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SOME FUNCTIONALS ASSOCIATED TO SEMI-INNER PRODUCTS ON COMPLEX BANACH SPACES

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Abstract. In this paper we introduce some functionals that are related to Schwarz's inequality for semi-inner products on complex linear spaces and study their properties such as superadditivity and monotonicity. Applications for particular semi-inner products generated by an element of norm one in Banach spaces as well as for some bounded operators that satisfy a Schwarz's type condition are given. Some suggestive examples in the case of complex Hilbert spaces are also provided.

1. Introduction

In what follows, we assume that X is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [8] but the main properties of it were discovered by J. R. Giles [9], P. L. Papini [15], P. M. Miličić [11]–[13], I. Roşca [16], B. Nath [14] and others, see [2].

In this introductory section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 1. *The mapping $[\cdot,\cdot] : X \times X \to \mathbb{K}$ will be called the semi-inner product in the sense of Lumer-Giles or L-G-s.i.p., for short, if the following properties are satisfied:*

(i) $[x+y,z] = [x,z] + [y,z]$ *for all* $x, y, z \in X$;
(ii) $[\lambda x, y] = \lambda [x,y]$ *for all* $x, y \in X$ *and* λ *a scalar in* $\mathbb{K}$;
(iii) $[x,x] \geq 0$ *for all* $x \in X$ *and* $[x,x] = 0$ *implies that* $x = 0$;
(iv) $|[x,y]|^2 \leq [x,x]\,[y,y]$ *for all* $x, y \in X$;
(v) $[x, \lambda y] = \bar{\lambda}\,[x,y]$ *for all* $x, y \in X$ *and* λ *a scalar in* $\mathbb{K}$.

The following results collects some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 1. *Let X be a linear space and $[\cdot,\cdot]$ a L-G-s.i.p on X. Then the following statements are true:*

(i) *The mapping* $X \ni x \xrightarrow{\|\cdot\|} [x,x]^{\frac{1}{2}} \in \mathbb{R}_+$ *is a norm on* X;
(ii) *For every $y \in X$ the functional* $X \ni x \xrightarrow{f_y} [x,y] \in \mathbb{K}$ *is a continuous linear functional on X endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality* $\|f_y\| = \|y\|$.

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Definition 2. *The mapping $J : X \to 2^{X^*}$, where X^* is the dual space of X, given by:*

$$J(x) := \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \ \|x^*\| = \|x\|\}, \ x \in X$$

will be called the normalised duality mapping of normed linear space $(X, \|\cdot\|)$.

Definition 3. *A mapping $\tilde{J} : X \to X^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in X.*

The following theorem due to I. Roşca [16] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 1. *Let $(X, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \text{ for all } x, y \text{ in } X,$$

where $\tilde{J}$ is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *X is smooth;*
(ii) *There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.*

A mapping $[\cdot, \cdot] : X \times X \to \mathbb{K}$ will be called a *sub-semi-inner product in the sense of Lumer-Giles* or s-L-G-*s.i.p.*, for short, if in the Definition 1 instead of condition (iii) we have the relaxed condition:

(iii') $[x, x] \geq 0$ *for all $x \in X$.*

We denote by $\mathcal{SS}(X)$ the class of all sub-semi-inner products defined on the linear space X.

We can introduce the following order relation amongst the elements of $\mathcal{SS}(X)$. For $[\cdot, \cdot]_1, [\cdot, \cdot]_2 \in \mathcal{SS}(X)$, we say that

$$[\cdot, \cdot]_2 \succeq [\cdot, \cdot]_1 \text{ iff } [\cdot, \cdot]_2 - [\cdot, \cdot]_1 \in \mathcal{SS}(X). \tag{1.1}$$

In the recent paper [4] we obtained amongst other the following inequalities for $[\cdot, \cdot]_1, [\cdot, \cdot]_2 \in \mathcal{SS}(X)$ with $[\cdot, \cdot]_2 \succeq [\cdot, \cdot]_1$

$$\begin{aligned} &\|x\|_2^2 \|y\|_2^2 - |[x, y]_2|^2 + \|x\|_1^2 \|y\|_1^2 - |[x, y]_1|^2 \\ &\geq \left[\det \begin{pmatrix} \|x\|_1 & \|y\|_1 \\ \|x\|_2 & \|y\|_2 \end{pmatrix}\right]^2 \geq 0 \end{aligned} \tag{1.2}$$

and the inequality

$$|[x, y]_1| \leq \frac{1}{2} [\|x\|_2 \|y\|_2 + |[x, y]_2|], \tag{1.3}$$

for any $x, y \in X$.

Some applications for particular semi-inner products generated by an element of norm one in the given Banach spaces as well as for some bounded operators that satisfy a Schwarz's type condition were given. Some norm and numerical radius inequalities for operators acting on smooth uniformly convex Banach spaces with more suggestive examples in the case of complex Hilbert spaces were also provided [4].

In this paper we introduce some functionals that are related to Schwarz's inequality for semi-inner products on complex linear spaces and study their properties such as superadditivity and monotonicity. Applications for particular semi-inner products generated by an element of norm one in Banach spaces as well as for some bounded operators that satisfy a Schwarz's type condition are given. Some suggestive examples in the case of complex Hilbert spaces are also provided.

2. Some Properties of Mapping σ

Let us consider the following mapping $\sigma : \mathcal{SS}(X) \times X^2 \to \mathbb{R}_+$,

$$\sigma\left([\cdot,\cdot]\,;x,y\right) := [x,x]^{1/2}\,[y,y]^{1/2} - |[x,y]| = \|x\|\,\|y\| - |[x,y]|$$

where $x,\, y \in X$, which is closely related to Schwarz's inequality (iv) from Definition 1.

