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HYPO-q-NORMS ON CARTESIAN PRODUCTS OF ALGEBRAS OF BOUNDED LINEAR OPERATORS ON HILBERT SPACES

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Abstract. In this paper we introduce the hypo-q-norms on a Cartesian product of algebras of bounded linear operators on Hilbert spaces. A representation of these norms in terms of inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given. Several bounds for the norms δ_p, ϑ_p and the real norms $\eta_{r,p}$ and $\theta_{r,p}$ are provided as well.

1. Introduction

In [11], the author has introduced the following norm on the Cartesian product $B^{(n)}(H) := B(H) \times \cdots \times B(H)$, where $B(H)$ denotes the Banach algebra of all bounded linear operators defined on the complex Hilbert space H :

$$\|(T_1, \ldots, T_n)\|_{n,e} := \sup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{B}_n} \|\lambda_1 T_1 + \cdots + \lambda_n T_n\|, \tag{1.1}$$

where $(T_1, \ldots, T_n) \in B^{(n)}(H)$ and $\mathbb{B}_n := \left\{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n \left| \sum_{i=1}^n |\lambda_i|^2 \le 1\right.\right\}$ is the Euclidean closed ball in $\mathbb{C}^n$. It is clear that $\|\cdot\|_{n,e}$ is a norm on $B^{(n)}(H)$ and for any $(T_1, \ldots, T_n) \in B^{(n)}(H)$ we have

$$\|(T_1, \ldots, T_n)\|_{n,e} = \|(T_1^*, \ldots, T_n^*)\|_{n,e}, \tag{1.2}$$

where T_i^* is the adjoint operator of T_i, $i \in \{1, \ldots, n\}$.

It has been shown in [11] that the following inequality holds true:

$$\frac{1}{\sqrt{n}}\left\|\sum_{j=1}^n T_j T_j^*\right\|^{\frac{1}{2}} \le \|(T_1, \ldots, T_n)\|_{n,e} \le \left\|\sum_{j=1}^n T_j T_j^*\right\|^{\frac{1}{2}} \tag{1.3}$$

for any n-tuple $(T_1, \ldots, T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{\sqrt{n}}$ and 1 are best possible.

In the same paper [11] the author has introduced the *Euclidean operator radius* of an n-tuple of operators $(T_1, \ldots, T_n)$ by

$$w_{n,e}(T_1, \ldots, T_n) := \sup_{\|x\|=1}\left(\sum_{j=1}^n |\langle T_j x, x\rangle|^2\right)^{\frac{1}{2}} \tag{1.4}$$

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and proved that $w_{n,e}(\cdot)$ is a norm on $B^{(n)}(H)$ and satisfies the double inequality:

$$\frac{1}{2}\left\|(T_1,\dots,T_n)\right\|_{n,e} \le w_{n,e}(T_1,\dots,T_n) \le \left\|(T_1,\dots,T_n)\right\|_{n,e} \tag{1.5}$$

for each n-tuple $(T_1,\dots,T_n) \in B^{(n)}(H)$.

As pointed out in [11], the Euclidean numerical radius also satisfies the double inequality:

$$\frac{1}{2\sqrt{n}}\left\|\sum_{j=1}^{n} T_j T_j^*\right\|^{\frac{1}{2}} \le w_{n,e}(T_1,\dots,T_n) \le \left\|\sum_{j=1}^{n} T_j T_j^*\right\|^{\frac{1}{2}} \tag{1.6}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$ and the constants $\frac{1}{2\sqrt{n}}$ and 1 are best possible.

Now, let $(E, \|\cdot\|)$ be a normed linear space over the complex number field $\mathbb{C}$. On $\mathbb{C}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$. As an example of such norms we should mention the usual *p-norms*

$$\|\lambda\|_{n,p} := \begin{cases} \max\{|\lambda_1|,\dots,|\lambda_n|\} & \text{if } p=\infty; \\ \left(\sum_{k=1}^{n}|\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p\in[1,\infty). \end{cases}$$

The *Euclidean norm* is obtained for $p=2$, i.e.,

$$\|\lambda\|_{n,2} := \left(\sum_{k=1}^{n}|\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E\times\cdots\times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|x\|_{n,p} := \begin{cases} \max\{\|x_1\|,\dots,\|x_n\|\} & \text{if } p=\infty; \\ \left(\sum_{k=1}^{n}\|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p\in[1,\infty); \end{cases}$$

where $x=(x_1,\dots,x_n)\in E^n$.

Following the paper [4], for a given norm $\|\cdot\|_n$ on $\mathbb{C}^n$, we define the functional $\|\cdot\|_{h,n}: E^n \to [0,\infty)$ by

$$\|x\|_{h,n} := \sup_{\|\lambda\|_n\le 1}\left\|\sum_{j=1}^{n}\lambda_j x_j\right\|, \tag{1.7}$$

where $x=(x_1,\dots,x_n)\in E^n$ and $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{C}^n$.

