	2	3	6	7	36
36	5	8	1	4	6	7	2	3	36
36	4	1	8	5	3	2	7	6	36
36	8	5	4	1	7	6	3	2	36
36	1	4	5	8	2	3	6	7	36
	36	36	36	36	36	36	36	36	36

The magic square given in Example 2.20 is obtained by the operation $8\times(A-1)+B$. Moreover, A and B are **pair of mutually orthogonal diagonal Latin squares**. Based on A and B, the composite magic square of order 8 is given in example below.

Example 2.22. *Composite* ***pan magic square*** *of order 8 applying the operation* $10\times A+B$ *in Example 2.21 is given by*

C		396	396	396	396	396	396	396	396
	45	58	11	84	46	57	12	83	396
396	14	81	48	55	13	82	47	56	396
396	88	15	54	41	87	16	53	42	396
396	51	44	85	18	52	43	86	17	396
396	35	68	21	74	36	67	22	73	396
396	24	71	38	65	23	72	37	66	396
396	78	25	64	31	77	26	63	32	396
396	61	34	75	28	62	33	76	27	396
	396	396	396	396	396	396	396	396	396

Since it is impossible to make magic rectangle of order (3,8), let us consider following **semi-magic rectangle** order (3,8), i.e., equality only in rows:

Example 2.23. *Let's consider the following* ***semi-magic rectangle*** *of order (3,8):*

(3,8)	1	2	3	4	5	6	7	8	Total
R1	1	6	7	12	13	18	19	24	100
R2	2	5	8	11	14	17	20	23	100
R3	3	4	9	10	15	16	21	22	100
Total	6	15	24	33	42	51	60	69	

Note 2.2. *If we add the numbers from 1 to 24, we have total sum as 300. It is impossible to divided 300 in eight equal parts, i.e.,* $\frac{300}{8} = 37.5$*. This is the reason, why we are unable to make magic rectangle of order (3,8).*

Applying the columns values given in Example 2.23 over the Example 2.1, we get 64 blocks of magic squares of order 3, where the operation used is $AB := 24 \times (A-1) + B$. See below some examples:

• Block 24

2			15
5	4	6	15
6	5	4	15
4	6	5	15
15	15	15	15

4			33
12	10	11	33
10	11	12	33
11	12	10	33
33	33	33	33

24			321
108	82	131	321
130	107	84	321
83	132	106	321
321	321	321	321

• Block 75

7			60
20	21	19	60
19	20	21	60
21	19	20	60
60	60	60	60

5			42
13	15	14	42
15	14	13	42
14	13	15	42
42	42	42	42

75			1410
469	495	446	1410
447	470	493	1410
494	445	471	1410
1410	1410	1410	1410

• Block 86

8			69
23	22	24	69
24	23	22	69
22	24	23	69
69	69	69	69

6			51
18	16	17	51
16	17	18	51
17	18	16	51
51	51	51	51

86			1635
546	520	569	1635
568	545	522	1635
521	570	544	1635
1635	1635	1635	1635

Above are only 3 blocks of magic squares of order 3, but in total we have 64 blocks. Put these 64 blocks of magic squares of order 3 according to composite magic square given in Example 2.22 we get a pan diagonal magic square of order 24.

Example 2.24. . *The pan magic square of order 24 constructed according to distribution 2.23 applied over the Example 2.1 and put according to Example 2.22 with the operation* $AB := 24 \times (A-1) + B$ *is given by*

I	1	2	3	4	5	6	7	8	9	10	11	12
		6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924
	253	231	278	336	358	311	25	51	2	540	514	563
6924	279	254	229	310	335	360	3	26	49	562	539	516
6924	230	277	255	359	312	334	50	1	27	515	564	538
6924	36	58	11	529	507	554	264	238	287	325	351	302
6924	10	35	60	555	530	505	286	263	240	303	326	349
6924	59	12	34	506	553	531	239	288	262	350	301	327
6924	552	526	575	37	63	14	324	346	299	241	219	266
6924	574	551	528	15	38	61	298	323	348	267	242	217
6924	527	576	550	62	13	39	347	300	322	218	265	243
6924	313	339	290	252	226	275	541	519	566	48	70	23
6924	291	314	337	274	251	228	567	542	517	22	47	72
6924	338	289	315	227	276	250	518	565	543	71	24	46
6924	181	207	158	408	382	431	97	75	122	468	490	443
6924	159	182	205	430	407	384	123	98	73	442	467	492
6924	206	157	183	383	432	406	74	121	99	491	444	466
6924	108	82	131	457	483	434	192	214	167	397	375	422
6924	130	107	84	435	458	481	166	191	216	423	398	373
6924	83	132	106	482	433	459	215	168	190	374	421	399
6924	480	502	455	109	87	134	396	370	419	169	195	146
6924	454	479	504	135	110	85	418	395	372	147	170	193
6924	503	456	478	86	133	111	371	420	394	194	145	171
6924	385	363	410	180	202	155	469	495	446	120	94	143
6924	411	386	361	154	179	204	447	470	493	142	119	96
6924	362	409	387	203	156	178	494	445	471	95	144	118
	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924

13	14	15	16	17	18	19	20	21	22	23	24	II
6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924
258	232	281	331	357	308	30	52	5	535	513	560	6924
280	257	234	309	332	355	4	29	54	561	536	511	6924
233	282	256	356	307	333	53	6	28	512	559	537	6924
31	57	8	534	508	557	259	237	284	330	352	305	6924
9	32	55	556	533	510	285	260	235	304	329	354	6924
56	7	33	509	558	532	236	283	261	353	306	328	6924
547	525	572	42	64	17	319	345	296	246	220	269	6924
573	548	523	16	41	66	297	320	343	268	245	222	6924
524	571	549	65	18	40	344	295	321	221	270	244	6924
318	340	293	247	225	272	546	520	569	43	69	20	6924
292	317	342	273	248	223	568	545	522	21	44	67	6924
341	294	316	224	271	249	521	570	544	68	19	45	6924
186	208	161	403	381	428	102	76	125	463	489	440	6924
160	185	210	429	404	379	124	101	78	441	464	487	6924
209	162	184	380	427	405	77	126	100	488	439	465	6924
103	81	128	462	484	437	187	213	164	402	376	425	6924
129	104	79	436	461	486	165	188	211	424	401	378	6924
80	127	105	485	438	460	212	163	189	377	426	400	6924
475	501	452	114	88	137	391	369	416	174	196	149	6924
453	476	499	136	113	90	417	392	367	148	173	198	6924
500	451	477	89	138	112	368	415	393	197	150	172	6924
390	364	413	175	201	152	474	496	449	115	93	140	6924
412	389	366	153	176	199	448	473	498	141	116	91	6924
365	414	388	200	151	177	497	450	472	92	139	117	6924
6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924	6924

In this case, the magic sum is $S_{24\times 24} = 6924$. The above **pan magic square** of order 24 has some extra properties given in note below.

Note 2.3. . *The pan magic square of order 24 given in Example 2.24 has following properties:*

(i) The 16 blocks of order 6×6 are of equal sums entries, i.e., $S_{36} := 10386$;

*(ii) The 4 corner blocks of order 12 are **pan magic squares** with equal magic sums, i.e., $S_{12\times 12} := 3462$;*

*(iii) Each 3×3 blocks are magic squares of order 3 with different magic sums forming again a **pan magic square** of order 8.*

Example 2.25. *The **pan diagonal square** of order 8 formed by magic square sums of order 3 of Example 2.24 is given by*

		6924	6924	6924	6924	6924	6924	6924	6924
	762	1005	78	1617	771	996	87	1608	6924
6924	105	1590	789	978	96	1599	780	987	6924
6924	1653	114	969	726	1644	123	960	735	6924
6924	942	753	1626	141	951	744	1635	132	6924
6924	546	1221	294	1401	555	1212	303	1392	6924
6924	321	1374	573	1194	312	1383	564	1203	6924
6924	1437	330	1185	510	1428	339	1176	519	6924
6924	1158	537	1410	357	1167	528	1419	348	6924
	6924	6924	6924	6924	6924	6924	6924	6924	6924

Each 4×4 block is a **pan magic square** of order 4 with equal magic sums given as $S_{4\times4} := 3462$. Also sum of all four members of each 2×2 blocks are the same as of magic square of order 4, i.e., 4362.

2.7 Pan Magic Square of Order 27

In previous work [22], the author constructed a magic square of order 27, where 9×9 blocks are magic squares with equal magic sums. Here our aim is to construct pan magic square of order 21 with 3×3 blocks of equal sums entries. We shall use the idea of magic rectangle of order (3,9) to construct this magic square. Let's rewrite the magic rectangle of order (3,9) given in Example 1.3 as below:

Example 2.26. *The magic rectangle of order (3,9) is given by*

(3,9)	1	2	3	4	5	6	7	8	9	Total
R1	1	15	5	16	21	22	9	26	11	126
R2	24	25	18	20	14	8	10	3	4	126
R3	17	2	19	6	7	12	23	13	27	126
Total	42	42	42	42	42	42	42	42	42	

Let's construct 81 blocks of order 3 using the columns of above magic rectangle 2.26 in a magic square of order 3 given in Example 2.1 by applying the operation $AB := 27 \times (A-1) + B$. See below some examples of these blocks:

• Block 52

5			42
14	7	21	42
21	14	7	42
7	21	14	42
42	42	42	42

2			75
15	2	25	42
2	25	15	42
25	15	2	42
42	42	42	42

52			1128
366	164	565	1095
542	376	177	1095
187	555	353	1095
1095	1095	1095	1095

- **Block 73**

7			42
10	23	9	42
9	10	23	42
23	9	10	42
42	42	42	42

3			54
5	19	18	42
19	18	5	42
18	5	19	42
42	42	42	42

73			1160
248	613	234	1095
235	261	599	1095
612	221	262	1095
1095	1095	1095	1095

- **Block 94**

9			42
4	27	11	42
11	4	27	42
27	11	4	42
42	42	42	42

4			60
16	6	20	42
6	20	16	42
20	16	6	42
42	42	42	42

94			1113
97	708	290	1095
276	101	718	1095
722	286	87	1095
1095	1095	1095	1095

Let's put these 81 blocks according to following structure:

Structure 2.4. *Let's consider following structure:*

11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

Example 2.27. *According to magic triangle Example 1.3, Structure 2.4 and 81 blocks of order 3 of equal sums entries, we have a **pan magic square** of order 27 given by*

