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CHOI-DAVIS-JENSEN'S TYPE TRACE INEQUALITIES FOR CONVEX FUNCTIONS OF SELF-ADJOINT OPERATORS IN HILBERT SPACES

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Abstract. Some Choi-Davis-Jensen's type trace inequalities for convex functions of self-adjoint operators in Hilbert spaces are proved. Applications for some convex functions are also given.

1. Introduction and Preliminaries

Let $B(\mathcal{H})$ stand for the algebra of all bounded linear operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ and $\{e_i\}_{i\in I}$ an orthonormal basis of $\mathcal{H}$.

For self-adjoint operators $A, B \in B(\mathcal{H})$ the order relation $A \le B$ means that $\langle Ax, x\rangle \le \langle Bx, x\rangle$ $(x \in H)$. In particular, if $0 \le A$, then A is called positive. If a positive operator A is invertible, then we say that it is strictly positive and write $0 < A$. Every positive operator B has a unique positive square root $B^{\frac{1}{2}}$, in particular, the absolute value of $A \in B(\mathcal{H})$ is defined to be $|A| = (A^*A)^{\frac{1}{2}}$.

Let A be a self-adjoint linear operator on a complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. The Gelfand map establishes a $*$-isometrically isomorphism Ψ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted by $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator $1_{\mathcal{H}}$ on $\mathcal{H}$ as follows: For any $f, g \in C(\mathrm{Sp}(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Psi(\alpha f + \beta g) = \alpha\Psi(f) + \beta\Psi(g)$;
(ii) $\Psi(fg) = \Psi(f)\Psi(g)$ and $\Psi(\bar{f}) = \Psi(f)^*$;
(iii) $\|\Psi(f)\| = \|f\| := \sup_{t\in \mathrm{Sp}(A)} |f(t)|$;
(iv) $\Psi(f_0) = 1_H$ and $\Psi(f_1) = A$ where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define

$$f(A) := \Psi(f), \qquad \Big(f \in C\big(\mathrm{Sp}(A)\big)\Big)$$

and we call it the continuous functional calculus for a self-adjoint operator A. If A is a self-adjoint operator and f is a real valued continuous function on $\mathrm{Sp}(A)$, then $f(t) \ge 0$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \ge 0$, a.e. $f(A)$ is a positive

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operator on $\mathcal{H}$. Moreover, if both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds:

$$f(t) \geq g(t) \text{ for any } t \in \mathrm{Sp}(A) \text{ implies that } f(A) \geq g(A) \tag{1.1}$$

in the operator order of $B(\mathcal{H})$.

Let A be an operator on a Hilbert space $\mathcal{H}$, and suppose that $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$. We define the Hilbert-Schmidt norm of A to be

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{\frac{1}{2}}. \tag{1.2}$$

Definition (1.2) is independent of the choice of basis. An operator A is a Hilbert-Schmidt operator if $\|A\|_2 < +\infty$. We denote the class of all Hilbert-Schmidt operators on $\mathcal{H}$ by $B_2(\mathcal{H})$.

For the proof of next results by the end of this section, we refer the reader to [5, Section 26.1].

Theorem 1.1. *We have*

(i) *$B_2(\mathcal{H})$ is a vector space which is invariant under taking adjoints and $\|\cdot\|_2$ is a norm on $B_2(\mathcal{H})$.*
(ii) *The Hilbert-Schmidt norm $\|\cdot\|_2$ dominates the operator norm $\|\cdot\|$*

$$\|A\| \leq \|A\|_2,$$

for all $A \in B_2(\mathcal{H})$.
(iii) *For all $A \in B_2(\mathcal{H})$ and all $B \in B(\mathcal{H})$ one has $AB \in B_2(\mathcal{H})$ and $BA \in B_2(\mathcal{H})$ with the estimates*

$$\|AB\|_2 \leq \|A\|_2\|B\|, \qquad \|BA\|_2 \leq \|B\|\|A\|_2$$

a.e. $B_2(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$.
(iv) *The vector space $B_2(\mathcal{H})$ is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I}\langle Ae_i, Be_i\rangle = \sum_{i\in I}\langle B^*Ae_i, e_i\rangle,$$

for any $A, B \in B_2(\mathcal{H})$ and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$.
(v) *$F(H)$, the space of all operators of finite rank, is a dense subspace of $B_2(\mathcal{H})$.*
(vi) *$B_2(\mathcal{H}) \subseteq K(\mathcal{H})$, where $K(\mathcal{H})$ denotes the algebra of all compact operators on H.*

If A is an operator on a Hilbert space $\mathcal{H}$, we define its trace-class norm to be $\|A\|_1 = \left\||A|^{\frac{1}{2}}\right\|_2^2$. If $\{e_i\}_{i\in I}$ is an orthonormal basis of H, then

$$\|A\|_1 = \sum_{i\in I}\langle |A|e_i, e_i\rangle.$$

If $\|A\|_1 < +\infty$, we call A a trace-class operator. The definition of $\|.\|_1$ dose not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $B_1(\mathcal{H})$ the set of all trace-class operators in $B(\mathcal{H})$

The connection between trace-class operators and Hilbert-Schmidt operators is given in the following result.

Proposition 1.2. *Let A be an operator on a Hilbert Space $\mathcal{H}$. The following conditions are equivalent:*

(i) *A is trace-class.*
(ii) *$|A|$ is trace-class.*
(iii) *$|A|^{\frac{1}{2}}$ is a Hilbert-Schmidt operator.*
(iv) *There exist Hilbert-Schmidt operators B_1 and B_2 on H such that $A = B_1 B_2$.*

There are similar results for the trace class norm as for the Hilbert-Schmidt norm.

Theorem 1.3. *We have*

(i) *$B_1(\mathcal{H})$ is a vector space which is invariant under taking adjoint, a.e. $A \in B_1(\mathcal{H})$ if and only if $A^* \in B_1(\mathcal{H})$; furthermore, for all $A \in B_1(H)$, $\|A^*\|_1 = \|A\|_1$.*
(ii) *$\|\cdot\|_1$ is a norm on $B_1(\mathcal{H})$ and the trace-class norm dominates the operator norm $\|\cdot\|$, a.e. $\|A\| \le \|A\|_1$ for all $A \in B_1(\mathcal{H})$.*
(iii) *For all $A \in B_1(\mathcal{H})$ and $B \in B(\mathcal{H})$ one has $AB \in B_1(\mathcal{H})$ and $BA \in B_1(\mathcal{H})$ with the estimates*

$$\|AB\|_1 \le \|A\|_1 \|B\|, \qquad \|BA\|_1 \le \|B\|\ \|A\|_1$$

a.e. $B_1(\mathcal{H})$ is a two-sided ideal in $B(\mathcal{H})$.

According to the definitions one has $\|A\|_2^2 = \|A^*A\|_1$. By $(iii), (ii)$ and (i) the space of trace-class operators is continuously embeded into the space of Hilbert-Schmidt operators. Let A be a trace-class operator, the trace of A is defined as

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{1.3}$$

where $\{e_i\}$ is any orthonormal basis of $\mathcal{H}$. The definition of tr is independent of the choice of orthonormal basis and the series (1.3) converges absolutely. Note that this definition coincides with the usual definition of the trace if $\mathcal{H}$ is finite-dimensional.