It is easy to see that [4]:

(i) $\|x\|_{h,n}\ge 0$ for any $x\in E^n$;
(ii) $\|x+y\|_{h,n}\le\|x\|_{h,n}+\|y\|_{h,n}$ for any x, $y\in E^n$;
(iii) $\|\alpha x\|_{h,n}=|\alpha|\,\|x\|_{h,n}$ for each $\alpha\in\mathbb{C}$ and $x\in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|x\|_{h,n}=0$ if and only if $\sum_{j=1}^{n}\lambda_j x_j=0$ for any $(\lambda_1,\dots,\lambda_n)\in B(\|\cdot\|_n)$. If there exists $\lambda_1^0,\dots,\lambda_n^0\neq 0$ such that $\left(\lambda_1^0,0,\dots,0\right)$, $\left(0,\lambda_2^0,\dots,0\right),\dots,\left(0,0,\dots,\lambda_n^0\right)\in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If $p \in [1, \infty]$ and we consider the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{C}^n$, then we can define the following *hypo-q-norms* on E^n :

$$\|x\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|, \tag{1.8}$$

with $q \in [1, \infty]$. If $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$, we have the *hypo-Euclidean norm* on E^n, i.e.,

$$\|x\|_{h,n,e} := \sup_{\|\lambda\|_{n,2} \le 1} \left\| \sum_{j=1}^{n} \lambda_j x_j \right\|. \tag{1.9}$$

If we consider now $E = B(H)$ endowed with the *operator norm* $\|\cdot\|$, then we can obtain the following *hypo-q-norms* on $B^{(n)}(H)$

$$\|(T_1, \dots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \text{ with } p,\ q \in [1, \infty], \tag{1.10}$$

with the convention that if $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm* $\|(\cdot, \dots, \cdot)\|_{n,e}$ defined in (1.2).

If we consider now $E = B(H)$ endowed with the *operator numerical radius* $w(\cdot)$, which is a norm on $B(H)$, then we can obtain the following *hypo-q-numerical radius* of $(T_1, \dots, T_n) \in B^{(n)}(H)$ defined by

$$w_{h,n,q}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,p} \le 1} w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \text{ with } p,\ q \in [1, \infty], \tag{1.11}$$

with the convention that if $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = 2$ we obtain the *hypo-Euclidian norm*

$$w_{h,n,e}(T_1, \dots, T_n) := \sup_{\|\lambda\|_{n,2} \le 1} w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \tag{1.12}$$

and will show further that it coincides with the Euclidean operator radius of an n-tuple of operators $(T_1, \dots, T_n)$ defined in (1.4).

Using the fundamental inequality between the operator norm and numerical radius $w(T) \le \|T\| \le 2w(T)$ for $T \in B(H)$ we have

$$w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \le \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \le 2w\left(\sum_{j=1}^{n} \lambda_j T_j \right)$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$ and any $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$. By taking the supremum over λ with $\|\lambda\|_{n,p} \le 1$ we get

$$w_{h,n,q}(T_1, \dots, T_n) \le \|(T_1, \dots, T_n)\|_{h,n,q} \le 2w_{h,n,q}(T_1, \dots, T_n) \tag{1.13}$$

with the convention that if $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

For $p = q = 2$ we recapture the inequality (1.5).

In 2012, [6] (see also [7] and [8]) the author have introduced the concept of *s-q-numerical radius* of an n-tuple of operators $(T_1, \ldots, T_n)$ for $q \geq 1$ as

$$w_{s,q}(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{\frac{1}{q}} \tag{1.14}$$

and established various inequalities of interest. For more recent results see also [10] and [12].

In the same paper [6] we also introduced the concept of *s-q-norm* of an n-tuple of operators $(T_1, \ldots, T_n)$ for $q \geq 1$ as

$$\|(T_1, \ldots, T_n)\|_{s,q} := \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{\frac{1}{q}}. \tag{1.15}$$

In [6], [7] and [8], by utilising *Kato's inequality* [9]

$$|\langle Tx, y \rangle|^2 \leq \left\langle |T|^{2\alpha} x, x \right\rangle \left\langle |T^*|^{2(1-\alpha)} y, y \right\rangle \tag{1.16}$$

for any $x, y \in H$, $\alpha \in [0,1]$, where *"absolute value"* operator of A is defined by $|A| := \sqrt{A^* A}$, the authors have obtained several inequalities for the s-q-numerical radius and s-q-norm.

In this paper we investigate the connections between these norms and establish some fundamental inequalities of interest in multivariate operator theory.

2. Representation Results

We start with the following lemma:

Lemma 1. *Let* $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$.

