Received 12/11/17

Two Mappings in Connection to Fejér Inequality with Applications

M. Rostamian Delavar[1]*, S. S. Dragomir[2]

[1] *Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran*

[2] *Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia*

m.rostamian@ub.ac.ir , sever.dragomir@vu.edu.au

Abstract

By the use of two h-convex mappings H_g and F_g, some results and refinements related to the h-convex version of Fejér inequality are established. Also some applications for obtained inequalities in connection with Beta function of Euler are given. [1]

1 Introduction

The following integral inequalities

$$f\Big(\frac{a+b}{2}\Big)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx, \qquad (1)$$

where $f:[a,b]\to\mathbb{R}$ is convex and $g:[a,b]\to[0,+\infty)$ is integrable and symmetric to $x=\frac{a+b}{2}\Big(g(x)=g(a+b-x),\forall x\in[a,b]\Big)$, known in the literature as Fejér inequality, has been proved in 1906 by L. Fejér [8].

In 2006, the concept of h-convex functions related to the nonnegative real functions has been introduced in [16] by S. Varošanec, although it was not a complete generalization of the concept of convexity. The class of h-convex functions is including a large class of nonnegative functions such

*Corresponding author.

[1] *Keywords and Phrases:* H-convex function, Fejér inequality.
2010 MSC: 26A51, 26D15, 52A01.

as nonnegative convex functions, Godunova-Levin functions [9], s-convex functions in the second sense [2] and P-functions [7].

Definition 1.1. *[16] Let $h : [0,1] \to \mathbb{R}^+$ be a function such that $h \not\equiv 0$. We say that $f : I \to \mathbb{R}^+$ is a h-convex function, if for all $x, y \in I$, $\lambda \in [0,1]$ we have*

$$f\big(\lambda x + (1-\lambda)y\big) \leq h(\lambda)f(x) + h(1-\lambda)f(y). \tag{2}$$

Also the function h is said to be supermultiplicative if

$$h(xy) \geq h(x)h(y),$$

for all $x, y \in [0,1]$.

The Fejér inequality related to h-convex functions has been introduced in [1] by M. Bombardelli et al. as the following without the assumption that h is nonnegative.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be h-convex, $w : [a,b] \to \mathbb{R}$, $w \geq 0$, symmetric with respect to $\frac{a+b}{2}$ with nonzero integral . Then*

$$\begin{aligned} &\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \int_a^b w(t)dt \leq \int_a^b f(t)w(t)dt \\ &\leq (b-a)[f(a)+f(b)] \int_0^1 h(t)w\big(ta+(1-t)b\big)dt. \end{aligned} \tag{3}$$

For other inequalities in connection to Fejér inequality see [1, 6, 10, 11, 13–15] and references therein.

In this paper, by the use of two h-convex mappings H_g (4) and F_g (13), we establish some inequalities and refinements related to the left part of (3). Also some applications for obtained results in connection with Beta function of Euler are given.

2 Main Results

2.1 The mapping H_g

The mapping $H_g : [0,1] \to \mathbb{R}$ defined by

$$H_g(t) := \int_a^b f\left(tu + (1-t)\frac{a+b}{2}\right) g(u)du, \tag{4}$$

has been introduced in [6] and some basic properties and applications related to the Fejér inequality in convex version have been obtained where symmetric function g enjoyed the density property on $[a,b]$, i.e.

$$\int_a^b g(u)du = 1.$$

This mapping reduces to $H(t)$ in the classical case if we consider $g(u) = \frac{1}{b-a}$ (see [5]).
The following theorem is h-convex version of Theorem 84 in [6] without density condition for g.

Theorem 2.1. *If $f : [a,b] \to \mathbb{R}$ is a h-convex function with $h(\frac{1}{2}) > 0$ and $g : [a,b] \to [0,\infty)$ is a symmetric function, then:*

(i) *H_g is h-convex on $[0,1]$.*
(ii) *For $t = 0$ and $t = 1$,*

$$H_g(0) = f\Big(\frac{a+b}{2}\Big)\int_a^b g(u)du \quad \text{and} \quad H_g(1) = \int_a^b f(u)g(u)du.$$

(iii) *For any $t \in (0,1]$,*

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big)\int_a^b g(u)du \leq H_g(t), \tag{5}$$

and for any $t \in (0,1)$,

$$H_g(t) \leq \Big[h(t) + 2h(\frac{1}{2})h(1-t)\Big]\int_a^b f(u)g(u)du. \tag{6}$$