The following simple properties of σ are obvious:

(s) $\sigma\left(\alpha\,[\cdot,\cdot]\,;x,y\right) = \alpha\sigma\left([\cdot,\cdot]\,;x,y\right);$

(ss) $\sigma\left([\cdot,\cdot]\,;y,x\right) = \sigma\left([\cdot,\cdot]\,;x,y\right);$

(sss) $\sigma\left([\cdot,\cdot]\,;x,y\right) \geq 0$ (by Schwarz's inequality);

for any $\alpha \geq 0$, $[\cdot,\cdot] \in \mathcal{SS}(X)$ and $x,\, y \in X$.

The following result concerning the functional properties of σ as a function depending on the sub-semi-inner product $[\cdot,\cdot] \in \mathcal{SS}(X)$:

Theorem 2. *The mapping σ satisfies the following statements:*

(i) *For every $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i = 1,2)$ one has the inequality*

$$\sigma\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2\,;x,y\right) \geq \sigma\left([\cdot,\cdot]_1\,;x,y\right) + \sigma\left([\cdot,\cdot]_2\,;x,y\right) \geq 0 \tag{2.1}$$

for all $x,\, y \in X$, i.e., the mapping $\sigma\left(\cdot;x,y\right)$ is superadditive on $\mathcal{SS}(X)$;

(ii) *For every $[\cdot,\cdot]_1\,,\ [\cdot,\cdot]_2 \in \mathcal{SS}(X)$ with $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$ one has*

$$\sigma\left([\cdot,\cdot]_2\,;x,y\right) \geq \sigma\left([\cdot,\cdot]_1\,;x,y\right) \geq 0 \tag{2.2}$$

for all $x,\, y \in X$, i.e., the mapping $\sigma\left(\cdot;x,y\right)$ is nondecreasing on $\mathcal{SS}(X)$.

Proof. (i) By the Cauchy-Bunyakovsky-Schwarz inequality for real numbers, we have

$$\left(a^2+b^2\right)^{\frac{1}{2}}\left(c^2+d^2\right)^{\frac{1}{2}} \geq ac+bd; \quad a,b,c,d \geq 0.$$

Therefore,

$$\begin{aligned}
&\sigma\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2\,;x,y\right)\\
&= \left([x,x]_1 + [x,x]_2\right)^{\frac{1}{2}}\left([y,y]_1 + [y,y]_2\right)^{\frac{1}{2}} - \left|[x,y]_1 + [x,y]_2\right|\\
&\geq [x,x]_1^{1/2}\,[y,y]_1^{1/2} + [x,x]_2^{1/2}\,[y,y]_2^{1/2} - \left|[x,y]_1\right| - \left|[x,y]_2\right|\\
&= \sigma\left((\cdot,\cdot)_1\,;x,y\right) + \sigma\left((\cdot,\cdot)_2\,;x,y\right),
\end{aligned}$$

for all $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i=1,2)$ and $x,\, y \in X$, and the statement is proved.

(ii) Suppose that $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$ and define $[\cdot,\cdot]_{2,1} := [\cdot,\cdot]_2 - [\cdot,\cdot]_1$. Then $[\cdot,\cdot]_{2,1}$ is a sub-semi-inner product and thus, by the above property one has,

$$\sigma\left([\cdot,\cdot]_2\,;x,y\right) = \sigma\left([\cdot,\cdot]_{2,1} + [\cdot,\cdot]_1\,;x,y\right) \geq \sigma\left([\cdot,\cdot]_{2,1}\,;x,y\right) + \sigma\left([\cdot,\cdot]_1\,;x,y\right)$$

from where we get:

$$\sigma\left([\cdot,\cdot]_2;x,y\right)-\sigma\left([\cdot,\cdot]_1;x,y\right)\geq\sigma\left([\cdot,\cdot]_{2,1};x,y\right)\geq 0$$

and the proof of the theorem is completed. □

Corollary 1. *Let* $[\cdot,\cdot]_1,\ [\cdot,\cdot]_2\in\mathcal{SS}(X)$ *and* $M>m>0$ *with* $M[\cdot,\cdot]_1\succeq[\cdot,\cdot]_2\succeq m[\cdot,\cdot]_1$, *then*

$$M\sigma\left([\cdot,\cdot]_1;x,y\right)\geq\sigma\left([\cdot,\cdot]_2;x,y\right)\geq m\sigma\left([\cdot,\cdot]_1;x,y\right)\geq 0. \tag{2.3}$$

Remark 1. *If we consider the related mapping*

$$\sigma_r\left([\cdot,\cdot];x,y\right):=[x,x]^{1/2}[y,y]^{1/2}-\operatorname{Re}[x,y],$$

then we can show, as above, that $\sigma(\cdot;x,y)$ *is superadditive and nondecreasing on* $\mathcal{SS}(X)$.

For $[\cdot,\cdot]\in\mathcal{SS}(X)$ and $e\in X$, $e\neq 0$ we consider the functional $[\cdot,\cdot]_e:X\times X\to\mathbb{K}$ defined by

$$[x,y]_e=[x,e]\overline{[y,e]}\text{ for }x,\ y\in X.$$

We observe that $[\cdot,\cdot]_e$ is linear in the first variable, anti-homogeneous in the second variable and

$$[x,x]_e=|[x,e]|^2\geq 0\text{ for any }x\in X.$$

Also, we have

$$\left|[x,y]_e\right|^2=\left|[x,e]\overline{[y,e]}\right|^2=|[x,e]|^2|[y,e]|^2=[x,x]_e[y,y]_e,$$

for $x,y\in X$, which shows that the Schwarz' s inequality (iv) is verified with equality. Therefore we conclude that $[\cdot,\cdot]_e\in\mathcal{SS}(X)$. We also observe that

$$\sigma\left([\cdot,\cdot]_e;x,y\right):=[x,x]_e^{1/2}[y,y]_e^{1/2}-\left|[x,y]_e\right|=0$$

for $x,\ y\in X$.