(i) *If* p, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,q}. \tag{2.1}$$

In particular,

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,2}. \tag{2.2}$$

(ii) *We have*

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,1} \quad \text{and} \quad \sup_{\|\alpha\|_{n,1} \leq 1} \left| \sum_{j=1}^{n} \alpha_j \beta_j \right| = \|\beta\|_{n,\infty}. \tag{2.3}$$

Proof. (i). Using Hölder's discrete inequality for p, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^{n} \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^{n} |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_{n,p}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_{n,q} \tag{2.4}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$ are n-tuples of complex numbers.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with

$$\alpha_j:=\frac{\overline{\beta_j}\left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We observe that

$$\begin{aligned}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|&=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}\left|\beta_j\right|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\beta_j\right|=\frac{\sum_{j=1}^{n}\left|\beta_j\right|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\\&=\left(\sum_{j=1}^{n}\left|\beta_j\right|^q\right)^{1/q}=\|\beta\|_{n,q}\end{aligned}$$

and

$$\begin{aligned}\|\alpha\|_{n,p}^p&=\sum_{j=1}^{n}|\alpha_j|^p=\sum_{j=1}^{n}\frac{\left|\overline{\beta_j}\left|\beta_j\right|^{q-2}\right|^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{\left(\left|\beta_j\right|^{q-1}\right)^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}\\&=\sum_{j=1}^{n}\frac{\left|\beta_j\right|^{qp-p}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{\left|\beta_j\right|^{q}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=1.\end{aligned}$$

Therefore, by (2.4) we have the representation (2.1).

(ii). Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\max_{j\in\{1,\ldots,n\}}|\alpha_j|\sum_{j=1}^{n}\left|\beta_j\right|,$$

which implies that

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_{n,1}, \tag{2.5}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with $\alpha_j:=\frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{\left|\beta_j\right|}\beta_j\right|=\sum_{j=1}^{n}\left|\beta_j\right|=\|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty}=\max_{j\in\{1,\ldots,n\}}|\alpha_j|=\max_{j\in\{1,\ldots,n\}}\left|\frac{\overline{\beta_j}}{\left|\beta_j\right|}\right|=1$$

and by (2.5) we get the first representation in (2.3).

Moreover, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \sum_{j=1}^{n}|\alpha_j| \max_{j\in\{1,\dots,n\}}\left|\beta_j\right|,$$

which implies that

$$\sup_{\|\alpha\|_{n,1}\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \le \|\beta\|_{n,\infty}, \tag{2.6}$$

where $\alpha = (\alpha_1,\dots,\alpha_n)$ and $\beta = (\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n) \ne 0$, let $j_0 \in \{1,...,n\}$ such that $\|\beta\|_\infty = \max_{j\in\{1,...,n\}}\left|\beta_j\right| = \left|\beta_{j_0}\right|$. Consider $\alpha = (\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}$ and $\alpha_j = 0$ for $j \ne j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j| = \frac{\left|\overline{\beta_{j_0}}\right|}{\left|\beta_{j_0}\right|} = 1 \text{ and } \left|\sum_{j=1}^{n}\alpha_j\beta_j\right| = \left|\frac{\overline{\beta_{j_0}}}{\left|\beta_{j_0}\right|}\beta_{j_0}\right| = \left|\beta_{j_0}\right| = \|\beta\|_{n,\infty},$$

therefore by (2.6) we obtain the second representation in (2). □

Theorem 1. *Let $(T_1,\dots,T_n) \in B^{(n)}(H)$ and $x,\ y \in H$, then for $p,\ q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$ we have*

$$\sup_{\|\alpha\|_{n,p}\le 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right\rangle\right| = \left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|^q\right)^{1/q} \tag{2.7}$$

and in particular

$$\sup_{\|\alpha\|_{n,2}\le 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right\rangle\right| = \left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|^2\right)^{1/2}. \tag{2.8}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty}\le 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right\rangle\right| = \sum_{j=1}^{n}|\langle T_jx,y\rangle| \tag{2.9}$$

and

$$\sup_{\|\alpha\|_{n,1}\le 1}\left|\left\langle\left(\sum_{j=1}^{n}\alpha_jT_j\right)x,y\right\rangle\right| = \max_{j\in\{1,\dots,n\}}\{|\langle T_jx,y\rangle|\}. \tag{2.10}$$

Proof. If we take $\beta = (\langle T_1x,y\rangle,\dots,\langle T_nx,y\rangle) \in \mathbb{C}^n$ in (2.1), then we get

$$\begin{aligned}\left(\sum_{j=1}^{n}|\langle T_jx,y\rangle|^q\right)^{1/q} &= \|\beta\|_{n,q} = \sup_{\|\alpha\|_p\le 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right| \\ &= \sup_{\|\alpha\|_{n,p}\le 1}\left|\sum_{j=1}^{n}\alpha_j\langle T_jx,y\rangle\right| = \sup_{\|\alpha\|_{n,p}\le 1}\left|\left\langle\sum_{j=1}^{n}\alpha_jT_jx,y\right\rangle\right|,\end{aligned}$$

which proves (2.7).