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
		9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855
	622	449	24	636	434	25	626	451	18	637	438	20	642	439	14
9855	17	645	433	2	646	447	19	639	437	6	641	448	7	635	453
9855	456	1	638	457	15	623	450	5	640	452	16	627	446	21	628
9855	649	44	402	663	29	403	653	46	396	664	33	398	669	34	392
9855	395	672	28	380	673	42	397	666	32	384	668	43	385	662	48
9855	51	379	665	52	393	650	45	383	667	47	394	654	41	399	655
9855	460	503	132	474	488	133	464	505	126	475	492	128	480	493	122
9855	125	483	487	110	484	501	127	477	491	114	479	502	115	473	507
9855	510	109	476	511	123	461	504	113	478	506	124	465	500	129	466
9855	514	152	429	528	137	430	518	154	423	529	141	425	534	142	419
9855	422	537	136	407	538	150	424	531	140	411	533	151	412	527	156
9855	159	406	530	160	420	515	153	410	532	155	421	519	149	426	520
9855	352	179	564	366	164	565	356	181	558	367	168	560	372	169	554
9855	557	375	163	542	376	177	559	369	167	546	371	178	547	365	183
9855	186	541	368	187	555	353	180	545	370	182	556	357	176	561	358
9855	190	314	591	204	299	592	194	316	585	205	303	587	210	304	581
9855	584	213	298	569	214	312	586	207	302	573	209	313	574	203	318
9855	321	568	206	322	582	191	315	572	208	317	583	195	311	588	196
9855	244	611	240	258	596	241	248	613	234	259	600	236	264	601	230
9855	233	267	595	218	268	609	235	261	599	222	263	610	223	257	615
9855	618	217	260	619	231	245	612	221	262	614	232	249	608	237	250
9855	55	341	699	69	326	700	59	343	693	70	330	695	75	331	689
9855	692	78	325	677	79	339	694	72	329	681	74	340	682	68	345
9855	348	676	71	349	690	56	342	680	73	344	691	60	338	696	61
9855	82	719	294	96	704	295	86	721	288	97	708	290	102	709	284
9855	287	105	703	272	106	717	289	99	707	276	101	718	277	95	723
9855	726	271	98	727	285	83	720	275	100	722	286	87	716	291	88
	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855

16	17	18	19	20	21	22	23	24	25	26	27	II
9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855
643	444	8	630	455	10	647	445	3	632	459	4	9855
12	629	454	23	631	441	13	624	458	27	625	443	9855
440	22	633	442	9	644	435	26	634	436	11	648	9855
670	39	386	657	50	388	674	40	381	659	54	382	9855
390	656	49	401	658	36	391	651	53	405	652	38	9855
35	400	660	37	387	671	30	404	661	31	389	675	9855
481	498	116	468	509	118	485	499	111	470	513	112	9855
120	467	508	131	469	495	121	462	512	135	463	497	9855
494	130	471	496	117	482	489	134	472	490	119	486	9855
535	147	413	522	158	415	539	148	408	524	162	409	9855
417	521	157	428	523	144	418	516	161	432	517	146	9855
143	427	525	145	414	536	138	431	526	139	416	540	9855
373	174	548	360	185	550	377	175	543	362	189	544	9855
552	359	184	563	361	171	553	354	188	567	355	173	9855
170	562	363	172	549	374	165	566	364	166	551	378	9855
211	309	575	198	320	577	215	310	570	200	324	571	9855
579	197	319	590	199	306	580	192	323	594	193	308	9855
305	589	201	307	576	212	300	593	202	301	578	216	9855
265	606	224	252	617	226	269	607	219	254	621	220	9855
228	251	616	239	253	603	229	246	620	243	247	605	9855
602	238	255	604	225	266	597	242	256	598	227	270	9855
76	336	683	63	347	685	80	337	678	65	351	679	9855
687	62	346	698	64	333	688	57	350	702	58	335	9855
332	697	66	334	684	77	327	701	67	328	686	81	9855
103	714	278	90	725	280	107	715	273	92	729	274	9855
282	89	724	293	91	711	283	84	728	297	85	713	9855
710	292	93	712	279	104	705	296	94	706	281	108	9855
9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855	9855

Combining Parts I and II we get a **pan magic square** of order 27 with magic sum $S_{27\times27} = 9855$. The 3×3 blocks are **semi-magic squares** with equal semi-magic sums $S_{3\times3} := 1095$ (in rows and columns).

2.8 Magic Square of Order 30

In the previous work, the author worked in magic square of order 30 with sub-blocks of order 10. Here the aim is to construct magic square of order 30, where each sub-blocks are magic squares of order 3 with different magic sums. Since we know that sum of all the numbers from 1 to 30 is 465. It is impossible to divided it in 10 equal parts, i.e., $\frac{465}{10} := 46.5$. This is the reason, we don't have magic rectangle of order (3,10) for sequential numbers from 1 to 30. Due to this we shall make magic squares of order 3 with different magic sums to complete a magic square of order 30. The construction is based on the magic square of order 10 and semi-magic rectangle of order (3,10). Both are given in examples below.

Example 2.28. *Let's consider a magic square of order 10 is given by*

AB										495
00	98	45	61	17	73	54	86	29	32	495
75	11	97	42	59	38	80	03	64	26	495
41	67	22	89	35	16	78	50	93	04	495
69	06	74	33	20	82	47	91	15	58	495
53	30	68	76	44	21	95	19	02	87	495
84	43	10	28	96	55	09	62	37	71	495
27	52	39	05	81	94	66	48	70	13	495
36	24	83	90	08	49	12	77	51	65	495
92	79	56	14	63	07	31	25	88	40	495
18	85	01	57	72	60	23	34	46	99	495
495	495	495	495	495	495	495	495	495	495	495

Example 2.29. *The Latin squares decompositions of magic square of order 10 given in Example 2.28 are given by*

A										45
0	9	4	6	1	7	5	8	2	3	45
7	1	9	4	5	3	8	0	6	2	45
4	6	2	8	3	1	7	5	9	0	45
6	0	7	3	2	8	4	9	1	5	45
5	3	6	7	4	2	9	1	0	8	45
8	4	1	2	9	5	0	6	3	7	45
2	5	3	0	8	9	6	4	7	1	45
3	2	8	9	0	4	1	7	5	6	45
9	7	5	1	6	0	3	2	8	4	45
1	8	0	5	7	6	2	3	4	9	45
45	45	45	45	45	45	45	45	45	45	45

B										45
0	8	5	1	7	3	4	6	9	2	45
5	1	7	2	9	8	0	3	4	6	45
1	7	2	9	5	6	8	0	3	4	45
9	6	4	3	0	2	7	1	5	8	45
3	0	8	6	4	1	5	9	2	7	45
4	3	0	8	6	5	9	2	7	1	45
7	2	9	5	1	4	6	8	0	3	45
6	4	3	0	8	9	2	7	1	5	45
2	9	6	4	3	7	1	5	8	0	45
8	5	1	7	2	0	3	4	6	9	45
45	45	45	45	45	45	45	45	45	45	45

The **magic square** of order 10 given in Example 2.28 is obtained as

$$AB := 10 \times A + B$$

where A and B are **mutually orthogonal diagonalize Latin squares**. In this case, we don't need to write a composite magic square. The magic given above itself serves as composite magic square, i.e., $AB = C$.

Example 2.30. *Let's consider the following **semi-magic rectangle** of order (3,10):*

(3,10)	0	1	2	3	4	5	6	7	8	9	Total
R1	1	6	7	12	13	18	19	24	25	30	155
R1	2	5	8	11	14	17	20	23	26	29	155
R1	3	4	9	10	15	16	21	22	27	28	155
Total	6	15	24	33	42	51	60	69	78	87	

For simplicity we have represented columns as 0 to 9 instead of 1 to 10. Applying the columns values given in Example 2.30 over the Example 2.1, we get 100 blocks of magic squares of order 3, where the operation used is $AB := 30 \times (A-1) + B$. Let us put these 100 blocks according composite magic square AB of order 10 given in Example **??** we get two magic squares. One with magic sums of order 3 and another a general magic square of order 30. Before below are examples of some blocks of order 3.

- **Block 30**

3			33
11	10	12	33
12	11	10	33
10	12	11	33
33	33	33	33

0			6
1	3	2	6
3	2	1	6
2	1	3	6
6	6	6	6

30			906
301	273	332	906
333	302	271	906
272	331	303	906
906	906	906	906

- **Block 74**

7			69
23	22	24	69
24	23	22	69
22	24	23	69
69	69	69	69

4			42
13	15	14	42
15	14	13	42
14	13	15	42
42	42	42	42

74			2022
673	645	704	2022
705	674	643	2022
644	703	675	2022
2022	2022	2022	2022

- **Block 92**

9			87
29	28	30	87
30	29	28	87
28	30	29	87
87	87	87	87

2			24
7	9	8	24
9	8	7	24
8	7	9	24
24	24	24	24

92			2544
847	819	878	2544
879	848	817	2544
818	877	849	2544
2544	2544	2544	2544

- **Block 08**

0			6
2	3	1	6
1	2	3	6
3	1	2	6
6	6	6	6

8			78
25	27	26	78
27	26	25	78
26	25	27	78
78	78	78	78

08			168
55	87	26	168
27	56	85	168
86	25	57	168
168	168	168	168

The 100 blocks constructed according to above examples, and keeping them according to Example 2.28 we get a magic square of order 30.

Example 2.31. *According to distribution given in Examples 2.30, 2.28 and 100 blocks of magic square of orders 3 lead us to following magic square of order 30:*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
I														
31	63	2	865	837	896	408	436	377	576	604	545	144	112	173
3	32	61	897	866	835	376	407	438	544	575	606	172	143	114
62	1	33	836	895	867	437	378	406	605	546	574	113	174	142
678	646	707	126	94	155	864	832	893	397	429	368	510	478	539
706	677	648	154	125	96	892	863	834	369	398	427	538	509	480
647	708	676	95	156	124	833	894	862	428	367	399	479	540	508
396	424	365	594	622	563	217	249	188	780	808	749	318	286	347
364	395	426	562	593	624	189	218	247	748	779	810	346	317	288
425	366	394	623	564	592	248	187	219	809	750	778	287	348	316
600	628	569	49	81	20	673	645	704	312	280	341	211	243	182
568	599	630	21	50	79	705	674	643	340	311	282	183	212	241
629	570	598	80	19	51	644	703	675	281	342	310	242	181	213
492	460	521	301	273	332	595	627	566	679	651	710	403	435	374
520	491	462	333	302	271	567	596	625	711	680	649	375	404	433
461	522	490	272	331	303	626	565	597	650	709	681	434	373	405
763	795	734	402	430	371	121	93	152	235	267	206	859	831	890
735	764	793	370	401	432	153	122	91	207	236	265	891	860	829
794	733	765	431	372	400	92	151	123	266	205	237	830	889	861
234	262	203	487	459	518	330	298	359	48	76	17	756	784	725
202	233	264	519	488	457	358	329	300	16	47	78	724	755	786
263	204	232	458	517	489	299	360	328	77	18	46	785	726	754
319	291	350	223	255	194	762	790	731	841	813	872	55	87	26
351	320	289	195	224	253	730	761	792	873	842	811	27	56	85
290	349	321	254	193	225	791	732	760	812	871	843	86	25	57
847	819	878	690	658	719	499	471	530	133	105	164	582	610	551
879	848	817	718	689	660	531	500	469	165	134	103	550	581	612
818	877	849	659	720	688	470	529	501	104	163	135	611	552	580
145	117	176	768	796	737	36	64	5	504	472	533	667	639	698
177	146	115	736	767	798	4	35	66	532	503	474	699	668	637
116	175	147	797	738	766	65	6	34	473	534	502	638	697	669
13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515