Corollary 2. *Let* $[\cdot,\cdot]\in\mathcal{SS}(X)$ *and* $e\in X$, $e\neq 0$, *then for any* $x,\ y\in X$ *we have*

$$\begin{aligned}&\left(\|x\|^2+|[x,e]|^2\right)^{1/2}\left(\|y\|^2+|[y,e]|^2\right)^{1/2}-\left|[x,y]+[x,e]\overline{[y,e]}\right|\\&\geq\|x\|\|y\|-|[x,y]|\geq 0.\end{aligned}\tag{2.4}$$

Proof. From (2.1) for $[\cdot,\cdot]_1=[\cdot,\cdot]$ and $[\cdot,\cdot]_2=[\cdot,\cdot]_e$ we have

$$\sigma\left([\cdot,\cdot]+[\cdot,\cdot]_e;x,y\right)\geq\sigma\left([\cdot,\cdot];x,y\right)+\sigma\left([x,y]_e;x,y\right)$$

for any $x,\ y\in X$, which is equivalent to (2.4). □

Corollary 3. *Let* $[\cdot,\cdot]\in\mathcal{SS}(X)$ *and* $f\in X$, $\|f\|=1$ *such that* $[\cdot,\cdot]\succeq[\cdot,\cdot]_f$. *Then for any* $x,\ y\in X$ *we have*

$$\begin{aligned}&\|x\|\|y\|-|[x,y]|\\&\geq\left(\|x\|^2-|[x,f]|^2\right)^{1/2}\left(\|y\|^2-|[y,f]|^2\right)^{1/2}-\left|[x,y]-[x,f]\overline{[y,f]}\right|\geq 0.\end{aligned}\tag{2.5}$$

Proof. From (2.1) for $[\cdot,\cdot]_2=[\cdot,\cdot]-[\cdot,\cdot]_f$ and $[\cdot,\cdot]_1=[\cdot,\cdot]_f$ we have

$$\sigma\left([\cdot,\cdot];x,y\right)\geq\sigma\left([\cdot,\cdot]-[\cdot,\cdot]_f;x,y\right)+\sigma\left([\cdot,\cdot]_f;x,y\right)$$

for any $x,\ y\in X$, which is equivalent to (2.5). □

3. Some Properties of Mapping δ

Now consider the following mapping naturally associated to Schwarz's inequality, namely $\delta : \mathcal{SS}(X) \times X^2 \to \mathbb{R}_+$,

$$\delta\left([\cdot,\cdot]\,;x,y\right) := [x,x]\,[y,y] - |[x,y]|^2 = \|x\|^2\,\|y\|^2 - |[x,y]|^2$$

for $x,\, y \in X$.

It is obvious that the following properties are valid:

(i) $\delta\left([\cdot,\cdot]\,;x,y\right) \geq 0$ (Schwarz's inequality);
(ii) $\delta\left([\cdot,\cdot]\,;x,y\right) = \delta\left([\cdot,\cdot]\,;y,x\right)$;
(iii) $\delta\left(\alpha\,[\cdot,\cdot]\,;x,y\right) = \alpha^2\delta\left([\cdot,\cdot]\,;x,y\right)$

for all $x,\, y \in X$, $\alpha \geq 0$ and $[\cdot,\cdot] \in \mathcal{SS}(X)$.

Theorem 3. *The mapping δ satisfies the following statements:*

(i) *For every $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i = 1,2)$ one has the inequality*

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2\,;x,y\right) - \delta\left([\cdot,\cdot]_1\,;x,y\right) - \delta\left([\cdot,\cdot]_2\,;x,y\right)\\
&\geq \left(\|x\|_2\,\|y\|_1 - \|x\|_1\,\|y\|_2\right)^2 + 2\left(\|x\|_2\,\|y\|_1\,\|x\|_1\,\|y\|_2 - |[x,y]_1|\,|[x,y]_2|\right)\\
&\geq 0
\end{aligned} \tag{3.1}$$

for all $x,\, y \in X$, i.e., the mapping $\delta\left(\cdot;x,y\right)$ is strongly superadditive on $\mathcal{SS}(X)$;

(ii) *For every $[\cdot,\cdot]_1\,,\ [\cdot,\cdot]_2 \in \mathcal{SS}(X)$ with $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$ one has*

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_2\,;x,y\right) - \delta\left([\cdot,\cdot]_1\,;x,y\right)\\
&\geq \left(\left(\|x\|_2^2 - \|x\|_1^2\right)^{1/2}\|y\|_1 - \|x\|_1\left(\|y\|_2^2 - \|y\|_1^2\right)^{1/2}\right)^2\\
&+ 2\left(\|y\|_1\,\|x\|_1\left(\|x\|_2^2 - \|x\|_1^2\right)^{1/2}\left(\|y\|_2^2 - \|y\|_1^2\right)^{1/2}\right.\\
&\left.- |[x,y]_1|\,|[x,y]_2 - [x,y]_1|\right)\\
&\geq 0
\end{aligned} \tag{3.2}$$

for all $x,\, y \in X$, i.e., the mapping $\delta\left(\cdot;x,y\right)$ is strongly nondecreasing on $\mathcal{SS}(X)$.