The function $\operatorname{tr} : B_1(\mathcal{H}) \to \mathbb{C}$, $A \mapsto \operatorname{tr}(A)$, is linear and for any $B \in B(\mathcal{H})$, we have $|\operatorname{tr}(BA)| \le \|B\| \|A\|_1$, then the function $\operatorname{tr}(A\cdot) : K(\mathcal{H}) \to \mathbb{C}$, $B \mapsto \operatorname{tr}(AB)$, is linear, bounded and $\|\operatorname{tr}(A\cdot)\| \le \|A\|_1$. We therefore have a map $B_1(\mathcal{H}) \to K(\mathcal{H})^*$, where $K(\mathcal{H})^*$ is the dual space of $K(\mathcal{H})$, is clearly linear and norm-decreasing. We call this map the canonical map from $B_1(\mathcal{H})$ to $K(\mathcal{H})^*$.

We know state some properties of the trace that are useful to us for the analysis of this paper.

Theorem 1.4. *With the above notations,*

(i) *If $A \in B_1(\mathcal{H})$, then $\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}$.*

(ii) If $A \in B_1(\mathcal{H})$ and $B \in B(\mathcal{H})$, then $AB, BA \in B_1(\mathcal{H})$ and

$$\operatorname{tr}(AB) = \operatorname{tr}(BA) \quad \textit{and} \quad |\operatorname{tr}(AB)| \leq \|A\|_1 \|B\|.$$

(iii) $\operatorname{tr}(\cdot)$ is a bounded linear functional on $B_1(\mathcal{H})$ with $\|\operatorname{tr}\| = 1$.

(iv) If $A, B \in B_2(\mathcal{H})$, then $AB, BA \in B_1(\mathcal{H})$ and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(v) The canonical map from $B_1(\mathcal{H})$ to $K(\mathcal{H})^*$, the dual space of $K(\mathcal{H})$, is an isometric linear isomorphism.

(vi) The canonical map from $B(\mathcal{H})$ to $B_1(\mathcal{H})^*$, the dual space of $B_1(\mathcal{H})$, is an isometric linear isomorphism.

Utilizing the trace notation we have

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \quad \textit{and} \quad \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}(|A|^2)$$

for any $A, B \in B_2(\mathcal{H})$.

The following Hölder's type inequality has been obtained by Ruskai in [29]:

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq [\operatorname{tr}(|A|^{1/\alpha})]^{\alpha}[\operatorname{tr}(|B|^{1/(1-\alpha)})]^{1-\alpha}$$

where $\alpha \in (0,1)$ and $A, B \in B(\mathcal{H})$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in B_1(\mathcal{H})$.
In particular, for $\alpha = \dfrac{1}{2}$ we get the Schwartz inequality

$$|\operatorname{tr}(AB)| \leq \operatorname{tr}(|AB|) \leq [\operatorname{tr}(|A|^2)]^{\frac{1}{2}}[\operatorname{tr}(|B|^2)]^{\frac{1}{2}} \tag{1.4}$$

with $A, B \in B_2(\mathcal{H})$.

If $A \geq 0$ and $P \in B_1(\mathcal{H})$ with $P \geq 0$ then

$$0 \leq \operatorname{tr}(PA) \leq \|A\|\operatorname{tr}(P). \tag{1.5}$$

This obviously implies the fact that, if A and B are self-adjoint operators with $A \leq B$ and $P \in B_1(\mathcal{H})$ with $P \geq 0$, then

$$\operatorname{tr}(PA) \leq \operatorname{tr}(PB). \tag{1.6}$$

For the theory of trace functionals and their applications the reader is referred to [31]. Some classical trace inequalities investigated in [7, 28, 34], which are continuations of [2]. For related works the reader can refer to [3, 19, 22, 23, 30].

Let $\varepsilon := \{e_i\}_{i\in I}$ be the orthonormal basis in the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ and $B \in B_2(\mathcal{H})$ a non-zero operator. We introduce the subset of indices from I as

$$I_{\varepsilon,B} := \{i \in I : Be_i \neq 0\}.$$

We observe that $I_{\varepsilon,B}$ is non-empty for any non-zero operator B and if $\ker(B) = 0$, a.e. B is injective, then $I_{\varepsilon,B} = I$. We also have for $B \in B_2(\mathcal{H})$ that

$$\operatorname{tr}(|B|^2) = \operatorname{tr}(B^*B) = \sum_{i\in I}\langle B^*Be_i, e_i\rangle = \sum_{i\in I}\|Be_i\|^2 = \sum_{i\in I_{\varepsilon,B}}\|Be_i\|^2.$$

A linear map $\Phi : \mathcal{A} \to \mathcal{B}$ between C^*-algebras is said to be positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is unital if Φ preserves the identity. The linear map Φ is called strictly positive if $\Phi(A)$ is strictly positive whenever A is strictly positive. It can be easily seen that a positive linear map Φ is strictly positive if and only if $\Phi(I) > 0$.

Since each self-adjoint operator is the difference of two positive operators with orthogonal supports, a positive map Φ carries self-adjoint operators to self-adjoint operators. If $A = B + iC$ with B and C self-adjoint in $\mathcal{A}$, we get

$$\Phi(A^*) = \Phi(B) - i\Phi(C) = \Phi(A)^*,$$

so Φ preserves adjoints, and is often referred to as a self-adjoint linear map.

If $\mathcal{K}$ and $\mathcal{H}$ are finite dimensional Hilbert spaces, then $B(\mathcal{K})$ and $B(\mathcal{H})$ with the Hilbert-Schmidt inner product $\langle A, B\rangle = \operatorname{tr}(B^*A)$ are Hilbert spaces. Thus a linear map $\Phi : B(\mathcal{K}) \to B(\mathcal{H})$ can be considers as a bounded operator between Hilbert spaces and therefore an adjoint map defined by

$$\operatorname{tr}\big(\Phi^*(B)A\big) = \operatorname{tr}\big(B\Phi(A)\big)$$

for all $A \in B(\mathcal{K})$ and $B \in B(\mathcal{H})$.

Now, for the infinite dimensional case, we express following results which is derived from [5], Sections 31.2.1 and 31.2.2.

Lemma 1.5. *If a positive map $\Phi : B_1(\mathcal{K}) \to B_1(\mathcal{H})$ satisfies*

$$\operatorname{tr}\big(\Phi(W)\big) \le 1 \tag{1.7}$$

for all $W \in B_1(\mathcal{K})$ with $W \ge 0$ and $\operatorname{tr}(W) = 1$, a.e., for all density operators on $\mathcal{K}$, then it is continuous with respect to the trace norm:

$$\|\Phi(A)\|_1 \le C\|A\|, \qquad C = \sup \operatorname{tr}\big(\Phi(W)\big) \le 1$$

where the sup *is taken over all density operators W on $\mathcal{K}$.*

The adjoint Φ^* of an operator Φ in the duality between trace-class operators and bounded linear operators (see Theorem 1.4 (vi)) is then a linear map $\Phi^* : B(\mathcal{H}) \to B(\mathcal{K})$ which is positive too (see Theorem 26·5 in [5]).