The equalities (2.9) and (2.10) follow by (2.3). $\square$

Corollary 1. *Let* $(T_1, \dots, T_n) \in B^{(n)}(H)$ *and* $x \in H$, *then for* p, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| = \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{1/q} \tag{2.11}$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| = \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{1/2}. \tag{2.12}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| = \sum_{j=1}^{n} |\langle T_j x, x \rangle| \tag{2.13}$$

and

$$\sup_{\|\alpha\|_{n,1} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| = \max_{j \in \{1,\dots,n\}} \{|\langle T_j x, x \rangle|\}. \tag{2.14}$$

Corollary 2. *Let* $(T_1, \dots, T_n) \in B^{(n)}(H)$ *and* $x \in H$, *then for* p, $q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} \tag{2.15}$$

and in particular

$$\sup_{\|\alpha\|_{n,2} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^2 \right)^{1/2}. \tag{2.16}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| = \sup_{\|y\|=1} \sum_{j=1}^{n} |\langle T_j x, y \rangle| \tag{2.17}$$

and

$$\sup_{\|\alpha\|_{n,1} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| = \max_{j \in \{1,\dots,n\}} \{\|T_j x\|\}. \tag{2.18}$$

Proof. By the properties of inner product, we have for any $u \in H$, $u \neq 0$ that

$$\|u\| = \sup_{\|y\|=1} |\langle u, y \rangle|.$$

Let $x \in H$, then by taking the supremum over $\|y\| = 1$ in (2.7) we get for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that

$$\begin{aligned}\sup_{\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|y\|=1} \left(\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left(\sup_{\|y\|=1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left\| \left(\sum_{j=1}^{n} \alpha_j T_j \right) x \right\|,\end{aligned}$$

which proves the equality (2.15).

The other equalities can be proved in a similar way by using Theorem 1, however the details are omitted. □

We can state and prove our main result.

Theorem 2. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$.

(i) *For* $q \ge 1$ *we have the representation for the hypo-q-norm*

$$\|(T_1, \ldots, T_n)\|_{h,n,q} = \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} = \|(T_1, \ldots, T_n)\|_{s,q} \tag{2.19}$$

and in particular

$$\|(T_1, \ldots, T_n)\|_{n,e} = \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^2 \right)^{1/2}. \tag{2.20}$$

We also have

$$\|(T_1, \ldots, T_n)\|_{h,n,\infty} = \max_{j \in \{1,\ldots,n\}} \{\|T_j\|\}. \tag{2.21}$$

(ii) *For* $q \ge 1$ *we have the representation for the hypo-q-numerical radius*

$$w_{h,n,q}(T_1, \ldots, T_n) = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{1/q} = w_{s,q}(T_1, \ldots, T_n) \tag{2.22}$$

and in particular

$$w_{n,e}(T_1, \ldots, T_n) := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^2 \right)^{\frac{1}{2}}. \tag{2.23}$$

We also have

$$w_{h,n,\infty}(T_1, \ldots, T_n) = \max_{j \in \{1,\ldots,n\}} \{w(T_j)\}. \tag{2.24}$$

Proof. (i) By using the equality (2.15) we have for $(T_1, \dots, T_n) \in B^{(n)}(H)$ that

$$\begin{aligned} \sup_{\|x\|=\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|y\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, y \rangle|^q \right)^{1/q} \right) \\ &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left(\sup_{\|x\|=1} \left\| \sum_{j=1}^{n} \alpha_j T_j x \right\| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left\| \sum_{j=1}^{n} \alpha_j T_j \right\| = \|(T_1, \dots, T_n)\|_{h,n,q}, \end{aligned}$$

which proves (2.19). The rest is obvious.

(ii) By using the equality (2.11) we have for $(T_1, \dots, T_n) \in B^{(n)}(H)$ that

$$\begin{aligned} \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle T_j x, x \rangle|^q \right)^{1/q} &= \sup_{\|x\|=1} \left(\sup_{\|\alpha\|_{n,p} \le 1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} \left(\sup_{\|x\|=1} \left| \left\langle \left(\sum_{j=1}^{n} \alpha_j T_j \right) x, x \right\rangle \right| \right) \\ &= \sup_{\|\alpha\|_{n,p} \le 1} w \left(\sum_{j=1}^{n} \alpha_j T_j \right) = w_{h,n,q}(T_1, \dots, T_n), \end{aligned}$$

which proves (2.22). The rest is obvious. □

Remark 1. *The case $q = 2$ was obtained in a different manner in* [4] *by utilising the rotation-invariant normalised positive Borel measure on the unit sphere.*