Proof. (i) We have by the definition of δ that

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2\,;x,y\right)\\
&= \left([x,x]_1 + [x,x]_2\right)\left([y,y]_1 + [y,y]_2\right) - |[x,y]_1 + [x,y]_2|^2\\
&= [x,x]_1\,[y,y]_1 + [x,x]_2\,[y,y]_1 + [x,x]_1\,[y,y]_2 + [x,x]_2\,[y,y]_2\\
&- |[x,y]_1|^2 - 2\operatorname{Re}\left([x,y]_1\,\overline{[x,y]_2}\right) - |[x,y]_2|^2\\
&= [x,x]_1\,[y,y]_1 - |[x,y]_1|^2 + [x,x]_2\,[y,y]_2 - |[x,y]_2|^2\\
&+ [x,x]_2\,[y,y]_1 + [x,x]_1\,[y,y]_2 - 2\operatorname{Re}\left([x,y]_1\,\overline{[x,y]_2}\right)
\end{aligned}$$

for every $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i = 1,2)$ and $x,\, y \in X$.

From this we have the equality of interest

$$\begin{aligned}(3.3)\qquad &\delta\left([\cdot,\cdot]_1+[\cdot,\cdot]_2\,;x,y\right)-\delta\left([\cdot,\cdot]_1\,;x,y\right)-\delta\left([\cdot,\cdot]_2\,;x,y\right)\\ &=\|x\|_2^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2-2\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\end{aligned}$$

for every $[\cdot,\cdot]_i\in\mathcal{SS}(X)$ $(i=1,2)$ and $x,\,y\in X$.

Now, observe that

$$\begin{aligned}(3.4)\qquad &\|x\|_2^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2-2\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\\ &=\|x\|_2^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2-2\|x\|_2\|y\|_1\|x\|_1\|y\|_2\\ &+2\|x\|_2\|y\|_1\|x\|_1\|y\|_2-2\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\\ &=\left(\|x\|_2\|y\|_1-\|x\|_1\|y\|_2\right)^2\\ &+2\left(\|x\|_2\|y\|_1\|x\|_1\|y\|_2-\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\right)\end{aligned}$$

for every $[\cdot,\cdot]_i\in\mathcal{SS}(X)$ $(i=1,2)$ and $x,\,y\in X$.

By the properties of modulus we also have

$$\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\le\left|\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\right|\le\left|[x,y]_1\overline{[x,y]_2}\right|=\left|[x,y]_1\right|\left|[x,y]_2\right|,$$

which, by the use of the equality (3.4), produces the inequality

$$\begin{aligned}&\|x\|_2^2\|y\|_1^2+\|x\|_1^2\|y\|_2^2-2\operatorname{Re}\left([x,y]_1\overline{[x,y]_2}\right)\\ &\ge\left(\|x\|_2\|y\|_1-\|x\|_1\|y\|_2\right)^2+2\left(\|y\|_1\|x\|_1\|x\|_2\|y\|_2-\left|[x,y]_1\right|\left|[x,y]_2\right|\right)\end{aligned}$$

and by (3.3) we get the first part of (3.1).

By Schwarz's inequality for the sub-semi-inner products we have

$$\|y\|_1\|x\|_1\ge\left|[x,y]_1\right|\ \text{ and }\ \|y\|_2\|x\|_2\ge\left|[x,y]_2\right|,$$

which by multiplication gives

$$\|y\|_1\|x\|_1\|x\|_2\|y\|_2\ge\left|[x,y]_1\right|\left|[x,y]_2\right|$$

for any $x,\,y\in X$, which proves the last inequality in (3.1).

(ii) If $[\cdot,\cdot]_2\succeq[\cdot,\cdot]_1$ then $[\cdot,\cdot]_{2,1}:=[\cdot,\cdot]_2-[\cdot,\cdot]_1$ is a sub-semi-inner product and if we write the inequality (3.1) for $[\cdot,\cdot]_1$ and $[\cdot,\cdot]_{2,1}$ then we have

$$\begin{aligned}&\delta\left([\cdot,\cdot]_2\,;x,y\right)-\delta\left([\cdot,\cdot]_1\,;x,y\right)-\delta\left([\cdot,\cdot]_{2,1}\,;x,y\right)\\ &\ge\left(\|x\|_{2,1}\|y\|_1-\|x\|_1\|y\|_{2,1}\right)^2\\ &+2\left(\|y\|_1\|x\|_1\|x\|_{2,1}\|y\|_{2,1}-\left|[x,y]_1\right|\left|[x,y]_{2,1}\right|\right)\\ &\ge 0,\end{aligned}$$

which implies that

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_2 ; x, y\right)-\delta\left([\cdot,\cdot]_1 ; x, y\right)\\
&\geq \delta\left([\cdot,\cdot]_{2,1} ; x, y\right)+\left(\|x\|_{2,1}\|y\|_1-\|x\|_1\|y\|_{2,1}\right)^2\\
&+2\left(\|y\|_1\|x\|_1\|x\|_{2,1}\|y\|_{2,1}-\left|[x, y]_1\right|\left|[x, y]_{2,1}\right|\right)\\
&\geq\left(\|x\|_{2,1}\|y\|_1-\|x\|_1\|y\|_{2,1}\right)^2\\
&+2\left(\|y\|_1\|x\|_1\|x\|_{2,1}\|y\|_{2,1}-\left|[x, y]_1\right|\left|[x, y]_{2,1}\right|\right)
\end{aligned}$$

for any x, $y \in X$. $\square$

Remark 2. *From the inequality (3.1) we have*

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_1+[\cdot,\cdot]_2 ; x, y\right)-\delta\left([\cdot,\cdot]_1 ; x, y\right)-\delta\left([\cdot,\cdot]_2 ; x, y\right)\\
&\geq\left(\det\begin{bmatrix}\|x\|_2 & \|x\|_1\\ \|y\|_2 & \|y\|_1\end{bmatrix}\right)^2 \geq 0
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_1+[\cdot,\cdot]_2 ; x, y\right)-\delta\left([\cdot,\cdot]_1 ; x, y\right)-\delta\left([\cdot,\cdot]_2 ; x, y\right)\\
&\geq 2\det\begin{bmatrix}\|x\|_2\|y\|_2 & \left|[x, y]_2\right|\\ \left|[x, y]_1\right| & \|y\|_1\|x\|_1\end{bmatrix} \geq 0
\end{aligned} \tag{3.6}$$

for any $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i=1,2)$ *and for any* x, $y \in X$.