(iv) *There exist bounds,*

$$\inf_{t\in[0,1]} H_g(t) \geq \min\Big\{\frac{1}{2h(\frac{1}{2})}, 1\Big\} H_g(0),$$

and

$$\sup_{t\in[0,1]} H_g(t) \leq \max\Big\{\sup_{t\in[0,1)}\big[h(t) + 2h(\frac{1}{2})h(1-t)\big], 1\Big\} H_g(1).$$

(v) *If h is nonnegative and supermultiplicative, then for any $0 < t_1 < t_2 < 1$ with $h(t_2) \neq 0$ we have*

$$H_g(t_1) \leq \alpha H_g(t_2),$$

where $\alpha = \frac{2h(\frac{1}{2})h(t_2-t_1)+h(t_1)}{h(t_2)}$.

Proof. (i) It follows from h-convexity of f that

$$H_g(\alpha t_1+\beta t_2)=\int_a^b f\left(\left[\alpha t_1+\beta t_2\right]u+\left[1-\alpha t_1-\beta t_2\right]\frac{a+b}{2}\right)g(u)du$$
$$=\int_a^b f\left(\alpha\left[t_1u+(1-t_1)\frac{a+b}{2}\right]+\beta\left[t_2u+(1-t_2)\frac{a+b}{2}\right]\right)g(u)du$$
$$\le h(\alpha)\int_a^b f\left(t_1u+(1-t_1)\frac{a+b}{2}\right)g(u)du$$
$$+h(\beta)\int_a^b f\left(t_2u+(1-t_2)\frac{a+b}{2}\right)g(u)du=h(\alpha)H_g(t_1)+h(\beta)H_g(t_2),$$

provided that $\alpha+\beta=1$.
(ii) It is obvious.
(iii) For inequality (5), consider the change of variable $x=tu+(1-t)\frac{a+b}{2}$ $(t>0)$ in (4). Then

$$H_g(t)=\frac{1}{t}\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(x)g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right)dx, \tag{7}$$

where

$$t=\frac{\left(tb+(1-t)\frac{a+b}{2}\right)-\left(ta+(1-t)\frac{a+b}{2}\right)}{b-a}.$$

On the other hand since g is symmetric to $\frac{a+b}{2}$ and

$$\frac{a+b}{2}=\frac{\left(ta+(1-t)\frac{a+b}{2}\right)+\left(tb+(1-t)\frac{a+b}{2}\right)}{2},$$

then g remains symmetric on interval $\left[ta+(1-t)\frac{a+b}{2},tb+(1-t)\frac{a+b}{2}\right]$ and so from Theorem 5 in [1] we have

$$\begin{aligned}
&\frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right)dx\\
&=\frac{1}{2h(\frac{1}{2})}f\left(\frac{ta+(1-t)\frac{a+b}{2}+tb+(1-t)\frac{a+b}{2}}{2}\right)\\
&\times\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right)dx\\
&\le\int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} f(x)g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right)dx.
\end{aligned} \tag{8}$$

The relations (7) and (8) imply that

$$H_g(t) \geq \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \frac{1}{t} \int_{ta+(1-t)\frac{a+b}{2}}^{tb+(1-t)\frac{a+b}{2}} g\left(\frac{x+(t-1)\frac{a+b}{2}}{t}\right) dx. \tag{9}$$

Now using the change of variable $u = \frac{x+(t-1)\frac{a+b}{2}}{t}$ in (9) we get desired inequality:

$$H_g(t) \geq \frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(u)du.$$

The case that $t = 1$, follows from inequality

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{a+b}{2}\right) \int_a^b g(u)du \leq \int_a^b f(u)g(u)du,$$

obtained from Theorem 5 in [1].
For inequality (6), using the h-convexity of f we have

$$H_g(t) \leq h(t) \int_a^b f(u)g(u)du + h(1-t) f\left(\frac{a+b}{2}\right) \int_a^b g(u)du. \tag{10}$$

Now from Theorem 5 in [1] and inequality (10) we get

$$\begin{aligned} H_g(t) &\leq h(t) \int_a^b f(u)g(u)du + h(1-t)2h(\frac{1}{2}) \int_a^b f(u)g(u)du \\ &= \left[h(t) + h(1-t)2h(\frac{1}{2})\right] \int_a^b f(u)g(u)du. \end{aligned}$$