16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	
II															13515
672	640	701	493	465	524	769	801	740	240	268	209	307	279	338	13515
700	671	642	525	494	463	741	770	799	208	239	270	339	308	277	13515
641	702	670	464	523	495	800	739	771	269	210	238	278	337	309	13515
325	297	356	751	783	722	42	70	11	583	615	554	229	261	200	13515
357	326	295	723	752	781	10	41	72	555	584	613	201	230	259	13515
296	355	327	782	721	753	71	12	40	614	553	585	260	199	231	13515
139	111	170	685	657	716	481	453	512	852	820	881	43	75	14	13515
171	140	109	717	686	655	513	482	451	880	851	822	15	44	73	13515
110	169	141	656	715	687	452	511	483	821	882	850	74	13	45	13515
757	789	728	414	442	383	846	814	875	138	106	167	505	477	536	13515
729	758	787	382	413	444	874	845	816	166	137	108	537	506	475	13515
788	727	759	443	384	412	815	876	844	107	168	136	476	535	507	13515
216	244	185	858	826	887	150	118	179	37	69	8	774	802	743	13515
184	215	246	886	857	828	178	149	120	9	38	67	742	773	804	13515
245	186	214	827	888	856	119	180	148	68	7	39	803	744	772	13515
498	466	527	60	88	29	577	609	548	324	292	353	666	634	695	13515
526	497	468	28	59	90	549	578	607	352	323	294	694	665	636	13515
467	528	496	89	30	58	608	547	579	293	354	322	635	696	664	13515
853	825	884	589	621	560	415	447	386	661	633	692	132	100	161	13515
885	854	823	561	590	619	387	416	445	693	662	631	160	131	102	13515
824	883	855	620	559	591	446	385	417	632	691	663	101	162	130	13515
420	448	389	127	99	158	684	652	713	486	454	515	588	616	557	13515
388	419	450	159	128	97	712	683	654	514	485	456	556	587	618	13515
449	390	418	98	157	129	653	714	682	455	516	484	617	558	586	13515
54	82	23	306	274	335	228	256	197	775	807	746	391	423	362	13515
22	53	84	334	305	276	196	227	258	747	776	805	363	392	421	13515
83	24	52	275	336	304	257	198	226	806	745	777	422	361	393	13515
571	603	542	222	250	191	313	285	344	409	441	380	870	838	899	13515
543	572	601	190	221	252	345	314	283	381	410	439	898	869	840	13515
602	541	573	251	192	220	284	343	315	440	379	411	839	900	868	13515
13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515	13515

Combining Parts I and II we get the required result. In this case, the magic square sum is $S_{30\times 30} = 13515$. Each 3×3 block a magic square of order 3 with different magic sums as given in Example 2.17. These magic sums again make a magic square of order 10 given in example below.

Example 2.32. *The magic square formed by magic sum 100 blocks of order 3 give us the following magic square of order 10:*

										13155
96	2598	1221	1725	429	2013	1482	2310	717	924	13155
2031	375	2589	1194	1527	978	2256	123	1752	690	13155
1185	1779	654	2337	951	420	2058	1446	2553	132	13155
1797	150	2022	933	636	2274	1239	2535	411	1518	13155
1473	906	1788	2040	1212	645	2571	447	114	2319	13155
2292	1203	366	708	2580	1491	177	1734	969	1995	13155
699	1464	987	141	2265	2562	1770	1248	1986	393	13155
960	672	2283	2526	168	1257	384	2049	1455	1761	13155
2544	2067	1500	402	1743	159	915	681	2328	1176	13155
438	2301	105	1509	2004	1716	663	942	1230	2607	13155
13155	13155	13155	13155	13155	13155	13155	13155	13155	13155	13155

2.9 Pan Magic Square of Order 33

In previous work [22], the author constructed a magic square of order 33, where 11×11 blocks are magic squares with equal magic sums. Here our aim is to construct pan magic square of order 33 with 3×3 blocks of equal sums entries. We shall use the idea of composite magic square of order 11 and magic rectangle of order (3,11) to construct this magic square. Just to recapitulate, let's rewrite below the magic rectangle of order (3,11) given in Example 1.4 as below:

(3,11)	1	2	3	4	5	6	7	8	9	10	11	Total
R1	22	29	3	7	24	9	26	13	16	32	6	187
R2	1	20	30	23	19	17	15	11	4	14	33	187
R3	28	2	18	21	8	25	10	27	31	5	12	187
Total	51	51	51	51	51	51	51	51	51	51	51	

The **pan magic square** of order 11 is given in example below.

Example 2.33. *Let's consider the following **pan magic square** of order 11:*

AB		671	671	671	671	671	671	671	671	671	671	671
	1	13	25	37	49	61	73	85	97	109	121	671
671	108	120	11	12	24	36	48	60	72	84	96	671
671	83	95	107	119	10	22	23	35	47	59	71	671
671	58	70	82	94	106	118	9	21	33	34	46	671
671	44	45	57	69	81	93	105	117	8	20	32	671
671	19	31	43	55	56	68	80	92	104	116	7	671
671	115	6	18	30	42	54	66	67	79	91	103	671
671	90	102	114	5	17	29	41	53	65	77	78	671
671	76	88	89	101	113	4	16	28	40	52	64	671
671	51	63	75	87	99	100	112	3	15	27	39	671
671	26	38	50	62	74	86	98	110	111	2	14	671
	671	671	671	671	671	671	671	671	671	671	671	671

Example 2.34. *The above **pan magic square** of order 11 is constructed based on a pair of **mutually diagonal Latin squares** A and B given by*

A		66	66	66	66	66	66	66	66	66	66	66
	1	2	3	4	5	6	7	8	9	10	11	66
66	10	11	1	2	3	4	5	6	7	8	9	66
66	8	9	10	11	1	2	3	4	5	6	7	66
66	6	7	8	9	10	11	1	2	3	4	5	66
66	4	5	6	7	8	9	10	11	1	2	3	66
66	2	3	4	5	6	7	8	9	10	11	1	66
66	11	1	2	3	4	5	6	7	8	9	10	66
66	9	10	11	1	2	3	4	5	6	7	8	66
66	7	8	9	10	11	1	2	3	4	5	6	66
66	5	6	7	8	9	10	11	1	2	3	4	66
66	3	4	5	6	7	8	9	10	11	1	2	66
	66	66	66	66	66	66	66	66	66	66	66	66

B		66	66	66	66	66	66	66	66	66	66	66
	1	2	3	4	5	6	7	8	9	10	11	66
66	9	10	11	1	2	3	4	5	6	7	8	66
66	6	7	8	9	10	11	1	2	3	4	5	66
66	3	4	5	6	7	8	9	10	11	1	2	66
66	11	1	2	3	4	5	6	7	8	9	10	66
66	8	9	10	11	1	2	3	4	5	6	7	66
66	5	6	7	8	9	10	11	1	2	3	4	66
66	2	3	4	5	6	7	8	9	10	11	1	66
66	10	11	1	2	3	4	5	6	7	8	9	66
66	7	8	9	10	11	1	2	3	4	5	6	66
66	4	5	6	7	8	9	10	11	1	2	3	66
	66	66	66	66	66	66	66	66	66	66	66	66

The **pan magic square** appearing in $\boldsymbol{AB}$ is constructed according to the following formula:

$$AB := 11 \times (A - 1) + B$$

where A and B are the **mutually orthogonal diagonal Latin squares** of order 11 as given above. Based on Latin squares decompositions, let's consider the following composite matrix using the pairs as (A, B).

Example 2.35. . *The composite matrix arising due to Example 2.34 is given by*

(1,1)	(2,2)	(3,3)	(4,4)	(5,5)	(6,6)	(7,7)	(8,8)	(9,9)	(10,10)	(11,11)
(10,9)	(11,10)	(1,11)	(2,1)	(3,2)	(4,3)	(5,4)	(6,5)	(7,6)	(8,7)	(9,8)
(8,6)	(9,7)	(10,8)	(11,9)	(1,10)	(2,11)	(3,1)	(4,2)	(5,3)	(6,4)	(7,5)
(6,3)	(7,4)	(8,5)	(9,6)	(10,7)	(11,8)	(1,9)	(2,10)	(3,11)	(4,1)	(5,2)
(4,11)	(5,1)	(6,2)	(7,3)	(8,4)	(9,5)	(10,6)	(11,7)	(1,8)	(2,9)	(3,10)
(2,8)	(3,9)	(4,10)	(5,11)	(6,1)	(7,2)	(8,3)	(9,4)	(10,5)	(11,6)	(1,7)
(11,5)	(1,6)	(2,7)	(3,8)	(4,9)	(5,10)	(6,11)	(7,1)	(8,2)	(9,3)	(10,4)
(9,2)	(10,3)	(11,4)	(1,5)	(2,6)	(3,7)	(4,8)	(5,9)	(6,10)	(7,11)	(8,1)
(7,10)	(8,11)	(9,1)	(10,2)	(11,3)	(1,4)	(2,5)	(3,6)	(4,7)	(5,8)	(6,9)
(5,7)	(6,8)	(7,9)	(8,10)	(9,11)	(10,1)	(11,2)	(1,3)	(2,4)	(3,5)	(4,6)
(3,4)	(4,5)	(5,6)	(6,7)	(7,8)	(8,9)	(9,10)	(10,11)	(11,1)	(1,2)	(2,3)

Let's construct 121 blocks of order 3 using the columns of above magic rectangle 1.4 in a magic square of order 3 given in Example 2.1 by applying the operation $AB := 33 \times (A-1) + B$. Below are few examples:

- **Block (10,9)**

10			*51*
14	5	32	*51*
32	14	5	*51*
5	32	14	*51*
51	*51*	*51*	*42*

9			*12*
16	31	4	*51*
31	4	16	*51*
4	16	31	*51*
51	*51*	*51*	*51*

(10,9)			*1596*
445	163	1027	*1635*
1054	433	148	*1635*
136	1039	460	*1635*
1635	*1635*	*1635*	*1338*

- **Block (5,11)**

5			*51*
19	8	24	*51*
24	19	8	*51*
8	24	19	*51*
51	*51*	*51*	*57*

11			*99*
6	12	33	*51*
12	33	6	*51*
33	6	12	*51*
51	*51*	*51*	*51*

(5,11)			*1683*
600	243	792	*1635*
771	627	237	*1635*
264	765	606	*1635*
1635	*1635*	*1635*	*1833*

- **Block (6,3)**

6			*51*
15	10	26	*51*
26	15	10	*51*
10	26	15	*51*
51	*51*	*51*	*45*

3			*69*
7	21	23	*51*
21	23	7	*51*
23	7	21	*51*
51	*51*	*51*	*51*

(6,3)			*1653*
469	318	848	*1635*
846	485	304	*1635*
320	832	483	*1635*
1635	*1635*	*1635*	*1437*

Proceeding on same procedure, we can construct the total 121 semi-magic squares of order 3 (in rows and columns). This lead us to following pan magic square of order 33.