If $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2 \in \mathcal{SS}(X)$ *with* $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$, *then by (3.2) we have*

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_2 ; x, y\right)-\delta\left([\cdot,\cdot]_1 ; x, y\right)\\
&\geq\left(\det\begin{bmatrix}\|x\|_1 & \left(\|x\|_2^2-\|x\|_1^2\right)^{1/2}\\ \|y\|_1 & \left(\|y\|_2^2-\|y\|_1^2\right)^{1/2}\end{bmatrix}\right)^2 \geq 0
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\delta\left([\cdot,\cdot]_2 ; x, y\right)-\delta\left([\cdot,\cdot]_1 ; x, y\right)\\
&\geq 2\det\begin{bmatrix}\|x\|_1\left(\|x\|_2^2-\|x\|_1^2\right)^{1/2} & \left|[x, y]_2-[x, y]_1\right|\\ \left|[x, y]_1\right| & \|y\|_1\left(\|y\|_2^2-\|y\|_1^2\right)^{1/2}\end{bmatrix} \geq 0
\end{aligned} \tag{3.8}$$

for any x, $y \in X$.

In particular, $\delta(\cdot; x, y)$ *is monotonic nondecreasing, namely*

$$\delta\left([\cdot,\cdot]_2 ; x, y\right) \geq \delta\left([\cdot,\cdot]_1 ; x, y\right) \tag{3.9}$$

provided that $[\cdot,\cdot]_1$, $[\cdot,\cdot]_2 \in \mathcal{SS}(X)$ *with* $[\cdot,\cdot]_2 \succeq [\cdot,\cdot]_1$.

Corollary 4. *Let $[\cdot,\cdot] \in \mathcal{SS}(X)$ and $e \in X$, $e \neq 0$, then for any $x, y \in X$ we have*

$$\begin{aligned}(3.10)\qquad & \left(\|x\|^2 + |[x,e]|^2\right)\left(\|y\|^2 + |[y,e]|^2\right) - \left|[x,y] + [x,e]\,\overline{[y,e]}\right|^2 \\ & - \|x\|^2\|y\|^2 + |[x,y]|^2 \\ & \geq \left(\det\begin{bmatrix} \|x\| & |[x,e]| \\ \|y\| & |[y,e]| \end{bmatrix}\right)^2 \geq 0\end{aligned}$$

and

$$\begin{aligned}(3.11)\qquad & \left(\|x\|^2 + |[x,e]|^2\right)\left(\|y\|^2 + |[y,e]|^2\right) - \left|[x,y] + [x,e]\,\overline{[y,e]}\right|^2 \\ & \geq 2\left(\|x\|\|y\| - |[x,y]|\right)\left[|[x,e]|\,|[y,e]| + \frac{1}{2}\left(\|x\|\|y\| + |[x,y]|\right)\right] \\ & \geq 0.\end{aligned}$$

The proof follows (3.5) and (3.6) for the sub-semi-inner products $[\cdot,\cdot]_2 = [\cdot,\cdot]$ and $[\cdot,\cdot]_1 = [\cdot,\cdot]_e$.

Corollary 5. *Let $[\cdot,\cdot] \in \mathcal{SS}(X)$ and $f \in X$, $\|f\| = 1$ such that $[\cdot,\cdot] \succeq [\cdot,\cdot]_f$. Then for any $x, y \in X$ we have*

$$(3.12)\qquad \|x\|^2\|y\|^2 - |[x,y]|^2 \geq \left(\det\begin{bmatrix} |[x,f]| & \left(\|x\|^2 - |[x,f]|^2\right)^{1/2} \\ |[y,f]| & \left(\|y\|^2 - |[y,f]|^2\right)^{1/2} \end{bmatrix}\right)^2 \geq 0$$

and

$$\begin{aligned}(3.13)\qquad & \|x\|^2\|y\|^2 - |[x,y]|^2 \geq 2\,|[x,f]|\,|[y,f]| \\ & \times\left[\left(\|x\|^2 - |[x,f]|^2\right)^{1/2}\left(\|y\|^2 - |[y,f]|^2\right)^{1/2} - \left|[x,y] - [x,f]\,\overline{[y,f]}\right|\right] \geq 0\end{aligned}$$

for any $x, y \in X$.

We also have:

Corollary 6. *Let $[\cdot,\cdot]_1, [\cdot,\cdot]_2 \in \mathcal{SS}(X)$ and $M > m > 0$ with $M[\cdot,\cdot]_1 \succeq [\cdot,\cdot]_2 \succeq m[\cdot,\cdot]_1$, then*

$$(3.14)\qquad M^2\delta\left([\cdot,\cdot]_1; x, y\right) \geq \delta\left([\cdot,\cdot]_2; x, y\right) \geq m^2\delta\left([\cdot,\cdot]_1; x, y\right) \geq 0.$$

Proof. From (3.9) we have that

$$\delta\left([\cdot,\cdot]_2; x, y\right) \geq \delta\left(m[\cdot,\cdot]_1; x, y\right) = m^2\delta\left([\cdot,\cdot]_1; x, y\right)$$

and the corresponding inequality for M, which proves (3.14). □

4. Some Properties of Mapping β

The following mapping associated to Schwarz's inequality can also be considered $\beta : \mathcal{SS}(X) \times X^2 \to \mathbb{R}_+$,

$$\beta\left([\cdot,\cdot]; x, y\right) := \left(\|x\|^2\|y\|^2 - |[x,y]|^2\right)^{1/2} = \delta^{1/2}\left([\cdot,\cdot]; x, y\right)$$

for $x, y \in X$.