Lemma 1.6. *Let $\Phi : B_1(\mathcal{K}) \to B_1(\mathcal{H})$ be a positive linear mapping such that $\operatorname{tr}\big(\Phi(W)\big) \le 1$ for all density operators W on $\mathcal{K}$. Then its dual map Φ^* is a linear map $B(\mathcal{H}) \to B(\mathcal{K})$ which is well defined by*

$$\operatorname{tr}\big(\Phi^*(B)A\big) = \operatorname{tr}\big(B\Phi(A)\big) \tag{1.8}$$

for all $B \in B(\mathcal{H})$ and $A \in B_1(\mathcal{K})$.

Corollary 1.7. *For a positive linear mapping $\Phi : B_1(\mathcal{K}) \to B_1(\mathcal{H})$ the following statements are equivalent:*

(i) $\operatorname{tr}\big(\Phi(W)\big) \le 1$ *for all density operators W on $\mathcal{K}$.*
(ii) *Φ is continuous and $\Phi^*(I) \le I$.*

Theorem 1.8. *[First Representation Theorem of Kraus] Given an operator $\Phi : B_1(\mathcal{H}) \to B_1(\mathcal{H})$, there exists a finite or countable family $\{A_j : j \in J\}$ of bounded linear operators on $\mathcal{H}$, satisfying*

$$\sum_{j\in J_0} A_j^* A_j \le I. \tag{1.9}$$

for all finite $J_0 \subset J$, such that for every $A \in B_1(\mathcal{H})$ and every $B \in B(\mathcal{H})$ one has

$$\Phi(A) = \sum_{j \in J} A_j A A_j^* \tag{1.10}$$

respectively

$$\Phi^*(B) = \sum_{j \in J} A_i^* B A_j \tag{1.11}$$

and

$$\Phi^*(I) = \sum_{j \in J} A_j^* A_j. \tag{1.12}$$

Conversely, if a countable family $\{A_j : j \in J\}$ of bounded linear operators on $\mathcal{H}$ is given which satisfies (1.9) then equation (1.10) defines an operation Φ whose adjoint Φ^* is given by (1.11) and $\Phi^*(I)$ defines by (1.3).

For a comprehensive account on positive linear maps see [4, 5, 33].

2. Jensen's Operator Inequality

In this section, we review some results related to Jensen's operator inequality for convex functons. A continuous real function f defined on an interval J is called operator convex if

$$f\big(\lambda A + (1-\lambda)B\big) \leq \lambda f(A) + (1-\lambda) f(B)$$

for all $0 \leq \lambda \leq 1$ and all self-adjoint operators A and B with spectra in J. A function f is called operator concave if $-f$ is operator convex. The classical Jensen inequality states if f is a convex function on an interval J then for elements $x_1, \ldots, x_n \in J$, we have

$$f\left(\sum_{i=1}^{n} t_i x_i\right) \leq \sum_{i=1}^{n} t_i f(x_i) \tag{2.1}$$

where $t_1, \ldots, t_n$ are positive real number with $\sum_{i=1}^{n} t_i = 1$.

To find an operator version of Jensen's inequality, let us consider the matrices $A = \begin{pmatrix} x_1 & & O \\ & \ddots & \\ O & & x_n \end{pmatrix}$ and $x = \begin{pmatrix} \sqrt{t_1} \\ \vdots \\ \sqrt{t_n} \end{pmatrix}$. Then inequality (2.1) can be stated as

$$f\left(\langle Ax, x\rangle\right) \leq \langle f(A)x, x\rangle.$$

using this approach, Mond and Pečarić [25, 27] proved that if f is a convex function on an interval J and A is a self-adjoint operator on a Hilbert space $\mathcal{H}$ with spectrum in J, then

$$f\left(\langle Ax, x\rangle\right) \leq \langle f(A)x, x\rangle, \tag{2.2}$$

for every unit vector $x \in \mathcal{H}$. If we put $A = \begin{pmatrix} A_1 & & O \\ & \ddots & \\ O & & A_n \end{pmatrix}$ and $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in (2.2), we obtain a multiple operator version of the Jensen inequality as follows:

$$f\left(\sum_{i=1}^{n} \langle A_i x_i, x_i \rangle\right) \le \sum_{i=1}^{n} \langle f(A_i) x_i, x_i \rangle,$$

where $\sum_{i=1}^{n} \|x_i\|^2 = 1$.

Now, if A is a self-adjoint operator, then

$$|\langle Ax, x\rangle| \le \langle |A|x, x\rangle \qquad (x \in \mathcal{H}).$$

This inequality follows by Jensen's inequality for the convex function $f(t) = |t|$ defined on a closed interval containing the spectrum of A.

If $\{e_i\}_{i\in I}$ is an orthonormal basis of $\mathcal{H}$, then

$$\begin{aligned} |\mathrm{tr}(PA)| &= |\sum_{i\in I} \langle AP^{\frac{1}{2}} e_i, P^{\frac{1}{2}} e_i\rangle| \le \sum_{i\in I} |\langle AP^{\frac{1}{2}} e_i, P^{\frac{1}{2}} e_i\rangle| \\ &\le \sum_{i\in I} \langle |A| P^{\frac{1}{2}} e_i, P^{\frac{1}{2}} e_i\rangle = \mathrm{tr}(P|A|) \end{aligned} \tag{2.3}$$

for a self-adjoint operator A and $P \in B_1(\mathcal{H})$ with $P \ge 0$.

Similar to inequality (2.2), it can be proved that if $\mathcal{A}$ is a C^*-algebra and φ is a state on $\mathcal{A}$, then for every convex function f, the inequality

$$f\big(\varphi(a)\big) \le \varphi\big(f(a)\big),$$

holds, for each $a \in \mathcal{A}$. But it is not generally true when the state φ is replaced by an arbitrary positive linear map between C^*-algebras. For some inequalities for convex functions see [10, 15, 32]. For inequalities for functions of self-adjoint operators, see [11, 12, 24] and the books [13, 14, 20].

However, by using the Stinespring decomposition theorem, Davis [8] and Choi [6] showed that if Φ is a normalized positive linear map on $B(\mathcal{H})$ and if f is an operator convex function on an interval J, then so-called the Choi-Davis-Jensen inequality

$$f\big(\Phi(A)\big) \le \Phi\big(f(A)\big),$$

holds for every self-adjoint operator A on $\mathcal{H}$ whose spectrum is contained in J.

We need the following theorems in section 3.

Theorem 2.1. *[1, Theorem 2.1] Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive unital map between unital C^*-algebras $\mathcal{A}, \mathcal{B}$ and f is a convex function. If $A \in \mathcal{A}$, such that $\Phi(f(A))$ and $\Phi(A)$ commute, then*

$$f\big(\Phi(A)\big) \le \Phi\big(f(A)\big).$$

Theorem 2.2. *[21, Theorem 2.2] If f is an operator convex function on an interval J, then*

$$f\big(\Phi(A)\big) \le \Phi\big(f(A)\big),$$

for every self-adjoint operator A in a C^*-algebra $\mathcal{A}$ with spectrum in J and every positive linear map $\Phi : \mathcal{A} \to B(\mathcal{H})$ with $\mathbf{0} < \Phi(I) \le I$.