Example 2.36. *According to magic rectangle given in Example 1.4, Example 2.35 and 121 blocks of order 3 of equal sums entries, we have a* ***pan magic square*** *of order 33 given by*

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985
17985	22	919	694	656	35	944	960	579	96	733	681	221	618	239	778	537	817	281
17985	721	1	913	926	647	62	84	987	564	219	749	667	767	613	255	289	545	801
17985	892	715	28	53	953	629	591	69	975	683	205	747	250	783	602	809	273	553
17985	445	163	1027	1088	368	179	6	903	726	649	61	925	986	563	86	729	678	228
17985	1054	433	148	170	1070	395	705	33	897	952	628	55	68	977	590	216	756	663
17985	136	1039	460	377	197	1061	924	699	12	34	946	655	581	95	959	690	201	744
17985	339	883	413	125	1000	510	442	159	1034	1072	394	169	32	896	707	633	45	957
17985	421	347	867	505	114	1016	1050	440	145	196	1060	379	698	14	923	936	660	39
17985	875	405	355	1005	521	109	143	1036	456	367	181	1087	905	725	5	66	930	639
17985	531	810	294	469	318	848	354	866	415	108	1015	512	455	142	1038	1069	390	176
17985	282	558	795	846	485	304	404	349	882	520	116	999	1033	444	158	192	1067	376
17985	822	267	546	320	832	483	877	420	338	1007	504	124	147	1049	439	374	178	1083
17985	732	672	231	616	259	760	557	794	284	465	315	855	337	879	419	123	998	514
17985	210	759	666	787	595	253	266	548	821	843	492	300	417	353	865	503	118	1014
17985	693	204	738	232	781	622	812	293	530	327	828	480	881	403	351	1009	519	107
17985	640	60	935	973	592	70	758	665	212	600	243	792	550	820	265	491	299	845
17985	951	638	46	97	961	577	203	740	692	771	627	237	292	529	814	827	482	326
17985	44	937	654	565	82	988	674	230	731	264	765	606	793	286	556	317	854	464
17985	1080	371	184	9	916	710	653	43	939	970	588	77	742	691	202	626	236	773
17985	173	1075	387	718	17	900	934	642	59	93	968	574	229	730	676	764	608	263
17985	382	189	1064	908	702	25	48	950	637	572	79	984	664	214	757	245	791	599
17985	128	992	515	432	150	1053	1063	384	188	24	899	712	636	58	941	983	571	81
17985	497	119	1019	1041	459	135	186	1079	370	701	19	915	949	644	42	76	972	587
17985	1010	524	101	162	1026	447	386	172	1077	910	717	8	50	933	652	576	92	967
17985	494	302	839	336	870	429	121	1018	496	458	134	1043	1059	381	195	7	912	716
17985	830	476	329	408	363	864	523	100	1012	1025	449	161	183	1086	366	714	23	898
17985	311	857	467	891	402	342	991	517	127	152	1052	431	393	168	1074	914	700	21
17985	620	241	774	541	819	275	478	328	829	362	863	410	105	1002	528	451	160	1024
17985	769	609	257	291	539	805	856	466	313	401	344	890	507	132	996	1051	430	154
17985	246	785	604	803	277	555	301	841	493	872	428	335	1023	501	111	133	1045	457
17985	964	582	89	750	668	217	603	256	776	554	802	279	475	324	836	346	889	400
17985	87	980	568	206	745	684	784	611	240	274	543	818	852	473	310	427	334	874
17985	584	73	978	679	222	734	248	768	619	807	290	538	308	838	489	862	412	361
	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	II
17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985
488	307	840	343	885	407	115	1021	499	461	137	1037	1062	375	198	17985
835	477	323	423	341	871	526	103	1006	1028	443	164	177	1089	369	17985
312	851	472	869	409	357	994	511	130	146	1055	434	396	171	1068	17985
601	252	782	552	800	283	471	322	842	356	868	411	112	1017	506	17985
780	617	238	272	547	816	850	479	306	406	345	884	522	110	1003	17985
254	766	615	811	288	536	314	834	487	873	422	340	1001	508	126	17985
979	589	67	755	662	218	597	249	789	535	813	287	486	305	844	17985
94	958	583	200	746	689	777	624	234	285	551	799	833	481	321	17985
562	88	985	680	227	728	261	762	612	815	271	549	316	849	470	17985
16	922	697	659	38	938	963	573	99	748	688	199	623	233	779	17985
724	4	907	929	641	65	78	990	567	226	727	682	761	614	260	17985
895	709	31	47	956	632	594	72	969	661	220	754	251	788	596	17985
438	157	1040	1082	373	180	13	918	704	643	64	928	989	566	80	17985
1048	446	141	175	1071	389	720	11	904	955	631	49	71	971	593	17985
149	1032	454	378	191	1066	902	706	27	37	940	658	575	98	962	17985
333	876	426	106	1011	518	453	140	1042	1065	388	182	26	901	708	17985
414	360	861	516	122	997	1031	448	156	190	1073	372	703	15	917	17985
888	399	348	1013	502	120	151	1047	437	380	174	1081	906	719	10	17985
534	804	297	484	325	826	359	860	416	102	1008	525	436	153	1046	17985
276	561	798	853	463	319	398	350	887	513	129	993	1044	452	139	17985
825	270	540	298	847	490	878	425	332	1020	498	117	155	1030	450	17985
739	687	209	610	262	763	560	797	278	468	309	858	352	886	397	17985
225	737	673	790	598	247	269	542	824	837	495	303	424	331	880	17985
671	211	753	235	775	625	806	296	533	330	831	474	859	418	358	17985
651	41	943	966	586	83	752	670	213	607	258	770	544	823	268	17985
932	646	57	91	974	570	208	741	686	786	605	244	295	532	808	17985
52	948	635	578	75	982	675	224	736	242	772	621	796	280	559	17985
1085	365	185	3	909	723	634	54	947	981	569	85	735	685	215	17985
167	1076	392	711	30	894	945	650	40	74	976	585	223	743	669	17985
383	194	1058	921	696	18	56	931	648	580	90	965	677	207	751	17985
131	995	509	435	144	1056	1078	391	166	29	893	713	630	51	954	17985
500	113	1022	1035	462	138	193	1057	385	695	20	920	942	657	36	17985
1004	527	104	165	1029	441	364	187	1084	911	722	2	63	927	645	17985
17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985	17985

Combining parts I and II we get required result. In this case, the magic sum is $S_{33\times 33} = 17985$. The 3×3 blocks are **semi-magic squares** with equal semi-magic sums $S_{3\times 3} := 1635$ (in rows and columns).

2.10 Pan Magic Square of Order 36

In [23], the author constricted block-wise pan magic square of order 36, where each block of order 4 is a pan magic square of order 4 with equal magic sums. Also in [25] author constructed block-wise pan magic square of order 36, where each block of order 9 a pan magic square with different magic sums. Here, the aim is to construct a **block-wise pan magic square** of order 36, where each block of order 3 a magic square with different magic sums. It is well known that a magic square of order 36 with 1296 numbers, i.e., 1-1296 has a magic sum $S_{24\times 24} := 23346$.

If we divide 23346 by 12 we get a fraction value, i.e., $\frac{23396}{12} := 1945.5$. It implies that we are unable to construct block-wise magic square of order 36 with equal sum blocks of order 3. In this case, we shall construct block-wise magic square of order 36 with each block of order 3 having different magic sums. This shall be done by use of magic square of order 3 given in Example 2.1 with composite matrix of order 12 based on Example 2.8. Below is a Latin square decomposition of Example 2.8.

Example 2.37. *The Latin square decompositions of a magic square of order 12 given in Example 2.8 are given by*

		78	78	78	78	78	78	78	78	78	78	78	78
	5	4	6	8	9	7	2	3	1	11	10	12	78
78	6	5	4	7	8	9	1	2	3	12	11	10	78
78	4	6	5	9	7	8	3	1	2	10	12	11	78
78	2	3	1	11	10	12	5	4	6	8	9	7	78
78	1	2	3	12	11	10	6	5	4	7	8	9	78
78	3	1	2	10	12	11	4	6	5	9	7	8	78
78	11	10	12	2	3	1	8	9	7	5	4	6	78
78	12	11	10	1	2	3	7	8	9	6	5	4	78
78	10	12	11	3	1	2	9	7	8	4	6	5	78
78	8	9	7	5	4	6	11	10	12	2	3	1	78
78	7	8	9	6	5	4	12	11	10	1	2	3	78
78	9	7	8	4	6	5	10	12	11	3	1	2	78
	78	78	78	78	78	78	78	78	78	78	78	78	78

		78	78	78	78	78	78	78	78	78	78	78	78
	7	9	8	12	10	11	1	3	2	6	4	5	78
78	9	8	7	10	11	12	3	2	1	4	5	6	78
78	8	7	9	11	12	10	2	1	3	5	6	4	78
78	6	4	5	1	3	2	12	10	11	7	9	8	78
78	4	5	6	3	2	1	10	11	12	9	8	7	78
78	5	6	4	2	1	3	11	12	10	8	7	9	78
78	12	10	11	7	9	8	6	4	5	1	3	2	78
78	10	11	12	9	8	7	4	5	6	3	2	1	78
78	11	12	10	8	7	9	5	6	4	2	1	3	78
78	1	3	2	6	4	5	7	9	8	12	10	11	78
78	3	2	1	4	5	6	9	8	7	10	11	12	78
78	2	1	3	5	6	4	8	7	9	11	12	10	78
	78	78	78	78	78	78	78	78	78	78	78	78	78

The magic square given in Example 2.8 is obtained by the operation $12 \times (A - 1) + B$. Moreover, A and B are not diagonalize. Based on A and B, the composite matrix of order 12 by considering just (A, B) is given in the example below.