(iv) It is a consequence of (iii).
(v) According to Proposition 16 in [16], assertions (i) and (iii), if we Consider $0 < t_1 < t_2 < 1$ and $h(t_2) \neq 0$ then

$$\begin{aligned} h(t_2)H_g(t_1) &\leq h(t_2 - t_1)H_g(0) + h(t_1)H_g(t_2) \\ &\leq h(\frac{1}{2})h(t_2 - t_1)H_g(t_2) + h(t_1)H_g(t_2) \\ &= [2h(\frac{1}{2})h(t_2 - t_1) + h(t_1)]H_g(t_2). \end{aligned}$$

Then

$$H_g(t_1) \leq \frac{2h(\frac{1}{2})h(t_2 - t_1) + h(t_1)}{h(t_2)} H_g(t_2).$$

□

If in Theorem 2.1, we consider $h(t)=t$ and $g(u)=\frac{1}{b-a}$ for $a<b$ we recapture the following result.

Corollary 2.1. *(Theorem 71 in [6])(see also [3, 5])*
For a given convex mapping $f:[a,b]\to\mathbb{R}$, let $H:[0,1]\to\mathbb{R}$ be defined by

$$H(t):=\frac{1}{b-a}\int_a^b f\Big(tu+(1-t)\frac{a+b}{2}\Big)du.$$

Then
(i) *H is convex on [0, 1].*
(ii) *One has the bounds:*

$$\inf_{t\in[0,1]} H(t)=H(0)=f\Big(\frac{a+b}{2}\Big),$$

and

$$\sup_{t\in[0,1]} H(t)=H(1)=\frac{1}{b-a}\int_a^b f(u)du.$$

(iii) *H increases monotonically on $[0,1]$.*

Corollary 2.2. *In Theorem 2.1, for $0\le a\le b$ consider*

$$\begin{cases} f(u)=u^r, & r\in(-\infty,-1)\cup(-1,0]\cup[1,\infty); \\ h(t)=t^s, & s\le 1; \\ g\equiv 1. \end{cases}$$

From Example 7 in [16], f is h-convex and then from inequalities (5) and (6) we have

$$\begin{aligned} &2^{s-1}\Big(\frac{a+b}{2}\Big)^r(b-a) \\ &\le \frac{1}{t(r+1)}\Bigg[\Big(\frac{(1-t)a+(1+t)b}{2}\Big)^{r+1}-\Big(\frac{(1+t)a+(1-t)b}{2}\Big)^{r+1}\Bigg] \\ &\le \big[t^s+2^{1-s}(1-t)^s\big]\Big(\frac{b^{r+1}-a^{r+1}}{r+1}\Big), \end{aligned} \tag{11}$$

for all $t\in(0,1]$. In more special case if we consider

$$\begin{cases} f(u)=u^r, & r\in[1,\infty); \\ h(t)=t, & \\ g\equiv 1, & \end{cases}$$

then we get the following inequalities obtained in [5].

$$\begin{aligned}
&\left(\frac{a+b}{2}\right)^r (b-a) \qquad (12)\\
&\leq \frac{1}{t(r+1)}\left[\left(\frac{a+b}{2}-t\Big(\frac{b-a}{2}\Big)\right)^{r+1}-\left(\frac{a+b}{2}-t\Big(\frac{b-a}{2}\Big)\right)^{r+1}\right]\\
&\leq \frac{b^{r+1}-a^{r+1}}{r+1},
\end{aligned}$$

for all $t \in (0,1]$.

Remark 1. *Assertion (iii) in Theorem 2.1 can be stated as*

$$\frac{1}{h(\frac{1}{2})}H_g(0) \leq H_g(t) \leq [h(t)+2h(\frac{1}{2})h(1-t)]H_g(1)$$

for all $t \in (0,1)$ *which gives a refinement for the left part of (3). Also assertions (i) and (iii) of Theorem 2.1 together give generalized form of Theorem 12 in [16] for* $t \in (0,1)$ *in general case.*

2.2 The mapping F_g

Now we consider the second mapping $F_g : [0,1] \to \mathbb{R}$ given by

$$F_g(t) := \int_a^b \int_a^b f\big(tx+(1-t)y\big)g(x)g(y)dxdy, \qquad (13)$$

which has been introduced in [6], where the function g assumed to be symmetric to $\frac{a+b}{2}$ with density property on $[a,b]$. Clearly, it reduces to F in the classical case when $g(u) = \frac{1}{b-a}$ (see [5]). The following theorem involved some results related to the mapping F_g when f is h-convex without density property for g.