We can consider on $B^{(n)}(H)$ the following usual operator and numerical radius q-norms, for $q \ge 1$

$$\|(T_1, \dots, T_n)\|_{n,q} := \left(\sum_{j=1}^{n} \|T_j\|^q \right)^{1/q} \text{ and } w_{n,q}(T_1, \dots, T_n) := \left(\sum_{j=1}^{n} w^q(T_j) \right)^{1/q}$$

where $(T_1, \dots, T_n) \in B^{(n)}(H)$. For $q = \infty$ we put

$$\|(T_1, \dots, T_n)\|_{n,\infty} := \max_{j \in \{1,\dots,n\}} \{\|T_j\|\} \text{ and } w_{n,\infty}(T_1, \dots, T_n) := \max_{j \in \{1,\dots,n\}} \{w(T_j)\}.$$

Corollary 3. *With the assumptions of Theorem 2 we have for $q \ge 1$ that*

$$\frac{1}{n^{1/q}} \|(T_1, \dots, T_n)\|_{n,q} \le \|(T_1, \dots, T_n)\|_{h,n,q} \le \|(T_1, \dots, T_n)\|_{n,q} \tag{2.25}$$

and

$$\frac{1}{n^{1/q}} w_{n,q}(T_1, \dots, T_n) \le w_{h,n,q}(T_1, \dots, T_n) \le w_{n,q}(T_1, \dots, T_n) \tag{2.26}$$

for any $(T_1, \dots, T_n) \in B^{(n)}(H)$.

In particular, we have [4]

$$\frac{1}{\sqrt{n}}\left\|(T_1,\dots,T_n)\right\|_{n,2}\leq\left\|(T_1,\dots,T_n)\right\|_{h,n,e}\leq\left\|(T_1,\dots,T_n)\right\|_{n,2} \tag{2.27}$$

and

$$\frac{1}{\sqrt{n}}w_{n,2}\left(T_1,\dots,T_n\right)\leq w_{h,n,e}\left(T_1,\dots,T_n\right)\leq w_{n,2}\left(T_1,\dots,T_n\right) \tag{2.28}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(H)$.

Proof. Let $(T_1,\dots,T_n)\in B^{(n)}(H)$ and $x,\ y\in H$ with $\|x\|=\|y\|=1$. Then by Schwarz's inequality we have

$$\left(\sum_{j=1}^{n}\left|\langle T_jx,y\rangle\right|^q\right)^{1/q}\leq\left(\sum_{j=1}^{n}\|T_jx\|^q\|y\|^q\right)^{1/q}=\left(\sum_{j=1}^{n}\|T_jx\|^q\right)^{1/q}.$$

By the operator norm inequality we also have

$$\left(\sum_{j=1}^{n}\|T_jx\|^q\right)^{1/q}\leq\left(\sum_{j=1}^{n}\|T_j\|^q\|x\|^q\right)^{1/q}=\left\|(T_1,\dots,T_n)\right\|_{n,q}.$$

Therefore

$$\left(\sum_{j=1}^{n}\left|\langle T_jx,y\rangle\right|^q\right)^{1/q}\leq\left\|(T_1,\dots,T_n)\right\|_{n,q}$$

and by taking the supremum over $\|x\|=\|y\|=1$ we get the second inequality in (2.25).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\dots,n\}}\left\{\left|\langle T_jx,y\rangle\right|\right\}\leq\left(\sum_{j=1}^{n}\left|\langle T_jx,y\rangle\right|^q\right)^{1/q}$$

$x,\ y\in H$ with $\|x\|=\|y\|=1$.

By taking the supremum over $\|x\|=\|y\|=1$ we get

$$\sup_{\|x\|=\|y\|=1}\left(\max_{j\in\{1,\dots,n\}}\left\{\left|\langle T_jx,y\rangle\right|\right\}\right)\leq\left\|(T_1,\dots,T_n)\right\|_{h,n,q} \tag{2.29}$$

and since

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\left(\max_{j\in\{1,\dots,n\}}\left\{\left|\langle T_jx,y\rangle\right|\right\}\right)&=\max_{j\in\{1,\dots,n\}}\left\{\sup_{\|x\|=\|y\|=1}\left|\langle T_jx,y\rangle\right|\right\}\\&=\max_{j\in\{1,\dots,n\}}\left\{\|T_j\|\right\}=\left\|(T_1,\dots,T_n)\right\|_{n,\infty},\end{aligned}$$

then by (2.29) we get

$$\left\|(T_1,\dots,T_n)\right\|_{n,\infty}\leq\left\|(T_1,\dots,T_n)\right\|_{h,n,q} \tag{2.30}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(H)$.