Example 2.38. *The composite matrix* (A, B) *based on* A *and* B *given in Example 2.37 is given by*

(5,7)	(4,9)	(6,8)	(8,12)	(9,10)	(7,11)	(2,1)	(3,3)	(1,2)	(11,6)	(10,4)	(12,5)
(6,9)	(5,8)	(4,7)	(7,10)	(8,11)	(9,12)	(1,3)	(2,2)	(3,1)	(12,4)	(11,5)	(10,6)
(4,8)	(6,7)	(5,9)	(9,11)	(7,12)	(8,10)	(3,2)	(1,1)	(2,3)	(10,5)	(12,6)	(11,4)
(2,6)	(3,4)	(1,5)	(11,1)	(10,3)	(12,2)	(5,12)	(4,10)	(6,11)	(8,7)	(9,9)	(7,8)
(1,4)	(2,5)	(3,6)	(12,3)	(11,2)	(10,1)	(6,10)	(5,11)	(4,12)	(7,9)	(8,8)	(9,7)
(3,5)	(1,6)	(2,4)	(10,2)	(12,1)	(11,3)	(4,11)	(6,12)	(5,10)	(9,8)	(7,7)	(8,9)
(11,12)	(10,10)	(12,11)	(2,7)	(3,9)	(1,8)	(8,6)	(9,4)	(7,5)	(5,1)	(4,3)	(6,2)
(12,10)	(11,11)	(10,12)	(1,9)	(2,8)	(3,7)	(7,4)	(8,5)	(9,6)	(6,3)	(5,2)	(4,1)
(10,11)	(12,12)	(11,10)	(3,8)	(1,7)	(2,9)	(9,5)	(7,6)	(8,4)	(4,2)	(6,1)	(5,3)
(8,1)	(9,3)	(7,2)	(5,6)	(4,4)	(6,5)	(11,7)	(10,9)	(12,8)	(2,12)	(3,10)	(1,11)
(7,3)	(8,2)	(9,1)	(6,4)	(5,5)	(4,6)	(12,9)	(11,8)	(10,7)	(1,10)	(2,11)	(3,12)
(9,2)	(7,1)	(8,3)	(4,5)	(6,6)	(5,4)	(10,8)	(12,7)	(11,9)	(3,11)	(1,12)	(2,10)

Since it is impossible to make magic rectangle of order (3,12), let us consider following **semi-magic rectangle** order (3,12), i.e., equality only in rows:

Example 2.39. *Let's consider the following* ***semi-magic rectangle*** *of order (3,8):*

(3,12)	1	2	3	4	5	6	7	8	9	10	11	12	Total
R1	1	6	7	12	13	18	19	24	25	30	31	36	222
R2	2	5	8	11	14	17	20	23	26	29	32	35	222
R3	3	4	9	10	15	16	21	22	27	28	33	34	222
Total	6	15	24	33	42	51	60	69	78	87	96	105	

Note 2.4. *If we add the numbers from 1 to 36, we have total sum as 666. It is impossible to divide 666 in 12 equal parts, i.e.,* $\frac{666}{12} = 55.5$*. This is the reason, why we are unable to make magic rectangle of order (3,12).*

Applying the columns values given in Example 2.39 over the Example 2.1, we get 144 blocks of magic squares of order 3, where the operation used is $AB := 36 \times (A - 1) + B$. See below some examples:

• Block (5,8)

5			42
14	15	13	42
13	14	15	42
15	13	14	42
42	42	42	42

8			69
24	22	23	69
22	23	24	69
23	24	22	69
69	69	69	69

(5,8)			1473
492	526	455	1473
454	491	528	1473
527	456	490	1473
1473	1473	1473	1473

• Block (12,11)

12			105
35	34	36	105
36	35	34	105
34	36	35	105
105	105	105	105

11			96
31	33	32	96
33	32	31	96
32	31	33	96
96	96	96	96

(12,11)			3768
1255	1221	1292	3768
1293	1256	1219	3768
1220	1291	1257	3768
3768	3768	3768	3768

Above are only 2 blocks of magic squares of order 3, but in total we have 144 blocks. Put these 144 blocks of magic squares of order 3 according to composite matrix given in Example 2.38 we get a **pan diagonal magic square** of order 36.

Example 2.40. . *The **pan magic square** of order 36 constructed according to distribution 2.39 applied over the Example 2.1 and put according to Example 2.38 with the operation* $AB := 36 \times (A-1) + B$ *is given by*

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346
	487	525	452	385	351	422	600	562	635	828	790	863	930	964	893	715	753	680
23346	453	488	523	423	386	349	634	599	564	862	827	792	892	929	966	681	716	751
23346	524	451	489	350	421	387	563	636	598	791	864	826	965	894	928	752	679	717
23346	601	567	638	492	526	455	379	345	416	714	748	677	823	789	860	936	970	899
23346	639	602	565	454	491	528	417	380	343	676	713	750	861	824	787	898	935	972
23346	566	637	603	527	456	490	344	415	381	749	678	712	788	859	825	971	900	934
23346	384	346	419	595	561	632	493	531	458	931	969	896	720	754	683	822	784	857
23346	418	383	348	633	596	559	459	494	529	897	932	967	682	719	756	856	821	786
23346	347	420	382	560	631	597	530	457	495	968	895	933	755	684	718	785	858	820
23346	162	124	197	264	298	227	49	87	14	1117	1155	1082	1015	981	1052	1230	1192	1265
23346	196	161	126	226	263	300	15	50	85	1083	1118	1153	1053	1016	979	1264	1229	1194
23346	125	198	160	299	228	262	86	13	51	1154	1081	1119	980	1051	1017	1193	1266	1228
23346	48	82	11	157	123	194	270	304	233	1231	1197	1268	1122	1156	1085	1009	975	1046
23346	10	47	84	195	158	121	232	269	306	1269	1232	1195	1084	1121	1158	1047	1010	973
23346	83	12	46	122	193	159	305	234	268	1196	1267	1233	1157	1086	1120	974	1045	1011
23346	265	303	230	54	88	17	156	118	191	1014	976	1049	1225	1191	1262	1123	1161	1088
23346	231	266	301	16	53	90	190	155	120	1048	1013	978	1263	1226	1189	1089	1124	1159
23346	302	229	267	89	18	52	119	192	154	977	1050	1012	1190	1261	1227	1160	1087	1125
23346	1152	1186	1115	1038	1000	1073	1255	1221	1292	163	129	200	277	315	242	60	94	23
23346	1114	1151	1188	1072	1037	1002	1293	1256	1219	201	164	127	243	278	313	22	59	96
23346	1187	1116	1150	1001	1074	1036	1220	1291	1257	128	199	165	314	241	279	95	24	58
23346	1254	1216	1289	1147	1185	1112	1044	1006	1079	61	99	26	168	130	203	271	309	236
23346	1288	1253	1218	1113	1148	1183	1078	1043	1008	27	62	97	202	167	132	237	272	307
23346	1217	1290	1252	1184	1111	1149	1007	1080	1042	98	25	63	131	204	166	308	235	273
23346	1039	1005	1076	1260	1222	1295	1146	1180	1109	276	310	239	55	93	20	169	135	206
23346	1077	1040	1003	1294	1259	1224	1108	1145	1182	238	275	312	21	56	91	207	170	133
23346	1004	1075	1041	1223	1296	1258	1181	1110	1144	311	240	274	92	19	57	134	205	171
23346	793	759	830	907	945	872	690	724	653	486	520	449	372	334	407	589	555	626
23346	831	794	757	873	908	943	652	689	726	448	485	522	406	371	336	627	590	553
23346	758	829	795	944	871	909	725	654	688	521	450	484	335	408	370	554	625	591
23346	691	729	656	798	760	833	901	939	866	588	550	623	481	519	446	378	340	413
23346	657	692	727	832	797	762	867	902	937	622	587	552	447	482	517	412	377	342
23346	728	655	693	761	834	796	938	865	903	551	624	586	518	445	483	341	414	376
23346	906	940	869	685	723	650	799	765	836	373	339	410	594	556	629	480	514	443
23346	868	905	942	651	686	721	837	800	763	411	374	337	628	593	558	442	479	516
23346	941	870	904	722	649	687	764	835	801	338	409	375	557	630	592	515	444	478
	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	II
23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346
145	111	182	259	297	224	42	76	5	1134	1168	1097	1020	982	1055	1237	1203	1274	23346
183	146	109	225	260	295	4	41	78	1096	1133	1170	1054	1019	984	1275	1238	1201	23346
110	181	147	296	223	261	77	6	40	1169	1098	1132	983	1056	1018	1202	1273	1239	23346
43	81	8	150	112	185	253	291	218	1236	1198	1271	1129	1167	1094	1026	988	1061	23346
9	44	79	184	149	114	219	254	289	1270	1235	1200	1095	1130	1165	1060	1025	990	23346
80	7	45	113	186	148	290	217	255	1199	1272	1234	1166	1093	1131	989	1062	1024	23346
258	292	221	37	75	2	151	117	188	1021	987	1058	1242	1204	1277	1128	1162	1091	23346
220	257	294	3	38	73	189	152	115	1059	1022	985	1276	1241	1206	1090	1127	1164	23346
293	222	256	74	1	39	116	187	153	986	1057	1023	1205	1278	1240	1163	1092	1126	23346
504	538	467	390	352	425	607	573	644	811	777	848	925	963	890	708	742	671	23346
466	503	540	424	389	354	645	608	571	849	812	775	891	926	961	670	707	744	23346
539	468	502	353	426	388	572	643	609	776	847	813	962	889	927	743	672	706	23346
606	568	641	499	537	464	396	358	431	709	747	674	816	778	851	919	957	884	23346
640	605	570	465	500	535	430	395	360	675	710	745	850	815	780	885	920	955	23346
569	642	604	536	463	501	359	432	394	746	673	711	779	852	814	956	883	921	23346
391	357	428	612	574	647	498	532	461	924	958	887	703	741	668	817	783	854	23346
429	392	355	646	611	576	460	497	534	886	923	960	669	704	739	855	818	781	23346
356	427	393	575	648	610	533	462	496	959	888	922	740	667	705	782	853	819	23346
810	772	845	912	946	875	697	735	662	469	507	434	367	333	404	582	544	617	23346
844	809	774	874	911	948	663	698	733	435	470	505	405	368	331	616	581	546	23346
773	846	808	947	876	910	734	661	699	506	433	471	332	403	369	545	618	580	23346
696	730	659	805	771	842	918	952	881	583	549	620	474	508	437	361	327	398	23346
658	695	732	843	806	769	880	917	954	621	584	547	436	473	510	399	362	325	23346
731	660	694	770	841	807	953	882	916	548	619	585	509	438	472	326	397	363	23346
913	951	878	702	736	665	804	766	839	366	328	401	577	543	614	475	513	440	23346
879	914	949	664	701	738	838	803	768	400	365	330	615	578	541	441	476	511	23346
950	877	915	737	666	700	767	840	802	329	402	364	542	613	579	512	439	477	23346
1135	1173	1100	1033	999	1070	1248	1210	1283	180	142	215	282	316	245	67	105	32	23346
1101	1136	1171	1071	1034	997	1282	1247	1212	214	179	144	244	281	318	33	68	103	23346
1172	1099	1137	998	1069	1035	1211	1284	1246	143	216	178	317	246	280	104	31	69	23346
1249	1215	1286	1140	1174	1103	1027	993	1064	66	100	29	175	141	212	288	322	251	23346
1287	1250	1213	1102	1139	1176	1065	1028	991	28	65	102	213	176	139	250	287	324	23346
1214	1285	1251	1175	1104	1138	992	1063	1029	101	30	64	140	211	177	323	252	286	23346
1032	994	1067	1243	1209	1280	1141	1179	1106	283	321	248	72	106	35	174	136	209	23346
1066	1031	996	1281	1244	1207	1107	1142	1177	249	284	319	34	71	108	208	173	138	23346
995	1068	1030	1208	1279	1245	1178	1105	1143	320	247	285	107	36	70	137	210	172	23346
23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346

Combining Parts I and II, we get a **pan magic square** of order 36 with magic sum is $S_{36\times 36} = 23346$. The 3×3 blocks are **magic squares** of order 3 with different magic sums. These magic sums of order 3 again forms a magic square of order 12 given in example below.