Theorem 2.2. *If* $f : [a,b] \to \mathbb{R}$ *is* h*-convex with* $h(\frac{1}{2}) > 0$ *and* $g : [a,b] \to [0,\infty)$ *a symmetric function, then*

(i) F_g *is* h*-convex on* $[0,1]$.

(ii) *For any* $t \in [0,1]$ *we have*

$$F_g(t) = F_g(1-t).$$

Specially

$$\begin{aligned}
F_g(0) = F_g(1) &= \int_a^b \int_a^b f(y)g(y)g(x)dxdy\\
&= \int_a^b \int_a^b f(x)g(x)g(y)dxdy.
\end{aligned}$$

(iii) *For any* $t \in (0,1)$,

$$\frac{1}{2h(\frac{1}{2})} F_g\big(\frac{1}{2}\big) \leq F_g(t) \leq \big[h(t) + h(1-t)\big] F_g(0) \tag{14}$$
$$= \big[h(t) + h(1-t)\big] F_g(1).$$

Also for $t = 0$ *and* $t = 1$,

$$\frac{1}{2h(\frac{1}{2})} F_g\big(\frac{1}{2}\big) \leq F_g(0) = F_g(1).$$

(iv) *For any* $t \in [0,1]$,

$$\frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \int_a^b \int_a^b g(x)g(y)dxdy \leq F_g(t). \tag{15}$$

(v) *If* g *has density property, then for any* $t \in [0,1]$

$$F_g(t) \geq \frac{1}{2h(\frac{1}{2})} \max\Big\{H_g(t), H_g(1-t)\Big\}. \tag{16}$$

(vi) *There exist bounds,*

$$\inf_{t\in[0,1]} F_g(t) \geq \frac{1}{2h(\frac{1}{2})} F_g\big(\frac{1}{2}\big),$$

and

$$\sup_{t\in[0,1]} F_g(t) \leq \max\Big\{\sup_{t\in(0,1)} \big[h(t) + h(1-t)\big], 1\Big\} F_g(1)$$
$$= \max\Big\{\sup_{t\in(0,1)} \big[h(t) + h(1-t)\big], 1\Big\} F_g(0).$$

Proof. (i) It follows from h-convexity of f.
(ii) It is obvious.
(iii) For any $x, y \in [a,b]$ and $t \in (0,1)$ we have

$$f\Big(\frac{x+y}{2}\Big) = f\Big(\frac{tx + (1-t)x + ty + (1-t)y}{2}\Big) \tag{17}$$
$$\leq h(\frac{1}{2})\Big[f\big(tx + (1-t)y\big) + f\big(ty + (1-t)x\big)\Big].$$

Multiplication by $g(x)g(y)$ and integration over $[a,b]\times[a,b]$ we get

$$\int_a^b\int_a^b f\left(\frac{x+y}{2}\right)g(x)g(y)dxdy \le h(\frac{1}{2})\int_a^b\int_a^b f\big(tx+(1-t)y\big)g(x)g(y)dxdy$$
$$+\,h(\frac{1}{2})\int_a^b\int_a^b f\big(ty+(1-t)x\big)g(x)g(y)dxdy = 2h(\frac{1}{2})F_g(t),$$

which proves the left side of (14).
For the right side of (14), using the h-convexity of f we have

$$F_g(t) \le \int_a^b\int_a^b \big[h(t)f(x)g(x)g(y)+h(1-t)f(y)g(y)g(x)\big]dxdy \tag{18}$$
$$= [h(t)+h(1-t)]\int_a^b\int_a^b f(x)g(y)g(x)dxdy = [h(t)+h(1-t)]F_g(0)$$
$$= [h(t)+h(1-t)]F_g(1).$$

(iv) For any $t\in(0,1]$ and constant $y\in[a,b]$ define the function

$$F_g^y(t)=\int_a^b f\big(tx+(1-t)y\big)g(x)dx.$$

Using the change of variable $u=tx+(1-t)y$ we obtain

$$F_g^y(t)=\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} f(u)g\left(\frac{u+(t-1)y}{t}\right)du. \tag{19}$$