It is obvious that the following properties are valid:

(i) $\beta([\cdot,\cdot];x,y)\geq 0$;
(ii) $\beta([\cdot,\cdot];x,y)=\beta([\cdot,\cdot];y,x)$;
(iii) $\beta(\alpha[\cdot,\cdot];x,y)=\alpha\beta([\cdot,\cdot];x,y)$

for all $x,\ y\in X$, $\alpha\geq 0$ and $[\cdot,\cdot]\in\mathcal{SS}(X)$.

Theorem 4. *For every $[\cdot,\cdot]_i\in\mathcal{SS}(X)$ $(i=1,2)$ one has the inequality*

$$\begin{aligned}\beta^2&([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)\\&\geq[\beta([\cdot,\cdot]_1;x,y)+\beta([\cdot,\cdot]_2;x,y)]^2\\&+(\|x\|_2\|y\|_1-\|x\|_1\|y\|_2)^2\\&+2(\|x\|_2\|y\|_1\|x\|_1\|y\|_2-|[x,y]_1||[x,y]_2|)\\&-2\left(\|x\|_1^2\|y\|_1^2-|[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2-|[x,y]_1|^2\right)^{1/2}\end{aligned}\tag{4.1}$$

for any $x,\ y\in X$.

In particular, we have

$$\begin{aligned}\beta^2([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)&\geq[\beta([\cdot,\cdot]_1;x,y)+\beta([\cdot,\cdot]_2;x,y)]^2\\&\quad+(\|x\|_2\|y\|_1-\|x\|_1\|y\|_2)^2\end{aligned}\tag{4.2}$$

and

$$\begin{aligned}\beta^2&([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)\\&\geq[\beta([\cdot,\cdot]_1;x,y)+\beta([\cdot,\cdot]_2;x,y)]^2\\&+2(\|x\|_2\|y\|_1\|x\|_1\|y\|_2-|[x,y]_1||[x,y]_2|)\\&-2\left(\|x\|_1^2\|y\|_1^2-|[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2-|[x,y]_1|^2\right)^{1/2}\end{aligned}\tag{4.3}$$

for any $x,\ y\in X$.

Proof. We have

$$\begin{aligned}\beta^2&([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)-[\beta([\cdot,\cdot]_1;x,y)+\beta([\cdot,\cdot]_2;x,y)]^2\\&=\beta^2([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)-\beta^2([\cdot,\cdot]_1;x,y)-\beta^2([\cdot,\cdot]_2;x,y)^2\\&-2\beta([\cdot,\cdot]_1;x,y)\beta([\cdot,\cdot]_2;x,y)\\&=\delta([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)-\delta([\cdot,\cdot]_1;x,y)-\delta([\cdot,\cdot]_2;x,y)\\&-2\left(\|x\|_1^2\|y\|_1^2-|[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2-|[x,y]_1|^2\right)^{1/2}\end{aligned}\tag{4.4}$$

for any $x,\ y\in X$.

Using the inequality (3.1) we then have

$$\begin{aligned}&\delta([\cdot,\cdot]_1+[\cdot,\cdot]_2;x,y)-\delta([\cdot,\cdot]_1;x,y)-\delta([\cdot,\cdot]_2;x,y)\\&-2\left(\|x\|_1^2\|y\|_1^2-|[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2-|[x,y]_1|^2\right)^{1/2}\\&\geq(\|x\|_2\|y\|_1-\|x\|_1\|y\|_2)^2+2(\|x\|_2\|y\|_1\|x\|_1\|y\|_2-|[x,y]_1||[x,y]_2|)\\&-2\left(\|x\|_1^2\|y\|_1^2-|[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2-|[x,y]_1|^2\right)^{1/2}\end{aligned}\tag{4.5}$$

for any $x,\ y\in X$.

By using (4.4) and (4.5) we get (4.1).

By utilising the elementary inequality for real numbers a, b, c, d

$$\left(a^2 - b^2\right)\left(c^2 - d^2\right) \le (ac - bd)^2$$

we get

$$\left(\|x\|_1^2\|y\|_1^2 - |[x,y]_1|^2\right)\left(\|x\|_2^2\|y\|_2^2 - |[x,y]_1|^2\right) \le \left(\|x\|_2\|y\|_1\|x\|_1\|y\|_2 - |[x,y]_1|\,|[x,y]_2|\right)^2 \tag{4.6}$$

for any x, $y \in X$.

Since by Schwarz's inequality we have

$$\|x\|_1^2\|y\|_1^2 \ge |[x,y]_1|^2, \quad \|x\|_2^2\|y\|_2^2 \ge |[x,y]_1|^2$$

and

$$\|x\|_2\|y\|_1\|x\|_1\|y\|_2 \ge |[x,y]_1|\,|[x,y]_2|,$$

then by (4.6) we get

$$\left(\|x\|_1^2\|y\|_1^2 - |[x,y]_1|^2\right)^{1/2}\left(\|x\|_2^2\|y\|_2^2 - |[x,y]_1|^2\right)^{1/2} \le \|x\|_2\|y\|_1\|x\|_1\|y\|_2 - |[x,y]_1|\,|[x,y]_2| \tag{4.7}$$

for any x, $y \in X$.