3. Choi-Davis-Jensen's Type Trace Inequalities

Dragomir in [15–18] proved Jensen's type trace inequalities for convex functions and in this section we extend the Choi-Davis-Jensen inequality for trace. Also we present some examples of interest convex functions.
We recall the gradient inequality for the convex function $f : [m, M] \to \mathbb{R}$, namely

$$f(s) - f(t) \ge \delta_f(t)(s - t) \tag{3.1}$$

for any $s, t \in [m, M]$ where $\delta_f(t) \in [f'_-(t), f'_+(t)]$, (for $t = m$ we take $\delta_f(t) = f'_+(m)$ and for $t = M$, we take $\delta_f(t) = f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.
The following result holds.

Theorem 3.1. *Let $\Phi : B_1(\mathcal{K}) \to B_1(\mathcal{H})$ be a positive linear mapping satisfy (1.7), whose adjoint is Φ^*, A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$ and $B \in B_1(\mathcal{K}) \setminus \{0\}$ is a strictly positive operator, then we have $\dfrac{\operatorname{tr}\big(B(\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)} \in [m, M]$ and the Choi-Davis-Jensen inequality*

$$f\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right) \le \frac{\operatorname{tr}\big(B\Phi^*(f(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}. \tag{3.2}$$

Proof. Let $\varepsilon := \{e_i\}_{i \in I}$ be an orthonormal basis in H. Since A is a self-adjoint operator on the Hilbert Space $\mathcal{H}$ with $\operatorname{Sp}(A) \subseteq [m, M]$, then

$$m1_{\mathcal{H}} \le A \le M1_{\mathcal{H}}$$

and we have

$$\begin{aligned} m\big\langle \big(\Phi(B)\big)^{\frac{1}{2}} e_i, \big(\Phi(B)\big)^{\frac{1}{2}} e_i \big\rangle &\le \big\langle A\big(\Phi(B)\big)^{\frac{1}{2}} e_i, \big(\Phi(B)\big)^{\frac{1}{2}} e_i \big\rangle \\ &\le M\big\langle \big(\Phi(B)\big)^{\frac{1}{2}} e_i, \big(\Phi(B)\big)^{\frac{1}{2}} e_i \big\rangle \end{aligned} \tag{3.3}$$

for any $i \in I$, which by summation (3.3) we get

$$m\operatorname{tr}\big(\Phi(B)\big) \le \operatorname{tr}\left(\big(\Phi(B)\big)^{\frac{1}{2}} A\big(\Phi(B)\big)^{\frac{1}{2}}\right) \le M\operatorname{tr}\big(\Phi(B)\big). \tag{3.4}$$

Now, since $\Phi(B) \in B_1(\mathcal{H})$ then by Proposition 1.2, Theorem 1.4 and equality (1.8) we have

$$\begin{aligned} \operatorname{tr}\left(\big(\Phi(B)\big)^{\frac{1}{2}} A\big(\Phi(B)\big)^{\frac{1}{2}}\right) &= \operatorname{tr}\big(A\Phi(B)\big) \\ &= \operatorname{tr}\big(\Phi^*(A)B\big) = \operatorname{tr}\big(B\Phi^*(A)\big) \end{aligned} \tag{3.5}$$

and by inequality (3.4) and equality (3.5) we conclude that $\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)} \in [m, M]$. Utilising the gradient inequality (3.1) we have

$$\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\left(s-\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right) \leq f(s)-f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right) \tag{3.6}$$

for any $s \in [m, M]$, where

$$\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right) \in \left[f'_{-}\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right), f'_{+}\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\right].$$

The inequality (3.6) implies in the operator order of $B(\mathcal{H})$ that

$$\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\left(A-\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right) \leq f(A)-f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)1_{\mathcal{H}}$$

which can be written as

$$\begin{aligned}\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)&\left(\langle Ay, y\rangle-\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\langle y, y\rangle\right)\\ &\leq \langle f(A)y, y\rangle - f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\langle y, y\rangle \end{aligned} \tag{3.7}$$

for any $y \in \mathcal{H}$. If we take in (3.7), $y = \big(\Phi(B)\big)^{\frac{1}{2}}e_i$ we get

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\\ &\times\left(\langle A\big(\Phi(B)\big)^{\frac{1}{2}}e_i, \big(\Phi(B)\big)^{\frac{1}{2}}e_i\rangle-\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\langle\big(\Phi(B)\big)^{\frac{1}{2}}e_i, \big(\Phi(B)\big)^{\frac{1}{2}}e_i\rangle\right)\\ &\leq \langle f(A)\big(\Phi(B)\big)^{\frac{1}{2}}e_i, \big(\Phi(B)\big)^{\frac{1}{2}}e_i\rangle - f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\langle\big(\Phi(B)\big)^{\frac{1}{2}}e_i, \big(\Phi(B)\big)^{\frac{1}{2}}e_i\rangle,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\\ &\times\left(\langle\big(\Phi(B)\big)^{\frac{1}{2}}A\big(\Phi(B)\big)^{\frac{1}{2}}e_i, e_i\rangle-\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\langle\Phi(B)e_i, e_i\rangle\right)\\ &\leq \langle\big(\Phi(B)\big)^{\frac{1}{2}}f(A)\big(\Phi(B)\big)^{\frac{1}{2}}e_i, e_i\rangle - f\left(\frac{\operatorname{tr}\big(B\Phi^{*}(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\langle\Phi(B)e_i, e_i\rangle\end{aligned} \tag{3.8}$$

for any $i \in I$. Summing in (3.8) we get

$$\begin{aligned}
&\delta_f\left(\frac{\operatorname{tr}\big(y\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\\
&\times\left(\sum_{i\in I}\langle\big(\Phi(B)\big)^{\frac{1}{2}}A\big(\Phi(B)\big)^{\frac{1}{2}}e_i,e_i\rangle-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\sum_{i\in I}\langle\Phi(B)e_i,e_i\rangle\right)\\
&\leq\sum_{i\in I}\langle\big(\Phi(B)\big)^{\frac{1}{2}}f(A)\big(\Phi(B)\big)^{\frac{1}{2}}e_i,e_i\rangle-f\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\sum_{i\in I}\langle\Phi(B)e_i,e_i\rangle. \qquad (3.9)
\end{aligned}$$

However

$$\sum_{i\in I}\langle\big(\Phi(B)\big)^{\frac{1}{2}}A\big(\Phi(B)\big)^{\frac{1}{2}}e_i,e_i\rangle=\operatorname{tr}(A\Phi(B))$$

By (1.8) and Theorem 1.3 we have

$$\operatorname{tr}\big(A\Phi(B)\big)=\operatorname{tr}\big(\Phi^*(A)B\big)=\operatorname{tr}\big(B\Phi^*(A)\big).$$

Then

$$\sum_{i\in I}\langle\big(\Phi(B)\big)^{\frac{1}{2}}A\big(\Phi(B)\big)^{\frac{1}{2}}e_i,e_i\rangle=\operatorname{tr}\big(B\Phi^*(A)\big).$$