Example 2.41. *The pan diagonal magic square of order 12 formed by magic sums of order 3 of Example ?? is given by*

		23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346
	1464	1158	1797	2481	2787	2148	438	780	123	3399	3057	3714	23346
23346	1806	1473	1140	2139	2472	2805	132	447	762	3705	3390	3075	23346
23346	1149	1788	1482	2796	2157	2463	771	114	456	3066	3723	3381	23346
23346	483	789	150	3354	3048	3687	1509	1167	1824	2436	2778	2121	23346
23346	141	474	807	3696	3363	3030	1815	1500	1185	2130	2445	2760	23346
23346	798	159	465	3039	3678	3372	1176	1833	1491	2769	2112	2454	23346
23346	3453	3111	3768	492	834	177	2427	2733	2094	1410	1104	1743	23346
23346	3759	3444	3129	186	501	816	2085	2418	2751	1752	1419	1086	23346
23346	3120	3777	3435	825	168	510	2742	2103	2409	1095	1734	1428	23346
23346	2382	2724	2067	1455	1113	1770	3408	3102	3741	537	843	204	23346
23346	2076	2391	2706	1761	1446	1131	3750	3417	3084	195	528	861	23346
23346	2715	2058	2400	1122	1779	1437	3093	3732	3426	852	213	519	23346
	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346	23346

3 Pan Magic Square of Order 35

The previous section brings block-wise construction of magic squares of type $3k$ starting from order 9 to 36. In all the case, the blocks of order 3 are either magic squares with different magic sums or semi-magic squares with equal sums entries. This section works with magic square of order 35 in two different ways. One 49 blocks of pan magic squares of order 5 with equal magic sums. The second way is 25 blocks of pan magic squares of order 7 with equal magic sums. In both the cases, the magic square of order 35 is pan diagonal. In order to bring these magic squares, we have used the idea of magic triangle of order (5,7), a pan magic square of order 5 and a pan magic square of order 7. In both the approaches we shall used composite forms of magic squares of order 5 and 7 given in examples below.

Example 3.1. *The composite and* ***pan magic squares*** *of order 5 are given by*

A		15	15	15	15	15
	1	2	3	4	5	15
15	4	5	1	2	3	15
15	2	3	4	5	1	15
15	5	1	2	3	4	15
15	3	4	5	1	2	15
	15	15	15	15	15	15

B		15	15	15	15	15
	1	4	2	5	3	15
15	2	5	3	1	4	15
15	3	1	4	2	5	15
15	4	2	5	3	1	15
15	5	3	1	4	2	15
	15	15	15	15	15	15

AB		65	65	65	65	65
	1	9	12	20	23	65
65	17	25	3	6	14	65
65	8	11	19	22	5	65
65	24	2	10	13	16	65
65	15	18	21	4	7	65
	65	65	65	65	65	65

C		165	165	165	165	165
	11	22	33	44	55	165
165	43	54	15	21	32	165
165	25	31	42	53	14	165
165	52	13	24	35	41	165
165	34	45	51	12	23	165
	165	165	165	165	165	165

The **pan magic square** ***AB*** and composite magic square ***C*** are calculated by using the operations:

$$AB := 5 \times (A - 1) + B$$
$$C := 10 \times A + B.$$

where A and B are the **mutually orthogonal diagonal Latin squares** of order 5.

Example 3.2. *The composite and* ***pan diagonal magic squares*** *of order 7 are is given by*

A		28	28	28	28	28	28	28
	1	2	3	4	5	6	7	28
28	6	7	1	2	3	4	5	28
28	4	5	6	7	1	2	3	28
28	2	3	4	5	6	7	1	28
28	7	1	2	3	4	5	6	28
28	5	6	7	1	2	3	4	28
28	3	4	5	6	7	1	2	28
	28	28	28	28	28	28	28	28

B		28	28	28	28	28	28	28
	1	2	3	4	5	6	7	28
28	5	6	7	1	2	3	4	28
28	2	3	4	5	6	7	1	28
28	6	7	1	2	3	4	5	28
28	3	4	5	6	7	1	2	28
28	7	1	2	3	4	5	6	28
28	4	5	6	7	1	2	3	28
	28	28	28	28	28	28	28	28

AB		175	175	175	175	175	175	175
	1	9	17	25	33	41	49	175
175	40	48	7	8	16	24	32	175
175	23	31	39	47	6	14	15	175
175	13	21	22	30	38	46	5	175
175	45	4	12	20	28	29	37	175
175	35	36	44	3	11	19	27	175
175	18	26	34	42	43	2	10	175
	175	175	175	175	175	175	175	175

C		308	308	308	308	308	308	308
	11	22	33	44	55	66	77	308
308	65	76	17	21	32	43	54	308
308	42	53	64	75	16	27	31	308
308	26	37	41	52	63	74	15	308
308	73	14	25	36	47	51	62	308
308	57	61	72	13	24	35	46	308
308	34	45	56	67	71	12	23	308
	308	308	308	308	308	308	308	308

The **pan magic square** ***AB*** and composite magic square ***C*** are calculated by using the operations:

$$AB := 7 \times (A - 1) + B$$
$$C := 10 \times A + B.$$

where A and B are the **mutually orthogonal diagonal Latin squares** of order 7.

3.1 Firth Approach: 25 Blocks of Order 7

In this subsection, we shall present pan magic square of order 35 with 25 blocks of pan magic squares of order 7 with equal magic sums. To bring these blocks we shall use magic rectangle of order (7,5) given in Example **??** in a vertical way:

Example 3.3. *The magic rectangle of order (7,5) is given by*

(7,5)	1	2	3	4	5	Total
R1	26	20	3	9	32	90
R2	19	6	7	35	23	90
R3	8	34	15	22	11	90
R4	31	24	18	12	5	90
R5	25	14	21	2	28	90
R6	13	1	29	30	17	90
R7	4	27	33	16	10	90
Total	126	126	126	126	126	

We can make 25 blocks of order 7 applying the columns of Example 3.3 over the Latin squares distributions given in 3.2. See below some examples.

- **Block 43**

4		126	126	126	126	126	126	126
	9	35	22	12	2	30	16	126
126	30	16	9	35	22	12	2	126
126	12	2	30	16	9	35	22	126
126	35	22	12	2	30	16	9	126
126	16	9	35	22	12	2	30	126
126	2	30	16	9	35	22	12	126
126	22	12	2	30	16	9	35	126
	126	126	126	126	126	126	126	126

3		126	126	126	126	126	126	126
	3	7	15	18	21	29	33	126
126	21	29	33	3	7	15	18	126
126	7	15	18	21	29	33	3	126
126	29	33	3	7	15	18	21	126
126	15	18	21	29	33	3	7	126
126	33	3	7	15	18	21	29	126
126	18	21	29	33	3	7	15	126
	126	126	126	126	126	126	126	126

43		4291	4291	4291	4291	4291	4291	4291
	283	1197	750	403	56	1044	558	4291
4291	1036	554	313	1193	742	400	53	4291
4291	392	50	1033	546	309	1223	738	4291
4291	1219	768	388	42	1030	543	301	4291
4291	540	298	1211	764	418	38	1022	4291
4291	68	1018	532	295	1208	756	414	4291
4291	753	406	64	1048	528	287	1205	4291
	4291	4291	4291	4291	4291	4291	4291	4291

- **Block 15**

1		126	126	126	126	126	126	126
	26	19	8	31	25	13	4	126
126	13	4	26	19	8	31	25	126
126	31	25	13	4	26	19	8	126
126	19	8	31	25	13	4	26	126
126	4	26	19	8	31	25	13	126
126	25	13	4	26	19	8	31	126
126	8	31	25	13	4	26	19	126
	126	126	126	126	126	126	126	126

5		126	126	126	126	126	126	126
	32	23	11	5	28	17	10	126
126	28	17	10	32	23	11	5	126
126	23	11	5	28	17	10	32	126
126	17	10	32	23	11	5	28	126
126	11	5	28	17	10	32	23	126
126	10	32	23	11	5	28	17	126
126	5	28	17	10	32	23	11	126
	126	126	126	126	126	126	126	126

15		4291	4291	4291	4291	4291	4291	4291
	907	653	256	1055	868	437	115	4291
4291	448	122	885	662	268	1061	845	4291
4291	1073	851	425	133	892	640	277	4291
4291	647	255	1082	863	431	110	903	4291
4291	116	880	658	262	1060	872	443	4291
4291	850	452	128	886	635	273	1067	4291
4291	250	1078	857	430	137	898	641	4291
	4291	4291	4291	4291	4291	4291	4291	4291

- **Block 31**

3		126	126	126	126	126	126	126
	3	7	15	18	21	29	33	126
126	29	33	3	7	15	18	21	126
126	18	21	29	33	3	7	15	126
126	7	15	18	21	29	33	3	126
126	33	3	7	15	18	21	29	126
126	21	29	33	3	7	15	18	126
126	15	18	21	29	33	3	7	126
	126	126	126	126	126	126	126	126

1		126	126	126	126	126	126	126
	26	19	8	31	25	13	4	126
126	25	13	4	26	19	8	31	126
126	19	8	31	25	13	4	26	126
126	13	4	26	19	8	31	25	126
126	8	31	25	13	4	26	19	126
126	4	26	19	8	31	25	13	126
126	31	25	13	4	26	19	8	126
	126	126	126	126	126	126	126	126

31		4291	4291	4291	4291	4291	4291	4291
	96	229	498	626	725	993	1124	4291
4291	1005	1133	74	236	509	603	731	4291
4291	614	708	1011	1145	83	214	516	4291
4291	223	494	621	719	988	1151	95	4291
4291	1128	101	235	503	599	726	999	4291
4291	704	1006	1139	78	241	515	608	4291
4291	521	620	713	984	1146	89	218	4291
	4291	4291	4291	4291	4291	4291	4291	4291

The 25 blocks of order 7 constructed according to above examples, and putting them according to Example 3.1 of composite magic square of order 5, we get a magic square of order 35 given in example below.