Since g is symmetric to $\frac{a+b}{2}$, then it remains symmetric on interval $\big[ta+(1-t)y, tb+(1-t)y]$ and so from Theorem 5 in [1] we have

$$F_g^y(t) \ge \frac{1}{2h(\frac{1}{2})}f\left(\frac{tb+(1-t)y+ta+(1-t)y}{2}\right) \tag{20}$$
$$\times\,\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} g\left(\frac{u+(t-1)y}{t}\right)du.$$

Using the change of variable $x=\frac{u+(t-1)y}{t}$ in (20), for any $y\in[a,b]$ we have

$$F_g^y(t) \ge \frac{1}{2h(\frac{1}{2})}f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx. \tag{21}$$

Multiplying (21) by $g(y)$ and then integrating over $[a,b]$ with respect to y, we obtain

$$F_g(t)=\int_a^b F_g^y(t)g(y)dy \ge \frac{1}{2h(\frac{1}{2})}f\Big(\frac{a+b}{2}\Big)\int_a^b\int_a^b g(x)g(y)dxdy,$$

for any $t \in (0,1]$.
For $t = 0$, using Theorem 5 in [1] we can obtain that

$$F_g(0) = \int_a^b \int_a^b f(y)g(x)g(y)dxdy = \int_a^b \Big[\int_a^b f(y)g(y)dy\Big] g(x)dx$$
$$\geq \frac{1}{2h(\frac{1}{2})} f\Big(\frac{a+b}{2}\Big) \int_a^b \int_a^b g(x)g(y)dxdy,$$

(v) From density of g, for any $t \in (0,1]$ we have

$$\frac{1}{t}\int_{ta+(1-t)y}^{tb+(1-t)y} g\bigg(\frac{u+(t-1)y}{t}\bigg)du = \int_a^b g(x)dx = 1.$$

So from inequality (20) we get

$$F_g(t) = \int_a^b \int_a^b f\big(tx+(1-t)y\big)g(x)g(y)dxdy = \int_a^b F_g^y(t)g(y)dy$$
$$\geq \frac{1}{2h(\frac{1}{2})}\int_a^b f\bigg(t\frac{a+b}{2}+(1-t)y\bigg)g(y)dy = \frac{1}{2h(\frac{1}{2})}H_g(t).$$

In the case that $t = 0$ we have

$$F_g(0) = \int_a^b \int_a^b f(y)g(x)g(y)dxdy = \int_a^b f(y)g(y)dy$$
$$\geq \frac{1}{2h(\frac{1}{2})}\int_a^b f\Big(\frac{a+b}{2}\Big)\int_a^b g(y)dy = \frac{1}{2h(\frac{1}{2})}H_g(0).$$

Also it is not hard to see that $F_g(t)$ is symmetric to $t = \frac{1}{2}$. So from assertion (ii) we obtain

$$F_g(t) \geq \frac{1}{2h(\frac{1}{2})}\max\Big\{H_g(t), H_g(1-t)\Big\}.$$

(vi) It immediately follows from relation (14).

□

If in Theorem 2.2, we consider $h(t) = t$ and $g(u) = \frac{1}{b-a}$ for $a < b$ we recapture the following result.

Corollary 2.3. *(Theorem 74 in [6])(see also [3, 4])*
Let $f : [a,b] \to \mathbb{R}$ be a convex function and $F : [0,1] \to \mathbb{R}$,

$$F(t) := \frac{1}{(b-a)^2}\int_a^b \int_a^b f\big(tx+(1-t)y\big)dxdy.$$

Then
(i) *F is convex on* $[0,1]$.
(ii) *For any $t\in[0,1]$ we have*

$$F(t)=F(1-t).$$

(iii) *The following inequality holds:*

$$f\Big(\frac{a+b}{2}\Big)\leq F(\frac{1}{2}).$$

(iv) *For any $t\in[0,1]$,*

$$F(t)\geq H(t).$$

(v) *We have the bounds:,*

$$\inf_{t\in[0,1]}F(t)=F\Big(\frac{1}{2}\Big)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f\Big(\frac{x+y}{2}\Big)dxdy,$$

and

$$\sup_{t\in[0,1]}F(t)=F(0)=F(1)=\frac{1}{b-a}\int_a^b f(x)dx.$$

Remark 2. *Assertions $(i)-(iii)$ and (v) in Theorem 2.2 together, give generalized form of Theorem 14 and Remark 15 in [16] for $t\in[0,1]$ in general case.*