Since

$$\|(T_1,\dots,T_n)\|_{n,q} := \left(\sum_{j=1}^{n}\|T_j\|^q\right)^{1/q} \le \left(n\,\|(T_1,\dots,T_n)\|_{n,\infty}^q\right)^{1/q} \tag{2.31}$$
$$= n^{1/q}\,\|(T_1,\dots,T_n)\|_{n,\infty}\,,$$

then by (2.30) and (2.31) we get

$$\frac{1}{n^{1/q}}\,\|(T_1,\dots,T_n)\|_{n,q} \le \|(T_1,\dots,T_n)\|_{h,n,q}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$.

The inequality (2.26) follows in a similar way and we omit the details. □

Corollary 4. *With the assumptions of Theorem 2 we have for $r \ge q \ge 1$ that*

$$\|(T_1,\dots,T_n)\|_{h,n,r} \le \|(T_1,\dots,T_n)\|_{h,n,q} \le n^{\frac{r-q}{rq}}\,\|(T_1,\dots,T_n)\|_{h,n,r} \tag{2.32}$$

and [12]

$$w_{h,n,r}(T_1,\dots,T_n) \le w_{h,n,q}(T_1,\dots,T_n) \le n^{\frac{r-q}{rq}}\,w_{h,n,r}(T_1,\dots,T_n) \tag{2.33}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1,...,n$ and $r \ge q > 0$ (see for instance [12])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \le \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \le n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{2.34}$$

Let $(T_1,\dots,T_n) \in B^{(n)}(H)$ and $x, y \in H$ with $\|x\| = \|y\| = 1$. Then by (2.34) we get

$$\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^r\right)^{1/r} \le \left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^q\right)^{1/q} \le n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n}|\langle T_j x, y\rangle|^r\right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (2.32).

The inequality (2.33) follows in a similar way and we omit the details. □

Remark 2. *For $q \ge 2$ we have by (2.32) and (2.33)*

$$\|(T_1,\dots,T_n)\|_{h,n,q} \le \|(T_1,\dots,T_n)\|_{h,n,e} \le n^{\frac{q-2}{2q}}\,\|(T_1,\dots,T_n)\|_{h,n,q} \tag{2.35}$$

and

$$w_{h,n,q}(T_1,\dots,T_n) \le w_{h,n,e}(T_1,\dots,T_n) \le n^{\frac{q-2}{2q}}\,w_{h,n,q}(T_1,\dots,T_n) \tag{2.36}$$

and for $1 \le q \le 2$ we have

$$\|(T_1,\dots,T_n)\|_{h,n,e} \le \|(T_1,\dots,T_n)\|_{h,n,q} \le n^{\frac{2-q}{2q}}\,\|(T_1,\dots,T_n)\|_{h,n,e} \tag{2.37}$$

and

$$w_{h,n,e}(T_1,\dots,T_n) \le w_{h,n,e}(T_1,\dots,T_n) \le n^{\frac{2-q}{2q}}\,w_{h,n,e}(T_1,\dots,T_n) \tag{2.38}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(H)$.

Also, if we take $q=1$ and $r\geq 1$ in (2.32) and (2.33), then we get

$$\|(T_1,\dots,T_n)\|_{h,n,r}\leq\|(T_1,\dots,T_n)\|_{h,n,1}\leq n^{\frac{r-1}{r}}\|(T_1,\dots,T_n)\|_{h,n,r}\tag{2.39}$$

and

$$w_{h,n,r}(T_1,\dots,T_n)\leq w_{h,n,1}(T_1,\dots,T_n)\leq n^{\frac{r-1}{r}}w_{h,n,r}(T_1,\dots,T_n)\tag{2.40}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(H)$.

In particular, for $r=2$ we get

$$\|(T_1,\dots,T_n)\|_{h,n,e}\leq\|(T_1,\dots,T_n)\|_{h,n,1}\leq\sqrt{n}\|(T_1,\dots,T_n)\|_{h,n,e}\tag{2.41}$$

and

$$w_{n,e}(T_1,\dots,T_n)\leq w_{h,n,1}(T_1,\dots,T_n)\leq\sqrt{n}w_{n,e}(T_1,\dots,T_n)\tag{2.42}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(H)$.

We have:

Proposition 1. *For any $(T_1,\dots,T_n)\in B^{(n)}(H)$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we have*

$$\|(T_1,\dots,T_n)\|_{h,n,q}\geq\frac{1}{n^{1/p}}\left\|\sum_{j=1}^n T_j\right\|\tag{2.43}$$

and

$$w_{h,n,q}(T_1,\dots,T_n)\geq\frac{1}{n^{1/p}}w\left(\sum_{j=1}^n T_j\right).\tag{2.44}$$