Now, by using (4.1) and (4.7) we get (4.2). The rest follows from these inequalities. □

Corollary 7. *For any x, $y \in X$ the mapping $\beta(\cdot; x, y)$ is superadditive, namely*

$$\beta\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2; x, y\right) \ge \beta\left([\cdot,\cdot]_1; x, y\right) + \beta\left([\cdot,\cdot]_2; x, y\right) \tag{4.8}$$

for any $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i = 1,2)$.

For all x, $y \in X$ the mapping $\beta(\cdot; x, y)$ is nondecreasing on $\mathcal{SS}(X)$.

Proof. From (4.2) we have for any $[\cdot,\cdot]_i \in \mathcal{SS}(X)$ $(i = 1,2)$ that

$$\begin{aligned}\beta^2\left([\cdot,\cdot]_1 + [\cdot,\cdot]_2; x, y\right) &\ge \left[\beta\left([\cdot,\cdot]_1; x, y\right) + \beta\left([\cdot,\cdot]_2; x, y\right)\right]^2 \\ &\quad + \left(\|x\|_2\|y\|_1 - \|x\|_1\|y\|_2\right)^2 \\ &\ge \left[\beta\left([\cdot,\cdot]_1; x, y\right) + \beta\left([\cdot,\cdot]_2; x, y\right)\right]^2\end{aligned}$$

for any x, $y \in X$, and by taking the square root we get (4.2).

The fact that the mapping $\beta(\cdot; x, y)$ is nondecreasing on $\mathcal{SS}(X)$ follows as above. □

5. Inequalities for Schwarz Type Operators

Following [6], on operator A on a complex Banach space $(X, \|\cdot\|)$ is said to be Hermitian if $[Ax, x]$ is real for any $x \in X$, where $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$.

Definition 4. *Let $(X, \|\cdot\|)$ be a complex Banach space and $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$. We say that the Hermitian operator $A : X \to X$ is of Schwarz type related to $[\cdot,\cdot]$ if $[Ax, x] \ge 0$ for any $x \in X$ and*

$$|[Ax, y]|^2 \le [Ax, x]\,[Ay, y] \tag{5.1}$$

for any x, $y \in X$. We write that $A \in \mathcal{S}_{[\cdot,\cdot]}(X)$.

We observe that the Hermitian operator $A : X \to X$ is of Schwarz type related to $[\cdot,\cdot]$ if and only if the functional $[\cdot,\cdot]_A : X \times X \to \mathbb{C}$, $[x,y]_A := [Ax, y]$ is a s-L-G-s.i.p. on X. Also notice that the identity operator I is of Schwarz type for any s-L-G-s.i.p. $[\cdot,\cdot]$ that generates the norm.

We observe that if $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, then $A + B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ and $\alpha A \in \mathcal{S}_{[\cdot,\cdot]}(X)$ for any $\alpha \geq 0$. This shows that $\mathcal{S}_{[\cdot,\cdot]}(X)$ is a cone in the Banach algebra $\mathcal{B}(X)$ of all bounded linear operators acting on X.

We can define on $\mathcal{S}_{[\cdot,\cdot]}(X)$ the following order relation $A \succeq B$ for $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ if $A - B \in \mathcal{S}_{[\cdot,\cdot]}(X)$. We observe that for $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, $A \succeq B$ iff $[\cdot,\cdot]_A \succeq [\cdot,\cdot]_B$ in the sense of the definition (1.1).

Proposition 3. *If $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, then*

$$\begin{aligned} &[(A+B)x,x]^{1/2}[(A+B)y,y]^{1/2} - |[(A+B)x,y]| \\ &\geq [Ax,x]^{1/2}[Ay,y]^{1/2} - |[Ax,y]| + [Bx,x]^{1/2}[By,y]^{1/2} - |[Bx,y]| \geq 0 \end{aligned} \tag{5.2}$$

for any $x, y \in X$.

If $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ with $MB \succeq A \succeq mB$ for some positive numbers $M > m$, then

$$\begin{aligned} &M\left[[Bx,x]^{1/2}[By,y]^{1/2} - |[Bx,y]|\right] \\ &\geq [Ax,x]^{1/2}[Ay,y]^{1/2} - |[Ax,y]| \\ &\geq m\left[[Bx,x]^{1/2}[By,y]^{1/2} - |[Bx,y]|\right] \geq 0 \end{aligned} \tag{5.3}$$

for any $x, y \in X$.

The proof follows by (2.1) and (2.3) written for the sub-semi-inner products $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$.

Proposition 4. *If $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, then*

$$\begin{aligned} &[(A+B)x,x][(A+B)y,y] - |[(A+B)x,y]|^2 \\ &\geq [Ax,x][Ay,y] - |[Ax,y]|^2 + [Bx,x][By,y] - |[Bx,y]|^2 \\ &+ \left(\det\begin{bmatrix} [Ax,x]^{1/2} & [Bx,x]^{1/2} \\ [Ay,y]^{1/2} & [By,y]^{1/2} \end{bmatrix}\right)^2 \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} &[(A+B)x,x][(A+B)y,y] - |[(A+B)x,y]|^2 \\ &\geq [Ax,x][Ay,y] - |[Ax,y]|^2 + [Bx,x][By,y] - |[Bx,y]|^2 \\ &+ 2\det\begin{bmatrix} [Bx,x]^{1/2}[By,y]^{1/2} & |[Ax,y]| \\ |[Bx,y]| & [Ax,x]^{1/2}[Ay,y]^{1/2} \end{bmatrix} \end{aligned} \tag{5.5}$$

for any $x, y \in X$.