Corollary 2.4. *In Theorem 2.2, for $0\leq a<b$ consider*

$$\begin{cases} f(u)=u^r, & r\in(-\infty,-2)\cup(-2,-1)\cup(-1,0]\cup[1,\infty);\\ \\ h(t)=t^s, & s\in(0,1);\\ \\ g\equiv 1. \end{cases}$$

Then

$$\begin{aligned} &2^{s-1}\Big(\frac{a+b}{2}\Big)^r(b-a)^2 \qquad (22)\\ &\leq\frac{1}{t(1-t)(r+1)(r+2)}\\ &\times\Big[b^{r+2}-\big(tb+(1-t)a\big)^{r+2}-\big(ta+(1-t)b\big)^{r+2}+a^{r+2}\Big]\\ &\leq\big[t^s+(1-t)^s\big](b-a)\frac{b^{r+1}-a^{r+1}}{r+1}, \end{aligned}$$

for all $t \in (0,1)$. In (22), if we consider $h(t) = t$, then we get

$$\begin{aligned}
&\left(\frac{a+b}{2}\right)^r (b-a)^2 \\
&\le \frac{1}{t(1-t)(r+1)(r+2)} \\
&\times \left[b^{r+2} - \big(tb + (1-t)a\big)^{r+2} - \big(ta + (1-t)b\big)^{r+2} + a^{r+2}\right] \\
&\le (b-a)\frac{b^{r+1} - a^{r+1}}{r+1}
\end{aligned}$$

for all $t \in (0,1]$. Furthermore in point $t = \frac{1}{2}$ we have

$$\begin{aligned}
&\left(\frac{a+b}{2}\right)^r (b-a)^2 \le \frac{4}{(r+1)(r+2)}\left[b^{r+2} - 2\left(\frac{a+b}{2}\right)^{r+2} + a^{r+2}\right] \\
&\le (b-a)\frac{b^{r+1} - a^{r+1}}{r+1},
\end{aligned}$$

which was obtained in [7].

3 Applications for the Beta Function

In this section as an application we find some relations between the results obtained in Theorem 2.1 and Theorem 2.2 and the Beta function of Euler. Consider the Beta function of Euler, that is,

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad p,q > -1,$$

and

$$H_B^r(t,p) = \int_0^1 \left(tu + \frac{1-t}{2}\right)^r u^{p-1}(1-u)^{p-1}du,$$

where $t \in [0,1], p > -1, r \ge 1$. Also for all $t \in [0,1]$ define the following functions

$$\begin{cases}
f(t) = \left(tu + \frac{1-t}{2}\right)^r, & r \ge 1,\ u \ge 0; \\
h(t) = t^k, & k \le 1; \\
g(t) = t^{p-1}(1-t)^{p-1}, & p > -1.
\end{cases}$$

According to Example 7 in [16], the function f is h-convex. Also the function g is symmetric to $t=\frac{1}{2}$. Then From Theorem 2.1, the function $H_B^r(.,p)$ is h-convex on $[0,1]$ and

$$\begin{aligned}&\frac{1}{2(\frac{1}{2})^k}\Big(\frac{1}{2}\Big)^r\int_0^1 u^{p-1}(1-u)^{p-1}du \le H_B^r(t,p)\\&\le \Big[t^k+2(\frac{1}{2})^k(1-t)^k\Big]\int_0^1 u^r u^{p-1}(1-u)^{p-1}du,\end{aligned}$$

which implies that

$$2^{k-r-1}B(p,p)\le H_B^r(t,p)\le\Big[t^k+2^{1-k}(1-t)^k\Big]B(r+p,p), \tag{23}$$

for all $t\in[0,1], r\ge 1, k\le 1$ and $p>-1$.

Now define the function

$$F_B^r(t,p)=\int_0^1\int_0^1 \big(tx+(1-t)y\big)^r x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}dxdy, \tag{24}$$

where $t\in[0,1], r\ge 1$ and $p>-1$ (also see [7]).
With assumptions

$$\begin{cases} f(t)=\big(tx+(1-t)y\big)^r, & r\ge 1,\ x,y\ge 0;\\ h(t)=t^k, & k\le 1;\\ g(t)=t^{p-1}(1-t)^{p-1}, & p>-1,\end{cases}$$

for all $t\in[0,1]$, From Example 7 in [16], the function f is h-convex. Therefore from Theorem 2.2, the function $F_B^r(.,p)$ is h-convex on $[0,1]$ and symmetric to $t=\frac{1}{2}$. Also we have the following inequalities:

$$\begin{aligned}&\frac{1}{2(\frac{1}{2})^k}\int_0^1\int_0^1\Big(\frac{x+y}{2}\Big)^r x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}dxdy\le F_B^r(t,p)\\&\le\Big[h(t)+h(1-t)\Big]\int_0^1\int_0^1 x^r x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}dxdy\\&=\Big[h(t)+h(1-t)\Big]\int_0^1\int_0^1 y^r x^{p-1}y^{p-1}(1-x)^{p-1}(1-y)^{p-1}dxdy,\end{aligned}$$

which implies that

$$2^{k-r-1}B^2(p,p) \le F_B^r(t,p) \le \left[t^k + (1-t)^k\right]B(r+p,p)B(p,p), \tag{25}$$

for all $t \in [0,1], r \ge 1, k \le 1$ and $p > -1$.

Furthermore since we have

$$\int_0^1 \frac{1}{B(p,p)} t^p (1-t)^{p-1} dt = 1,$$

then if we consider $g(t) = \frac{1}{B(p,p)} t^p (1-t)^{1-p}$, from inequality (16) we get

$$F_B^r(t,p) \ge 2^{k-1} \max\left\{H_B^r(t,p), H_B^r(1-t,p)\right\}B^2(p,p).$$

Remark 3. *Inequality (23) reduces to the convex version obtained in [6], if we consider* $k = 1$*,*

$$2^{-r}B(p,p) \le H_B^r(t,p) \le B(r+p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \ge 1$.

Also the convex version of inequality (25) can be stated as the following.

$$2^{-r}B^2(p,p) \le F_B^r(t,p) \le B(r+p,p)B(p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \ge 1$.

Furthermore we have

$$F_B^r(t,p) \ge \max\left\{H_B^r(t,p), H_B^r(1-t,p)\right\}B^2(p,p),$$

for all $t \in [0,1]$, $p > -1$, $r \ge 1$.

References

[1] M. Bombardelli and S. Varošanec, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Computers and Mathematics with Applications, **58** (2009) 1869–1877.

[2] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Räumen*, Publ. Inst. Math. **23** (1978), 13–20.

[3] S. S. Dragomir, *A mapping in connection to Hadamards inequalities*, An. Öster. Akad. Wiss. Math.-Natur., (Wien), **128** (1991), 17–20.

[4] S. S. Dragomir, *On Hadamards inequalities for convex functions*, Mat. Balkanica, **6** (1992), 215–222.

[5] S. S. Dragomir, *Two Mappings in Connection to Hadamard's Inequalities*, J. Math. Anal. Appl. **167** (1992) 49–56.

[6] S. S. Dragomir and C. E. M. Pearce, *Selected topics on Hermite-Hadamard inequalities and applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://ajmaa.org/RGMIA/monographs.php/)

[7] S. S. Dragomir, J. Pečarić, L. E. Persson, *Some inequalities of Hadamard type*, Soochow J. Math. **21** (1995), 335–341.

[8] L. Fejér, *Über die fourierreihen*, II, Math. Naturwise. Anz Ungar. Akad. Wiss. **24** (1906) 369–390.

[9] E. K. Godunova, V. I. Levin, *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i neko-torye drugie vidy funkcii*, in: Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, pp. 138–142, MGPI, Moskva, 1985.

[10] B. Micherda and T. Rajba, *On Some Hermite-Hadamard-Fejér inequalities for (k,h)-convex function*, Math. Inequal. Appl. **4** (2012), 931–940.

[11] J. Park, *Inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals*, Int. J. Math. Anal. **8**(59) (2014), 2927–2937.

[12] A. W. Robert and D. E. Varbeg, *Convex functions*, Academic Press, 1973.

[13] M. Rostamian Delavar and M. De La Sen, *On Generalization of Fejér Type Inequalities*, Communications in Mathematics and Applications, **8**(1) (2017), 31–43.

[14] M. Z. Sarikaya, *On new Hermite Hadamard Fejér type integral inequalities*, Stud. Univ. Babes-Bolyai Math. 57(3) (2012), 377–386.

[15] K-L. Tseng, G-S. Yang and K-C. Hsu, *Some Inequalities for Differentiable Mappings and Applications to Fejér Inequality and Weighted Trapezoidal Formula*, Taiwanese J. Math. **15**(4) (2011), 1737–1747.

[16] S. Varošanec, *On h-convexity*, J. Math. Anal. Appl. **326** (2007), 303–311.