Example 3.4. . *The* ***pan magic square*** *of order 36 constructed according to distribution 2.39 applied over the Example 2.1 and put according to Example 2.38 with the operation* $AB := 24 \times (A-1) + B$ *is given by*

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455
	901	649	253	1081	865	433	109	685	181	1189	829	469	1	937	73	217	505	613
21455	445	118	879	656	264	1058	871	14	911	692	195	1161	839	479	1001	1149	103	213
21455	1069	848	451	130	888	634	271	811	489	24	924	666	202	1175	602	715	998	1141
21455	643	249	1076	859	428	136	900	176	1182	825	461	34	934	679	239	523	598	707
21455	113	906	655	258	1054	866	439	944	689	189	1156	832	475	6	1135	88	231	519
21455	844	446	124	883	661	270	1063	482	20	916	699	199	1169	806	733	983	1127	85
21455	276	1075	853	424	131	894	638	1179	819	456	27	930	671	209	508	616	729	1013
21455	283	1197	750	403	56	1044	558	1094	805	372	152	947	590	331	907	653	256	1055
21455	1036	554	313	1193	742	400	53	562	345	1101	779	385	162	957	448	122	885	662
21455	392	50	1033	546	309	1223	738	175	967	572	317	1115	786	359	1073	851	425	133
21455	1219	768	388	42	1030	543	301	800	366	149	980	582	327	1087	647	255	1082	863
21455	540	298	1211	764	418	38	1022	337	1097	772	380	156	954	595	116	880	658	262
21455	68	1018	532	295	1208	756	414	961	569	350	1107	782	352	170	850	452	128	886
21455	753	406	64	1048	528	287	1205	362	142	975	576	324	1120	792	250	1078	857	430
21455	697	198	1166	810	483	17	920	96	229	498	626	725	993	1124	300	1196	769	409
21455	28	927	675	207	1178	816	460	1005	1133	74	236	509	603	731	1029	526	307	1210
21455	828	466	5	938	682	185	1187	614	708	1011	1145	83	214	516	391	69	1039	539
21455	192	1165	837	478	11	915	693	223	494	621	719	988	1151	95	1191	762	405	41
21455	921	670	203	1172	815	487	23	1128	101	235	503	599	726	999	559	304	1204	736
21455	465	32	933	676	180	1183	822	704	1006	1139	78	241	515	608	62	1035	531	314
21455	1160	833	472	10	942	688	186	521	620	713	984	1146	89	218	759	399	36	1042
21455	1105	776	384	164	959	561	342	878	637	260	1068	861	449	138	674	210	1177	817
21455	574	316	1112	790	356	174	969	441	134	908	633	252	1065	858	2	940	681	184
21455	146	979	584	329	1086	797	370	1057	855	438	126	904	663	248	840	477	12	912
21455	771	377	160	951	594	339	1099	659	278	1053	847	435	123	896	205	1171	814	490
21455	349	1109	784	351	167	965	566	120	893	651	274	1083	843	427	932	677	177	1185
21455	972	580	321	1119	794	364	141	873	423	112	890	648	266	1079	471	9	945	687
21455	374	154	946	587	335	1091	804	263	1071	869	453	108	882	645	1167	807	485	16
21455	79	245	512	607	702	1010	1136	312	1213	746	390	63	1032	535	1111	789	358	171
21455	982	1150	86	219	525	617	712	1043	542	290	1222	758	396	40	585	328	1089	796
21455	630	722	992	1122	100	226	499	408	46	1020	553	297	1200	767	159	953	591	340
21455	240	506	604	735	1002	1132	72	1207	745	417	58	1026	530	308	783	354	166	964
21455	1142	82	212	520	611	709	1015	536	285	1218	752	395	67	1038	323	1116	795	363
21455	716	989	1155	92	222	492	625	45	1047	548	291	1195	763	402	949	586	334	1093
21455	502	597	730	996	1129	105	232	740	413	52	1025	557	303	1201	381	165	958	564
	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	II
21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455
721	1009	1153	289	1225	757	397	37	1045	541	1117	793	361	145	973	577	325	21455
497	610	718	1017	555	296	1199	770	407	47	588	332	1095	802	373	151	950	21455
99	243	493	420	57	1027	527	310	1206	744	163	956	565	343	1102	780	382	21455
995	1138	91	1220	751	394	70	1037	537	282	787	360	172	968	571	320	1113	21455
628	703	987	547	292	1192	765	401	44	1050	326	1090	798	367	150	977	583	21455
228	511	624	51	1024	560	302	1202	737	415	955	592	338	1096	775	378	157	21455
1123	77	225	747	387	65	1031	534	315	1212	355	168	962	570	347	1108	781	21455
868	437	115	691	194	1163	836	480	13	914	90	216	524	619	714	981	1147	21455
268	1061	845	25	923	669	201	1174	813	486	994	1121	97	230	496	629	724	21455
892	640	277	824	463	31	935	678	179	1181	601	734	1004	1134	71	237	510	21455
431	110	903	188	1159	831	474	8	941	690	211	517	615	706	1014	1144	84	21455
1060	872	443	918	696	200	1168	809	481	19	1154	94	224	491	622	720	986	21455
635	273	1067	459	26	929	673	206	1180	818	727	1000	1126	104	234	504	596	21455
137	898	641	1186	830	468	4	936	684	183	514	609	701	1007	1140	76	244	21455
49	1016	552	1088	777	365	158	966	589	348	884	665	267	1062	842	450	121	21455
741	419	59	581	344	1118	773	357	155	963	422	135	891	639	280	1072	852	21455
281	1217	755	147	960	578	336	1114	803	353	1085	862	432	107	905	646	254	21455
1049	549	294	799	383	143	952	575	333	1106	660	261	1059	875	442	117	877	21455
412	55	1021	330	1103	791	379	173	948	567	127	887	632	275	1066	849	455	21455
1214	749	386	978	563	322	1100	788	371	169	856	429	140	897	642	247	1080	21455
545	286	1224	368	161	974	593	318	1092	785	257	1052	870	436	114	910	652	21455
457	30	926	102	233	501	600	728	997	1130	306	1209	743	416	60	1028	529	21455
1190	827	467	1008	1137	80	242	513	606	705	1040	538	284	1216	754	393	66	21455
695	191	1164	618	711	985	1148	87	220	522	404	43	1046	550	293	1194	761	21455
22	922	667	227	500	627	723	991	1125	98	1203	739	411	54	1023	556	305	21455
821	464	35	1131	75	238	507	605	732	1003	533	311	1215	748	389	61	1034	21455
187	1157	835	710	1012	1143	81	215	518	612	39	1041	544	288	1221	760	398	21455
919	700	197	495	623	717	990	1152	93	221	766	410	48	1019	551	299	1198	21455
970	573	319	895	636	279	1074	854	421	132	668	182	1170	823	476	29	943	21455
369	148	976	434	106	902	650	251	1084	864	21	939	698	178	1162	820	473	21455
1098	774	376	1056	874	444	119	876	657	265	812	470	18	931	694	208	1158	21455
568	346	1110	631	272	1070	846	454	129	889	204	1188	808	462	15	928	686	21455
144	971	579	139	899	644	246	1077	860	426	925	683	196	1184	838	458	7	21455
801	375	153	867	440	111	909	654	259	1051	488	3	917	680	193	1176	834	21455
341	1104	778	269	1064	841	447	125	881	664	1173	826	484	33	913	672	190	21455
21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455

Combining Parts I and II, we get a **pan magic square** of order 35 with magic sum $S_{35\times 35} = 21455$. All 25 blocks of order 7 are **pan magic square** with equal magic sums, $S_{7\times 7} = 4291$.

3.2 Second Approach: 49 Blocks of Order 5

In this subsection, we shall present **pan magic square** of order 35 with 49 blocks of **pan magic squares** of order 5 with equal magic sums. To bring these blocks we shall use the magic rectangle of order (5,7) given in Example ??. For simplicity, let's rewrite:

Example 3.5. *The magic rectangle of order (5,7) is given by*

(5,7)	1	2	3	4	5	6	7	Total
R1	26	19	8	31	25	13	4	126
R2	20	6	34	24	14	1	27	126
R3	3	7	15	18	21	29	33	126
R4	9	35	22	12	2	30	16	126
R5	32	23	11	5	28	17	10	126
Total	90	90	90	90	90	90	90	

We can make 49 blocks of order 5 applying the columns of Example 3.5 over the Latin squares distributions given in 3.1. See below some examples.

- **Block 26**

2		90	90	90	90	90
	19	6	7	35	23	90
90	35	23	19	6	7	90
90	6	7	35	23	19	90
90	23	19	6	7	35	90
90	7	35	23	19	6	90
	90	90	90	90	90	90

6		90	90	90	90	90
	13	30	1	17	29	90
90	1	17	29	13	30	90
90	29	13	30	1	17	90
90	30	1	17	29	13	90
90	17	29	13	30	1	90
	90	90	90	90	90	90

26		3065	3065	3065	3065	3065
	643	205	211	1207	799	3065
3065	1191	787	659	188	240	3065
3065	204	223	1220	771	647	3065
3065	800	631	192	239	1203	3065
3065	227	1219	783	660	176	3065
	3065	3065	3065	3065	3065	3065

- **Block 75**

7		90	90	90	90	90
	4	27	33	16	10	90
90	16	10	4	27	33	90
90	27	33	16	10	4	90
90	10	4	27	33	16	90
90	33	16	10	4	27	90
	90	90	90	90	90	90

5		90	90	90	90	90
	25	2	14	28	21	90
90	14	28	21	25	2	90
90	21	25	2	14	28	90
90	2	14	28	21	25	90
90	28	21	25	2	14	90
	90	90	90	90	90	90

75		3065	3065	3065	3065	3065
	130	912	1134	553	336	3065
3065	539	343	126	935	1122	3065
3065	931	1145	527	329	133	3065
3065	317	119	938	1141	550	3065
3065	1148	546	340	107	924	3065
	3065	3065	3065	3065	3065	3065

- **Block 23**

2		90	90	90	90	90
	19	6	7	35	23	90
90	35	23	19	6	7	90
90	6	7	35	23	19	90
90	23	19	6	7	35	90
90	7	35	23	19	6	90
	90	90	90	90	90	90

3		90	90	90	90	90
	8	22	34	11	15	90
90	34	11	15	8	22	90
90	15	8	22	34	11	90
90	22	34	11	15	8	90
90	11	15	8	22	34	90
	90	90	90	90	90	90

23		3065	3065	3065	3065	3065
	638	197	244	1201	785	3065
3065	1224	781	645	183	232	3065
3065	190	218	1212	804	641	3065
3065	792	664	186	225	1198	3065
3065	221	1205	778	652	209	3065
	3065	3065	3065	3065	3065	3065

The 49 blocks of order 5 constructed according to above examples, and putting them according to Example 3.2 of composite magic square of order 7, we get a magic square of order 35 given in example below.