Similarly by (1.8) and Theorem 1.3 we get

$$\begin{aligned}
\sum_{i\in I}\langle\big(\Phi(B)\big)^{\frac{1}{2}}f(A)\big(\Phi(B)\big)^{\frac{1}{2}}e_i,e_i\rangle&=\operatorname{tr}\big(f(A)\Phi(B)\big)\\
&=\operatorname{tr}\Big(B\Phi^*\big(f(A)\big)\Big).
\end{aligned}$$

By (3.9) we have

$$\begin{aligned}
&\delta_f\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\left(\operatorname{tr}\big(B\Phi^*(A)\big)-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\big(\Phi(B)\big)\right)\\
&\qquad\leq\operatorname{tr}\Big(B\Phi^*\big(f(A)\big)\Big)-f\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)\operatorname{tr}\big(\Phi(B)\big)
\end{aligned}$$

and the inequality (3.2) is thus proved. □

Lemma 3.2. *Let T be a self-adjoint operator such that $\alpha 1_{\mathcal{H}}\leq T\leq\beta 1_{\mathcal{H}}$ for some real constant $\beta\geq\alpha$ and assume that $\Phi:B_1(\mathcal{K})\to B_1(\mathcal{H})$ be a positive linear mapping satisfy (1.7), whose adjoint is Φ^*. Then for any strictly positive operator $S\in B_1(\mathcal{K})\setminus\{0\}$ we have*

$$\begin{aligned}
0&\leq\frac{\operatorname{tr}\big(S\Phi^*(T^2)\big)}{\operatorname{tr}\big(\Phi(S)\big)}-\left(\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\right)^2\\
&\leq\frac{1}{2}(\beta-\alpha)\frac{1}{\operatorname{tr}\big(\Phi(S)\big)}\operatorname{tr}\left(S\Phi^*\left(\left|T-\frac{\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}.1_{\mathcal{H}}\right|\right)\right)\\
&\leq\frac{1}{2}(\beta-\alpha)\left[\frac{\operatorname{tr}\big(S\Phi^*(T^2)\big)}{\operatorname{tr}\big(\Phi(S)\big)}-\left(\frac{\operatorname{tr}\big(S\Phi^*(T)\big)}{\operatorname{tr}\big(\Phi(S)\big)}\right)^2\right]^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{1}{4}(\beta-\alpha)^2. \tag{3.10}$$

Proof. The first inequality follows from Choi-Davis Jensen's inequality (3.2) for the convex function $f(t)=t^2$. Now observe that

$$\begin{aligned}
&\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(S\Phi^*\Big(\Big[T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big]\Big[T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big]\Big)\right)\\
&\quad=\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(S\Phi^*\Big(T\Big[T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big]\Big)\right)\\
&\qquad-\frac{\beta+\alpha}{2}\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(S\Phi^*\Big(\Big[T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big]\Big)\right)\\
&\quad=\frac{\mathrm{tr}\big(S\Phi^*(T^2)\big)}{\mathrm{tr}\big(\Phi(S)\big)}-\left(\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\right)^2. \qquad (3.11)
\end{aligned}$$

Since $\mathrm{tr}\big(S\Phi^*(1_{\mathcal{H}})\big)=\mathrm{tr}\big(\Phi^*(1_{\mathcal{H}})S\big)=\mathrm{tr}\big(\Phi(S)\big)$, we have

$$\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(S\Phi^*\Big(\Big[T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big]\Big)\right)=0.$$

Now, since $\alpha 1_{\mathcal{H}}\leq T\leq \beta 1_{\mathcal{H}}$ then

$$\left|T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\right|\leq\frac{1}{2}(\beta-\alpha)\cdot 1_{\mathcal{H}},$$

which implies that

$$\left\|T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\right\|\leq\frac{1}{2}(\beta-\alpha). \tag{3.12}$$

Taking the modulus in (3.11) and using the property (2.3), we get (3.13)

$$\begin{aligned}
&\frac{\mathrm{tr}\big(S\Phi^*(T^2)\big)}{\mathrm{tr}\big(\Phi(S)\big)}-\left(\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\right)^2\\
&=\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\left|\mathrm{tr}\left(S\Phi^*\Big(\Big[T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big]\Big[T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big]\Big)\right)\right|\\
&=\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\left|\mathrm{tr}\left(\Phi(S)\Big(T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big)\Big(T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big)\right)\right|\\
&\leq\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(\Phi(S)\left|\Big(T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big)\Big(T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big)\right|\right). \qquad (3.13)
\end{aligned}$$

Put $v=T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}$ and $u=T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}$. Let $vu=w'|vu|$ be the polar decomposition of vu, where w' is a unique partial isometry on $\mathcal{H}$, and $w''={w'}^*vw'$. Then $|vu|={w'}^*vu=w''|u|$. Hence

$$|vu|^2=|u|{w''}^*w''|u|\leq|u|^2\|w''\|^2\leq|u|^2\|v\|^2,$$

so

$$|vu| \leq |u|\|v\|. \tag{3.14}$$

Now by (3.14), (3.12) and using the property (1.6), we get

$$\begin{aligned} &\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(\Phi(S)\left|\Big(T-\frac{\beta+\alpha}{2}\cdot 1_{\mathcal{H}}\Big)\Big(T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big)\right|\right) \\ &\qquad \leq \frac{1}{2}(\beta-\alpha)\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(\Phi(S)\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right) \end{aligned} \tag{3.15}$$

that proves the first part of (3.10).
Using Schwartz's inequality (1.4) we have

$$\begin{aligned} &\mathrm{tr}\left(\Phi(S)\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right) \\ &\leq \mathrm{tr}\left(\left|\Phi(S)\Big|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big|\right|\right) \\ &= \mathrm{tr}\left(\left|\big(\Phi(S)\big)^{\frac{1}{2}}\big(\Phi(S)\big)^{\frac{1}{2}}\Big|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big|\right|\right) \\ &\leq \big[\mathrm{tr}\big(\Phi(S)\big)\big]^{\frac{1}{2}}\left[\mathrm{tr}\left(\left|\big(\Phi(S)\big)^{\frac{1}{2}}\Big|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big|\right|^2\right)\right]^{\frac{1}{2}}. \end{aligned} \tag{3.16}$$

Observe that

$$\begin{aligned} &\mathrm{tr}\left(\left|\big(\Phi(S)\big)^{\frac{1}{2}}\Big|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\Big|\right|\right)^2 \\ &= \mathrm{tr}\left(\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\big(\Phi(S)\big)^{\frac{1}{2}}\big(\Phi(S)\big)^{\frac{1}{2}}\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right) \\ &= \mathrm{tr}\left(\Phi(s)\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|^2\right) \\ &= \mathrm{tr}\left(\Phi(S)\left(T^2-\frac{\mathrm{tr}\big(\Phi(S)T\big)}{\mathrm{tr}\big(\Phi(S)\big)}T-\frac{\mathrm{tr}\big(\Phi(S)T\big)}{\mathrm{tr}\big(\Phi(S)\big)}T+\left(\frac{\mathrm{tr}\Phi(S)T\big)}{\mathrm{tr}\big(\Phi(S)\big)}\right)^2\cdot 1_{\mathcal{H}}\right)\right) \\ &= \left(\frac{\mathrm{tr}\big(\Phi(S)T^2\big)}{\mathrm{tr}\big(\Phi(S)\big)}-\left(\frac{\mathrm{tr}\big(\Phi(S)T\big)}{\mathrm{tr}\big(\Phi(S)\big)}\right)^2\right)\mathrm{tr}\big(\Phi(S)\big). \end{aligned} \tag{3.17}$$