Proof. Let $\lambda_j=\frac{1}{n^{1/p}}$ for $j\in\{1,...,n\}$, then $\sum_{j=1}^n|\lambda_j|^p=1$. Therefore by (1.8) we get

$$\|(T_1,\dots,T_n)\|_{h,n,q}=\sup_{\|\lambda\|_{n,p}\leq 1}\left\|\sum_{j=1}^n\lambda_jT_j\right\|\geq\left\|\sum_{j=1}^n\frac{1}{n^{1/p}}T_j\right\|=\frac{1}{n^{1/p}}\left\|\sum_{j=1}^n T_j\right\|.$$

The inequality (2.44) follows in a similar way. □

We can also introduce the following norms for $(T_1,\dots,T_n)\in B^{(n)}(H)$,

$$\|(T_1,\dots,T_n)\|_{s,n,p}:=\sup_{\|x\|=1}\left(\sum_{j=1}^n\|T_jx\|^p\right)^{1/p}$$

where $p\geq 1$ and

$$\|(T_1,\dots,T_n)\|_{s,n,\infty}:=\sup_{\|x\|=1}\left(\max_{j\in\{1,\dots,n\}}\|T_jx\|\right)=\max_{j\in\{1,\dots,n\}}\{\|T_j\|\}.$$

The triangle inequality $\|\cdot\|_{s,n,q}$ follows by Minkowski inequality, while the other properties of the norm are obvious.

Proposition 2. *Let* $(T_1,\dots,T_n)\in B^{(n)}(H)$.

(i) *We have for $p\geq 1$, that*

$$\|(T_1,\dots,T_n)\|_{h,n,p}\leq\|(T_1,\dots,T_n)\|_{s,n,p}\leq\|(T_1,\dots,T_n)\|_{n,p};\tag{2.45}$$

(ii) *For* $p \geq 2$ *we also have*

$$\left\|(T_1,\ldots,T_n)\right\|_{s,n,p} = \left[w_{h,n,p/2}\left(\left|T_1\right|^2,\ldots,\left|T_n\right|^2\right)\right]^{1/2}, \tag{2.46}$$

where the absolute value $|T|$ *is defined by* $|T| := (T^*T)^{1/2}$.

Proof. (i) We have for $p \geq 2$ and $x, y \in H$ with $\|x\| = \|y\| = 1$, that

$$|\langle T_j x, y\rangle|^p \leq \|T_j x\|^p \|y\|^p = \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p$$

for $j \in \{1, ..., n\}$.

This implies

$$\sum_{j=1}^n |\langle T_j x, y\rangle|^p \leq \sum_{j=1}^n \|T_j x\|^p \leq \sum_{j=1}^n \|T_j\|^p,$$

namely

$$\left(\sum_{j=1}^n |\langle T_j x, y\rangle|^p\right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j x\|^p\right)^{1/p} \leq \left(\sum_{j=1}^n \|T_j\|^p\right)^{1/p}, \tag{2.47}$$

for any $x, y \in H$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (2.47), we get the desired result (2.45).

(ii) We have

$$\begin{aligned}
&\left\|(T_1,\ldots,T_n)\right\|_{s,n,p} \\
&= \sup_{\|x\|=1}\left(\sum_{j=1}^n \|T_j x\|^p\right)^{1/p} = \sup_{\|x\|=1}\left(\sum_{j=1}^n \left(\|T_j x\|^2\right)^{p/2}\right)^{1/p} \\
&= \sup_{\|x\|=1}\left(\sum_{j=1}^n \langle T_j x, T_j x\rangle^{p/2}\right)^{1/p} = \sup_{\|x\|=1}\left(\sum_{j=1}^n \left\langle T_j^* T_j x, x\right\rangle^{p/2}\right)^{1/p} \\
&= \sup_{\|x\|=1}\left(\sum_{j=1}^n \left\langle \left|T_j\right|^2 x, x\right\rangle^{p/2}\right)^{1/p} = \left[\sup_{\|x\|=1}\left(\sum_{j=1}^n \left\langle \left|T_j\right|^2 x, x\right\rangle^{p/2}\right)^{1/(p/2)}\right]^{1/2} \\
&= \left[w_{h,n,p/2}\left(\left|T_1\right|^2,\ldots,\left|T_n\right|^2\right)\right]^{1/2},
\end{aligned}$$

which proves the equality (2.46). □

3. Some Reverse Inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [1] (see also [2, Theorem 5. 14]):

Lemma 2. *Let* $a,\ A \in \mathbb{R}$ *and* $\mathbf{z} = (z_1,\ldots,z_n)$, $\mathbf{y} = (y_1,\ldots,y_n)$ *be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \quad \text{for each} \quad j \in \{1,\ldots,n\}. \tag{3.1}$$

Then for any $\mathbf{w}=(w_1,\dots,w_n)$ *a sequence of positive real numbers, one has the inequality*

$$0\leq\sum_{j=1}^{n}w_jz_j^2\sum_{j=1}^{n}w_jy_j^2-\left(\sum_{j=1}^{n}w_jz_jy_j\right)^2\leq\frac{1}{4}\left(A-a\right)^2\left(\sum_{j=1}^{n}w_jy_j^2\right)^2. \tag{3.2}$$