Example 3.6. . *The* ***pan magic square*** *of order 36 constructed according to distribution 2.39 applied over the Example 2.1 and put according to Example 2.38 with the operation* $AB := 24 \times (A-1) + B$ *is given by*

I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
		21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455
	901	674	90	312	1088	649	210	216	1213	777	253	1177	524	746	365	1081	817	619
21455	300	1117	878	691	79	1196	793	637	194	245	769	361	260	1163	512	409	145	1068
21455	668	96	289	1105	907	182	229	1225	776	653	1170	498	757	384	256	823	626	397
21455	1094	895	697	73	306	805	636	198	217	1209	372	279	1166	505	743	152	1074	810
21455	102	283	1111	884	685	233	1197	789	665	181	501	750	358	267	1189	600	403	171
21455	445	2	994	1043	581	118	940	1121	542	344	879	681	97	290	1118	656	184	230
21455	1029	588	441	25	982	526	332	134	923	1150	307	1095	908	669	86	1210	802	633
21455	21	1005	1017	574	448	939	1133	555	316	122	698	74	296	1112	885	178	236	1199
21455	562	434	28	1001	1040	345	106	927	1149	538	1101	902	675	103	284	779	650	207
21455	1008	1036	585	422	14	1137	554	328	135	911	80	313	1089	891	692	242	1193	796
21455	1069	840	601	408	147	848	477	734	46	960	451	12	1004	1020	578	130	912	1134
21455	391	163	1057	824	630	69	956	855	463	722	1039	565	438	31	992	539	343	126
21455	812	614	420	146	1073	470	708	57	979	851	18	1011	1027	584	425	931	1145	527
21455	175	1056	828	602	404	967	874	466	715	43	572	444	5	998	1046	317	119	938
21455	618	392	159	1085	811	711	50	953	862	489	985	1033	591	432	24	1148	546	340
21455	643	205	211	1207	799	249	1171	517	745	383	1076	814	615	417	143	859	490	706
21455	1191	787	659	188	240	762	360	278	1159	506	405	172	1053	831	604	41	968	847
21455	204	223	1220	771	647	1188	494	751	377	255	808	621	394	160	1082	462	719	70
21455	800	631	192	239	1203	366	272	1165	523	739	149	1070	837	598	411	980	846	478
21455	227	1219	783	660	176	500	768	354	261	1182	627	388	166	1059	825	723	42	964
21455	113	932	1154	536	330	906	677	94	285	1103	655	177	224	1218	791	258	1185	491
21455	559	326	120	918	1142	304	1090	893	696	82	1204	798	651	200	212	736	367	274
21455	925	1128	547	349	116	683	101	292	1109	880	196	235	1192	784	658	1184	503	765
21455	337	139	921	1135	533	1097	899	670	88	311	772	644	203	231	1215	380	246	1172
21455	1131	540	323	127	944	75	298	1116	887	689	238	1211	795	632	189	507	764	363
21455	844	471	727	45	978	446	9	1000	1047	563	124	945	1126	548	322	883	687	104
21455	62	955	873	459	716	1035	592	423	26	989	531	338	112	929	1155	314	1096	890
21455	488	704	51	972	850	3	1006	1024	580	452	917	1139	560	321	128	680	78	302
21455	961	867	465	733	39	569	440	32	983	1041	350	111	933	1127	544	1107	909	676
21455	710	68	949	856	482	1012	1018	586	429	20	1143	532	334	140	916	81	295	1093
21455	276	1167	514	740	368	1075	807	609	413	161	853	485	701	52	974	424	16	1007
21455	759	355	263	1186	502	399	168	1071	830	597	36	962	869	468	730	1042	570	453
21455	1173	521	747	374	250	826	620	387	154	1078	484	713	65	946	857	33	984	1031
21455	362	269	1160	508	766	142	1064	833	616	410	975	841	472	729	48	576	447	10
21455	495	753	381	257	1179	623	406	165	1052	819	717	64	958	870	456	990	1048	564
	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455

19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	II
21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455
390	158	865	457	714	63	966	433	30	981	1032	589	109	926	1147	535	348	21455
836	607	49	973	861	480	702	1016	577	449	13	1010	552	325	138	914	1136	21455
164	1055	476	725	37	959	868	29	993	1045	561	437	943	1124	541	342	115	21455
613	416	947	854	483	721	60	590	421	17	1009	1028	331	132	920	1153	529	21455
1062	829	728	56	970	842	469	997	1044	573	450	1	1130	558	319	121	937	21455
1222	773	264	1190	496	758	357	1058	827	629	396	155	871	467	724	40	963	21455
201	219	741	373	252	1174	525	419	151	1065	813	617	59	950	858	486	712	21455
790	662	1162	509	770	356	268	820	603	407	174	1061	473	731	47	969	845	21455
213	1216	385	251	1178	497	754	162	1084	816	610	393	957	864	460	718	66	21455
639	195	513	742	369	280	1161	606	400	148	1072	839	705	53	976	852	479	21455
553	336	888	695	71	297	1114	634	191	237	1200	803	271	1164	510	767	353	21455
935	1122	281	1102	904	678	100	1217	780	663	179	226	755	382	248	1181	499	21455
329	133	694	83	310	1086	892	208	214	1206	797	640	1158	516	744	370	277	21455
1141	550	1115	876	682	99	293	786	657	185	243	1194	359	265	1187	493	761	21455
107	924	87	309	1098	905	666	220	1223	774	646	202	522	738	376	254	1175	21455
58	952	428	22	1014	1026	575	136	922	1144	530	333	900	667	84	308	1106	21455
474	735	1049	571	435	8	1002	549	320	123	941	1132	294	1113	896	690	72	21455
951	863	15	988	1037	594	431	928	1151	537	339	110	686	95	282	1099	903	21455
707	54	582	454	11	995	1023	327	129	915	1138	556	1087	889	693	91	305	21455
875	461	991	1030	568	442	34	1125	543	346	117	934	98	301	1110	877	679	21455
752	379	1054	821	622	395	173	866	464	720	67	948	439	35	986	1038	567	21455
1168	520	412	150	1083	809	611	55	977	843	481	709	1021	583	427	19	1015	21455
351	262	838	599	401	167	1060	458	726	44	965	872	7	999	1050	566	443	21455
519	748	156	1077	815	628	389	954	860	487	703	61	595	426	23	987	1034	21455
275	1156	605	418	144	1066	832	732	38	971	849	475	1003	1022	579	455	6	21455
291	1100	661	187	234	1195	788	270	1157	504	763	371	1063	835	596	402	169	21455
673	92	1214	775	648	206	222	749	378	266	1180	492	386	157	1079	818	625	21455
1119	886	193	241	1202	794	635	1176	515	737	364	273	834	608	415	141	1067	21455
85	288	782	654	180	228	1221	352	259	1183	511	760	170	1051	822	624	398	21455
897	699	215	1208	801	642	199	518	756	375	247	1169	612	414	153	1080	806	21455
1025	593	131	919	1140	557	318	894	700	76	303	1092	638	197	244	1201	785	21455
4	996	545	347	108	936	1129	286	1108	882	684	105	1224	781	645	183	232	21455
587	430	913	1146	534	335	137	672	89	315	1091	898	190	218	1212	804	641	21455
1013	1019	324	125	942	1123	551	1120	881	688	77	299	792	664	186	225	1198	21455
436	27	1152	528	341	114	930	93	287	1104	910	671	221	1205	778	652	209	21455
21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455	21455

Combining Parts I and II, we get a **pan magic square** of order 35 with magic sum is $S_{35\times 35} = 21455$. Each 5×5 block a **pan magic square** of order 5 with equal magic sums, $S_{5\times 5} = 3065$.

4 Final Comments

This work brings pan magic squares of orders 9, 12, 15, 21, 24, 27, 30, 33, 35 and 36. Except order 35 all other magic square are multiples of 3 including the orders 18 and 30. These are the only two that are not pan diagonal but multiple of 3. The magic squares multiples of 3 are in such a way that either they are blocks of magic squares of order 3 with different magic sums or semi-magic squares of equal sums (in rows and columns). In order to bring pan magic squares as blocks of semi-magic squares of order 3 (in rows and columns), we used the idea of **magic rectangles**, such as, of orders, 15, 21, 27 and 33. The other cases, such as of orders 9, 12, 18, 24, 30, 36 we used the idea of **semi-magic rectangles** (equality only in rows), except the order 9, where a magic square of order 9 is applied. The pan magic square of order 35 is obtained in two different ways, one with 25 blocks of order 7 equal sums magic squares, and second with 49 blocks of order 5 equal sums magic squares. Even though it is not necessary, but we have used the idea of **magic rectangle** of order $(5, 7)$. Some work in magic squares can seen in [2, 3, 4, 6]

During past years the author worked with magic squares in different situations. These are given in details below:

• Author's Contributions to Magic Squares

The item-wise author's work on magic squares is as follows:

(i) **Digital numbers** magic squares – [8, 9, 10, 11, 12, 13];
(ii) **Block-wise construction** of **bimagic squares** – [14];
(iii) Connections with **genetic tables** and **Shannon's entropy** – [15];
(iv) **Selfie** and **palindromic-type** magic squares – [16];
(v) **Intervally distributed** and **block-wise** magic squares – [17, 18, 19];
(vi) **Multi-digits** magic squares – [20];
(vii) **Perfect square sum** magic squares with **uniformity** and **minimum Sum** – [21, 22];
(viii) **Pythagorean triples** to generate **perfect square sum** magic squares – [22];
(ix) **Block-wise** equal sums **pan magic squares of order** $4k$ – [23];
(x) **Block-wise** equal sums **magic squares of order** $3k$– [24];
(xi) **Block-wise** unequal sums **magic squares of order** $3k$– [25].

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