In particular,

$$\begin{aligned} &[(A+B)x,x][(A+B)y,y] - |[(A+B)x,y]|^2 \\ &\geq [Ax,x][Ay,y] - |[Ax,y]|^2 + [Bx,x][By,y] - |[Bx,y]|^2 \end{aligned} \tag{5.6}$$

for any $x, y \in X$.

The proof follows by (3.5) and (3.6) written for the sub-semi-inner products $[\cdot,\cdot]_A$ and $[\cdot,\cdot]_B$.

By making use of (3.14) we also have:

Proposition 5. *If* $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$ *with* $MB \succeq A \succeq mB$ *for some positive numbers* $M > m$, *then*

$$M^2\left[[Bx,x][By,y] - |[Bx,y]|^2\right] \geq [Ax,x][Ay,y] - |[Ax,y]|^2 \geq m^2\left[[Bx,x][By,y] - |[Bx,y]|^2\right] \geq 0 \tag{5.7}$$

for any $x, y \in X$.

Finally, on making use of Corollary 7 we have:

Proposition 6. *If* $A, B \in \mathcal{S}_{[\cdot,\cdot]}(X)$, *then*

$$\begin{aligned}
&\left([(A+B)x,x][(A+B)y,y] - |[(A+B)x,y]|^2\right)^{1/2} \\
&\geq \left([Ax,x][Ay,y] - |[Ax,y]|^2\right)^{1/2} + \left([Bx,x][By,y] - |[Bx,y]|^2\right)^{1/2}
\end{aligned} \tag{5.8}$$

for any $x, y \in X$.

6. The Case of Hilbert Spaces

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $e \in H$ with $\|e\| = 1$. If we take $[\cdot,\cdot] = \langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]_e = \langle\cdot,e\rangle\langle e,\cdot\rangle$ then we observe that

$$(x,y)_e := [x,y] - [x,y]_e = \langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle,\ x,\ y \in H$$

is linear in the first variable and anti-linear in the second and, by Schwarz's inequality in the Hilbert space $(H, \langle\cdot,\cdot\rangle)$,

$$(x,x)_e = [x,x] - [x,x]_e = \|x\|^2 - |\langle x,e\rangle|^2 \geq 0 \text{ for any } x \in H.$$

Therefore $(\cdot,\cdot)_e$ is a nonnegative Hermitian from on the complex linear space H and thus satisfy the Schwarz inequality

$$|(x,y)_e|^2 \leq (x,x)_e\,(y,y)_e \text{ for any } x, y \in H.$$

Using the terminology introduced above, we then have $[\cdot,\cdot] \succeq [\cdot,\cdot]_e$ and by Corollary 2 and 2 we get

$$\begin{aligned}
&\left(\|x\|^2 + |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 + |\langle y,e\rangle|^2\right)^{1/2} - |\langle x,y\rangle + \langle x,e\rangle\langle e,y\rangle| \\
&\geq \|x\|\,\|y\| - |\langle x,y\rangle| \\
&\geq \left(\|x\|^2 - |\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2 - |\langle y,e\rangle|^2\right)^{1/2} - |\langle x,y\rangle - \langle x,e\rangle\langle e,y\rangle| \\
&\geq 0
\end{aligned} \tag{6.1}$$

for any $x,\ y \in H$.

Using Corollary 4 we also have

$$\begin{aligned}(6.2)\qquad &\left(\|x\|^2+|\langle x,e\rangle|^2\right)\left(\|y\|^2+|\langle y,e\rangle|^2\right)-|\langle x,y\rangle+\langle x,e\rangle\langle e,y\rangle|^2\\ &-\|x\|^2\|y\|^2+|\langle x,y\rangle|^2\\ &\geq\left(\det\begin{bmatrix}\|x\| & |\langle x,e\rangle|\\ \|y\| & |\langle y,e\rangle|\end{bmatrix}\right)^2\geq 0\end{aligned}$$

and

$$\begin{aligned}(6.3)\qquad &\left(\|x\|^2+|\langle x,e\rangle|^2\right)\left(\|y\|^2+|\langle y,e\rangle|^2\right)-|\langle x,y\rangle+\langle x,e\rangle\langle e,y\rangle|^2\\ &\geq 2\left(\|x\|\|y\|-|\langle x,y\rangle|\right)\left[|\langle x,e\rangle||\langle y,e\rangle|+\frac{1}{2}\left(\|x\|\|y\|+|\langle x,y\rangle|\right)\right]\\ &\geq 0,\end{aligned}$$

for any x, $y\in H$.

From Corollary 5 we have (see also [3])

$$(6.4)\qquad \|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq\left(\det\begin{bmatrix}|\langle x,e\rangle| & \left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\\ |\langle y,e\rangle| & \left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}\end{bmatrix}\right)^2\geq 0$$

and

$$\begin{aligned}(6.5)\qquad &\|x\|^2\|y\|^2-|\langle x,y\rangle|^2\geq 2|\langle x,e\rangle||\langle y,e\rangle|\\ &\times\left[\left(\|x\|^2-|\langle x,e\rangle|^2\right)^{1/2}\left(\|y\|^2-|\langle y,e\rangle|^2\right)^{1/2}-|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\right]\geq 0\end{aligned}$$

for any x, $y\in H$.

We recall the selfadjoint operator $P:H\to H$ is called nonnegative if $\langle Px,x\rangle\geq 0$ for any $x\in H$. If A, B are nonnegative operators with $A\geq B$, namely $A-B\geq 0$, then all inequalities in previous section hold with the inner product $\langle\cdot,\cdot\rangle$ instead of the $[\cdot,\cdot]$ the s-L-G-s.i.p. that generates the norm $\|\cdot\|$ and A, $B\geq 0$. The details are omitted.

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