By (3.16) and (3.17) we get

$$\frac{1}{\mathrm{tr}\big(\Phi(S)\big)}\mathrm{tr}\left(S\Phi^*\left(\left|T-\frac{\mathrm{tr}\big(S\Phi^*(T)\big)}{\mathrm{tr}\big(\Phi(S)\big)}\cdot 1_{\mathcal{H}}\right|\right)\right)$$

$$\leq \left[\frac{\operatorname{tr}\left(S\Phi^*(T^2)\right)}{\operatorname{tr}\left(\Phi(S)\right)} - \left(\frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\right)^2\right]^{\frac{1}{2}} \tag{3.18}$$

and by (3.13) and (3.18) we have

$$\frac{\operatorname{tr}\left(S\Phi^*(T^2)\right)}{\operatorname{tr}\left(\Phi(S)\right)} - \left(\frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\right)^2 \leq \frac{1}{2}(\beta-\alpha)\left[\frac{\operatorname{tr}\left(S\Phi^*(T^2)\right)}{\operatorname{tr}\left(\Phi(S)\right)} - \left(\frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\right)^2\right]^{\frac{1}{2}}$$

which implies that

$$\left[\frac{\operatorname{tr}\left(S\Phi^*(T^2)\right)}{\operatorname{tr}\left(\Phi(S)\right)} - \left(\frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2}(\beta-\alpha).$$

Then by (3.18) we get

$$\begin{aligned}
&\frac{1}{\operatorname{tr}\left(\Phi(S)\right)}\operatorname{tr}\left(S\Phi^*\left(\left|T - \frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\cdot 1_{\mathcal{H}}\right|\right)\right) \\
&\qquad \leq \left[\frac{\operatorname{tr}\left(S\Phi^*(T^2)\right)}{\operatorname{tr}\left(\Phi(S)\right)} - \left(\frac{\operatorname{tr}\left(S\Phi^*(T)\right)}{\operatorname{tr}\left(\Phi(S)\right)}\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2}(\beta-\alpha)
\end{aligned}$$

which proves the last part of (3.10). □

Theorem 3.3. *Let $\Phi : B_1(\mathcal{K}) \to B_1(\mathcal{H})$ be a positive linear mapping satisfy (1.7), whose adjoint is Φ^*, A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$.*

If f is a continuously differentiable convex function on $[m, M]$ and $B \in B_1(\mathcal{K}) \setminus \{0\}$ is a strictly positive operator, then we have

$$\begin{aligned}
0 &\leq \frac{\operatorname{tr}\left(B\Phi^*\left(f(A)\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right) \\
&\leq \frac{\operatorname{tr}\left(B\Phi^*\left(f'(A)A\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\cdot\frac{\operatorname{tr}\left(B\Phi^*\left(f'(A)\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\
&=: L(\Phi, \Phi^*, f', B, A)
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}
&L(\Phi, \Phi^*, f', B, A) \\
&\leq \begin{matrix}(i)\\(ii)\end{matrix}\begin{cases}
\dfrac{1}{2}[f'(M) - f'(m)]\dfrac{\operatorname{tr}\left(B\Phi^*\left(\left|A - \frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\cdot 1_{\mathcal{H}}\right|\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\[2ex]
\dfrac{1}{2}[f'(M) - f'(m)]\left[\dfrac{\operatorname{tr}\left(B\Phi^*(A^2)\right)}{\operatorname{tr}\left(\Phi(B)\right)} - \left(\dfrac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)^2\right]^{\frac{1}{2}}
\end{cases}
\end{aligned}$$

$$\leq \begin{matrix}(iii)\\(iv)\end{matrix} \begin{cases} \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(B\Phi^*\left(\left|f'(A)-\frac{\operatorname{tr}\left(B\Phi^*(f'(A))\right)}{\operatorname{tr}\left(\Phi(B)\right)}\cdot 1_{\mathcal{H}}\right|\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)} \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(B\Phi^*\left([f'(A)]^2\right)\right)}{\operatorname{tr}\left(\Phi(B)\right)}-\left(\frac{\operatorname{tr}\left(B\Phi^*(f'(A))\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

$$\leq \frac{1}{4}[f'(M)-f'(m)](M-m). \tag{3.20}$$

Proof. By the gradient inequality we have

$$f(s)-f(t)\leq f'(s)(s-t) \tag{3.21}$$

for any $s,t\in[m,M]$.
This inequality implies in the operator order

$$f(A)-f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\cdot 1_{\mathcal{H}}\leq f'(A)\left(A-\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\cdot 1_{\mathcal{H}}\right)$$

that is equivalent to

$$\langle f(A)y,y\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\langle y,y\rangle \leq \langle f'(A)Ay,y\rangle-\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\langle f'(A)y,y\rangle \tag{3.22}$$

for any $y\in\mathcal{H}$, which is of intrest in itself as well.

Let $\varepsilon:=\{e_i\}_{i\in I}$ be an orthonormal basis in $\mathcal{H}$. If we take $y=\left(\Phi(B)\right)^{\frac{1}{2}}e_i$ in (3.22), then we get

$$\sum_{i\in I}\langle f(A)\left(\Phi(B)\right)^{\frac{1}{2}}e_i,\left(\Phi(B)\right)^{\frac{1}{2}}e_i\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}}e_i,\left(\Phi(B)\right)^{\frac{1}{2}}e_i\rangle$$
$$\leq\sum_{i\in I}\langle f'(A)A\left(\Phi(B)\right)^{\frac{1}{2}}e_i,\left(\Phi(B)\right)^{\frac{1}{2}}e_i\rangle-\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\sum_{i\in I}\langle f'(A)\left(\Phi(B)\right)^{\frac{1}{2}}e_i,\left(\Phi(B)\right)^{\frac{1}{2}}e_i\rangle$$

which is equivalent to

$$\sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}}f(A)\left(\Phi(B)\right)^{\frac{1}{2}}e_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\right)\sum_{i\in I}\langle\left(\Phi(B)\right)e_i,e_i\rangle$$
$$\leq\sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}}f'(A)A\left(\Phi(B)\right)^{\frac{1}{2}}e_i,e_i\rangle-\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\sum_{i\in I}\langle\left(\Phi(B)\right)^{\frac{1}{2}}f'(A)\left(\Phi(B)\right)^{\frac{1}{2}}e_i,e_i\rangle$$

and the inequality (3.19) is obtained.

Now, since f is continuously convex on $[m,M]$, then f' is monotonic nondecreasing on $[m,M]$ and $f'(m)\leq f'(t)\leq f'(M)$ for any $t\in[m,M]$. We also observe that

$$\frac{1}{\operatorname{tr}\left(\Phi(B)\right)}\operatorname{tr}\left(B\Phi^*\left(\left[f'(A)-\frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right]\left[A-\frac{\operatorname{tr}\left(B\Phi^*(A)\right)}{\operatorname{tr}\left(\Phi(B)\right)}\cdot 1_{\mathcal{H}}\right]\right)\right)$$

$$
\begin{aligned}
&= \frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(B\Phi^*\left(f'(A)\left[A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right]\right)\right)\\
&\quad-\frac{f'(m)+f'(M)}{2}\frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(B\Phi^*\left[A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right]\right)\\
&= L(\Phi,\Phi^*,f',B,A).
\end{aligned}
\tag{3.23}
$$

Since

$$
\left|f'(A)-\frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right| \le \frac{1}{2}[f'(M)-f'(m)]1_{\mathcal{H}},
$$

we have

$$
\left\|f'(A)-\frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right\| \le \frac{1}{2}[f'(M)-f'(m)]. \tag{3.24}
$$

Then by taking the modulus in (3.23) and using the property (2.3), we get to the inequality (3.25).

$$
\begin{aligned}
&0 \le L(\Phi,\Phi^*,f',B,A)\\
&= \frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\left|\operatorname{tr}\left(B\Phi^*\left(\left[f'(A)-\frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right]\left[A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right]\right)\right)\right|\\
&\le \frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(\Phi(B)\left|\left(f'(A)-\frac{f'(m)+f'(M)}{2}\cdot 1_{\mathcal{H}}\right)\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right)\right|\right)\\
&\le \frac{1}{2}[f'(M)-f'(m)]\frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(B\Phi^*\left(\left|A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right|\right)\right).
\end{aligned}
\tag{3.25}
$$

From, (3.14) and (3.24), we have the part (i) of (3.20). Since A is a self-adjoint operator on the Hilbert space $\mathcal{H}$ with $\operatorname{Sp}(A)\subseteq[m,M]$ then $m1_{\mathcal{H}}\le A\le M1_{\mathcal{H}}$. Now if we apply Lemma 3.2 we have

$$
\begin{aligned}
&\frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(B\Phi^*\left(\left|A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\right|\right)\right)\\
&\le \left[\frac{\operatorname{tr}\big(B\Phi^*(A^2)\big)}{\operatorname{tr}\big(\Phi(B)\big)}-\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)^2\right]^{\frac{1}{2}}\\
&\le \frac{1}{2}(M-m).
\end{aligned}
\tag{3.26}
$$

By applying (3.25) and (3.26) we get the part (ii) of (3.20). We observe that $L(\Phi,\Phi^*,f',B,A)$ can be represented as

$$
\begin{aligned}
&L(\Phi,\Phi^*,f',B,A)\\
&= \frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\Big(B\Phi^*\Big(\Big[f'(A)-\frac{\operatorname{tr}\Big(B\Phi^*\big(f'(A)\big)\Big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_{\mathcal{H}}\Big]\Big[A-\frac{m+M}{2}1_{\mathcal{H}}\Big]\Big)\Big).
\end{aligned}
$$

Applying a similar argument as above for this representation, we get parts (iii) and (iv) of (3.20). $\square$

Theorem 3.4. *Let $\Phi : B_1(K) \to B_1(\mathcal{H})$ be a positive linear mapping satisfy (1.7), whose adjoint is Φ^*, A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $B \in B_1(K) \setminus \{0\}$ is a strictly positive operator, then we have*

$$\begin{aligned}
0 &\leq \frac{\mathrm{tr}\Big(B\Phi^*\big(f(A)\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} - f\left(\frac{\mathrm{tr}\Big(B\big(\Phi^*(A)\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)}\right) \\
&\leq \frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)A\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot \frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} \\
&\leq \inf_{\lambda\in\mathbb{C}} \|f'(A) - \lambda \cdot 1_H\| \frac{1}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(\left|\left(A - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right)\Phi(B)\right|\right) \\
&\leq \inf_{\lambda\in\mathbb{C}} \|f'(A) - \lambda \cdot 1_H\| \left[\frac{\mathrm{tr}\big(B\Phi^*(A^2)\big)}{\mathrm{tr}\big(\Phi(B)\big)} - \left(\frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)}\right)^2\right]^{\frac{1}{2}}
\end{aligned} \tag{3.27}$$

where $\|\cdot\|$ is the operator norm.

Proof. We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\begin{aligned}
&\frac{1}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(B\Phi^*\left(\Big[f'(A) - \lambda \cdot 1_H\Big]\left[A - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right]\right)\right) \\
&\quad= \frac{1}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(B\Phi^*\left(f'(A)\left[A - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right]\right)\right) \\
&\qquad - \frac{\lambda}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(B\Phi^*\left(A - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right)\right) \\
&\quad= \frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)A\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot \frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)}.
\end{aligned} \tag{3.28}$$

Taking the modulus in (3.28), utilizing the equality (1.8) and using the Theorem 1.4, part (ii) we get the inequality (3.20) for any $\lambda \in \mathbb{C}$. We also have

$$\begin{aligned}
&\frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)A\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot \frac{\mathrm{tr}\Big(B\Phi^*\big(f'(A)\big)\Big)}{\mathrm{tr}\big(\Phi(B)\big)} \\
&\quad= \frac{1}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(\Phi(B)\big(f'(A) - \lambda \cdot 1_H\big)\left(A - \frac{\mathrm{tr}\big(B\Phi^*(A)\big)}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right)\right) \\
&\quad= \frac{1}{\mathrm{tr}\big(\Phi(B)\big)} \mathrm{tr}\left(\big(f'(A) - \lambda \cdot 1_H\big)\left[\left(A - \frac{\mathrm{tr}(B\Phi^*(A))}{\mathrm{tr}\big(\Phi(B)\big)} \cdot 1_H\right)\Phi(B)\right]\right)
\end{aligned}$$

$$\leq \|f'(A)-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\Phi(B)\right|\right). \tag{3.29}$$

Utilizing Schwarz's inequality (1.4) we have

$$\begin{aligned}
&\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\Phi(B)\right|\right)\\
&\quad=\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\big(\Phi(B)\big)^{\frac{1}{2}}\big(\Phi(B)\big)^{\frac{1}{2}}\right|\right)\\
&\quad\leq\left[\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\big(\Phi(B)\big)^{\frac{1}{2}}\right|^2\right)\right]^{\frac{1}{2}}\big[\operatorname{tr}\big(\Phi(B)\big)\big]^{\frac{1}{2}}.
\end{aligned}\tag{3.30}$$

Observe that

$$\begin{aligned}
&\operatorname{tr}\left(\left(\left|A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\big(\Phi(B)\big)^{\frac{1}{2}}\right|^2\right)\\
&\quad=\operatorname{tr}\left(\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)^2\Phi(B)\right)\\
&\quad=\left[\frac{\operatorname{tr}\big(B\Phi^*(A^2)\big)}{\operatorname{tr}\big(\Phi(B)\big)}-\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)^2\right]\operatorname{tr}\big(\Phi(B)\big)
\end{aligned}\tag{3.31}$$

(3.30) and (3.31) imply that

$$\begin{aligned}
&\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\Phi(B)\right|\right)\\
&\quad\leq\left[\frac{\operatorname{tr}\big(B\Phi^*(A^2)\big)}{\operatorname{tr}\big(\Phi(B)\big)}-\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)^2\right]^{\frac{1}{2}}\operatorname{tr}\big(\Phi(B)\big).
\end{aligned}\tag{3.32}$$

Using the inequalities (3.29) and (3.32) we have

$$\begin{aligned}
&\frac{\operatorname{tr}\Big(B\Phi^*\big(f'(A)A\big)\Big)}{\operatorname{tr}\big(\Phi(B)\big)}-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot\frac{\operatorname{tr}\Big(B\Phi^*\big(f'(A)\big)\Big)}{\operatorname{tr}\big(\Phi(B)\big)}\\
&\quad\leq\|f'(A)-\lambda\cdot 1_H\|\frac{1}{\operatorname{tr}\big(\Phi(B)\big)}\operatorname{tr}\left(\left|\left(A-\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\cdot 1_H\right)\Phi(B)\right|\right)\\
&\quad\leq\|f'(A)-\lambda\cdot 1_H\|\left[\frac{\operatorname{tr}\big(B\Phi^*(A^2)\big)}{\operatorname{tr}\big(\Phi(B)\big)}-\left(\frac{\operatorname{tr}\big(B\Phi^*(A)\big)}{\operatorname{tr}\big(\Phi(B)\big)}\right)^2\right]^{\frac{1}{2}}
\end{aligned}\tag{3.33}$$

for any $\lambda\in\mathbb{C}$.

Taking the infimum over $\lambda\in\mathbb{C}$ in (3.33) we get the inequality (3.27). $\square$

4. Some examples

In this sction we give some examples of convex functions which satisfy in requierments of Theorem 3.3.

Example 4.1. Consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex and for $r\in(0,1)$, f is concave.

Let $r\geq 1$ and A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A)\subseteq[m,M]$ for some scalars m,M with $0\leq m<M$. If $B\in B_1(\mathcal{H})\setminus\{0\}$ is a strictly positive operator and $\{A_j : j\in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (1.9), then by Theorems 3.3 and 1.8 respectively, we get

$$\begin{aligned}
0 &\leq \frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^rA_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\left(\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*AA_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right)^r\\
&\leq r\left[\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^rA_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*AA_j\Big)}{r\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^{r-1}A_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right]\\
&\leq \begin{cases}
\frac{1}{2}r(M^{r-1}-m^{r-1})\dfrac{\mathrm{tr}\left(B\sum_{j\in J}A_j^*\left|A-\frac{\mathrm{tr}\left(B\sum_{j\in J}A_j^*AA_j\right)}{\mathrm{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\cdot 1_{\mathcal{H}}\right|A_j\right)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\\[2ex]
\frac{1}{2}r(M^{r-1}-m^{r-1})\left[\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^2A_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\left(\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*AA_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right)^2\right]^{\frac{1}{2}}
\end{cases}\\
&\leq \begin{cases}
\frac{1}{2}r(M-m)\dfrac{\mathrm{tr}\left(B\sum_{j\in J}A_j^*\left|A-\frac{\mathrm{tr}\left(B\sum_{j\in J}A_j^*A^{r-1}A_j\right)}{\mathrm{tr}\left(\sum_{j\in J}A_jBA_j^*\right)}\cdot 1_{\mathcal{H}}\right|A_j\right)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\\[2ex]
\frac{1}{2}r(M-m)\left[\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^{2(r-1)}A_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\left(\frac{\mathrm{tr}\Big(B\sum_{j\in J}A_j^*A^{r-1}A_j\Big)}{\mathrm{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right)^2\right]^{\frac{1}{2}}
\end{cases}\\
&\leq \frac{1}{4}r(M^{r-1}-m^{r-1})(M-m). \qquad (4.1)
\end{aligned}$$

Example 4.2. Consider the convex function $f:(0,\infty)\to(0,\infty)$, $f(t)=-\ln t$ and let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\mathrm{Sp}(A)\subseteq$

$[m, M]$ for some scalars m, M with $0 < m < M$. If $B \in B_1(\mathcal{H}) \setminus \{0\}$ is a strictly positive operator and $\{A_j : j \in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (1.9), then by Theorems 3.3 and 1.8, respectively we get

$$
\begin{aligned}
0 &\le \ln\left(\frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right) - \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* \ln A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\
&\le \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A^{-1} A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - 1 \\
&\le \begin{cases} \dfrac{M-m}{2mM} \dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* \left| A - \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot 1_{\mathcal{H}} \right| A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\[2ex] \dfrac{M-m}{2mM} \left[\dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A^2 A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\le \begin{cases} \dfrac{1}{2}(M-m) \dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* \left| A^{-1} - \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A^{-1} A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} a_J B A_j^*\right)} \cdot 1_{\mathcal{H}} \right| A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \\[2ex] \dfrac{1}{2}(M-m) \left[\dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A^{-2} A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \left(\dfrac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A^{-1} A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right)^2\right]^{\frac{1}{2}} \end{cases} \\
&\le \frac{(M-m)^2}{4mM}.
\end{aligned}
$$

Example 4.3. Consider the convex function $f(t) = t \ln t$ and let A be a self-adjoint operator on the Hilbert space $\mathcal{H}$ and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some m, M with $0 < m < M$. If $B \in B_1(\mathcal{H}) \setminus \{0\}$ is a strictly positive operator and $\{A_j : j \in J\}$ is a countable family of bounded linear operators on $\mathcal{H}$ which satisfies (1.9), then by Theorems 3.3 and 1.8 respectively, we have

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A \ln A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \ln\left(\frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}\right) \\
&\le \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A \ln A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} - \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* A A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)} \cdot \frac{\operatorname{tr}\left(B\sum_{j\in J} A_j^* \ln(eA) A_j\right)}{\operatorname{tr}\left(\sum_{j\in J} A_j B A_j^*\right)}
\end{aligned}
$$

$$\leq \begin{cases} \dfrac{1}{2}\ln(\dfrac{M}{m})\dfrac{\operatorname{tr}\Bigg(B\sum_{j\in J}A_j^*\left|A-\frac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*AA_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\cdot 1_{\mathcal{H}}\right|A_j\Bigg)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)} \\ \dfrac{1}{2}\ln(\dfrac{M}{m})\left[\dfrac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*A^2A_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\left(\dfrac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*AA_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

$$\leq \begin{cases} \dfrac{1}{2}(M-m)\dfrac{\operatorname{tr}\Bigg(B\sum_{j\in J}A_j^*\left|\ln(eA)-\frac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*\ln(eA)A_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\cdot 1_{\mathcal{H}}\right|A_j\Bigg)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)} \\ \dfrac{1}{2}(M-m)\left[\dfrac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*[\ln(eA)]^2A_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}-\left(\dfrac{\operatorname{tr}\Big(B\sum_{j\in J}A_j^*\ln(eA)A_j\Big)}{\operatorname{tr}\Big(\sum_{j\in J}A_jBA_j^*\Big)}\right)^2\right]^{\frac{1}{2}} \end{cases}$$

$$\leq \frac{1}{4}(M-m)\ln(\frac{M}{m}).$$

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