The constant $\frac{1}{4}$ *is sharp in (3.2).*

O. Shisha and B. Mond obtained in 1967 (see [13]) the following counterparts of (CBS)- inequality (see also [2, Theorem 5.20 & 5.21]):

Lemma 3. *Assume that* $\mathbf{a}=(a_1,\dots,a_n)$ *and* $\mathbf{b}=(b_1,\dots,b_n)$ *are such that there exists* a, A, b, B *with the property that:*

$$0\leq a\leq a_j\leq A\quad\text{and}\quad 0<b\leq b_j\leq B\quad\text{for any } j\in\{1,\dots,n\}, \tag{3.3}$$

then we have the inequality

$$\sum_{j=1}^{n}a_j^2\sum_{j=1}^{n}b_j^2-\left(\sum_{j=1}^{n}a_jb_j\right)^2\leq\left(\sqrt{\frac{A}{b}}-\sqrt{\frac{a}{B}}\right)^2\sum_{j=1}^{n}a_jb_j\sum_{j=1}^{n}b_j^2. \tag{3.4}$$

and

Lemma 4. *Assume that* $\mathbf{a}$, $\mathbf{b}$ *are nonnegative sequences and there exists* γ, Γ *with the property that*

$$0\leq\gamma\leq\frac{a_j}{b_j}\leq\Gamma<\infty\quad\text{for any}\quad j\in\{1,\dots,n\}. \tag{3.5}$$

Then we have the inequality

$$0\leq\left(\sum_{j=1}^{n}a_j^2\sum_{j=1}^{n}b_j^2\right)^{\frac{1}{2}}-\sum_{j=1}^{n}a_jb_j\leq\frac{\left(\Gamma-\gamma\right)^2}{4\left(\gamma+\Gamma\right)}\sum_{j=1}^{n}b_j^2. \tag{3.6}$$

We have:

Theorem 3. *Let* $(T_1,\dots,T_n)\in B^{(n)}(H)$.

(i) *We have*

$$0\leq\left\|(T_1,\dots,T_n)\right\|_{h,n,e}^2-\frac{1}{n}\left\|(T_1,\dots,T_n)\right\|_{h,n,1}^2\leq\frac{1}{4}n\left\|(T_1,\dots,T_n)\right\|_{n,\infty}^2 \tag{3.7}$$

and

$$0\leq w_{n,e}^2\left(T_1,\dots,T_n\right)-\frac{1}{n}w_{h,n,1}^2\left(T_1,\dots,T_n\right)\leq\frac{1}{4}n\left\|(T_1,\dots,T_n)\right\|_{n,\infty}^2. \tag{3.8}$$

(ii) *We have*

$$\begin{aligned}0&\leq\left\|(T_1,\dots,T_n)\right\|_{h,n,e}^2-\frac{1}{n}\left\|(T_1,\dots,T_n)\right\|_{h,n,1}^2\\&\leq\left\|(T_1,\dots,T_n)\right\|_{n,\infty}\left\|(T_1,\dots,T_n)\right\|_{h,n,1}\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}0&\leq w_{n,e}^2\left(T_1,\dots,T_n\right)-\frac{1}{n}w_{h,n,1}^2\left(T_1,\dots,T_n\right)\\&\leq\left\|(T_1,\dots,T_n)\right\|_{n,\infty}w_{h,n,1}\left(T_1,\dots,T_n\right).\end{aligned} \tag{3.10}$$

(iii) *We have*

$$0\leq\left\Vert\left(T_{1},\dots,T_{n}\right)\right\Vert_{h,n,e}-\frac{1}{\sqrt{n}}\left\Vert\left(T_{1},\dots,T_{n}\right)\right\Vert_{h,n,1}\leq\frac{1}{4}\sqrt{n}\left\Vert\left(T_{1},\dots,T_{n}\right)\right\Vert_{n,\infty} \tag{3.11}$$

and

$$0\leq w_{n,e}\left(T_{1},\dots,T_{n}\right)-\frac{1}{\sqrt{n}}w_{h,n,1}\left(T_{1},\dots,T_{n}\right)\leq\frac{1}{4}\sqrt{n}\left\Vert\left(T_{1},\dots,T_{n}\right)\right\Vert_{n,\infty}. \tag{3.12}$$

Proof. (i). Let $(T_1,\dots,T_n)\in B^{(n)}(H)$ and put $R=\max_{j\in\{1,\dots,n\}}\{\|T_j\|\}=\|(T_1,\dots,T_n)\|_{n,\infty}$. If $x,\ y\in H$, with $\|x\|=\|y\|=1$ then $|\langle T_jx,y\rangle|\leq\|T_jx\|\leq\|T_j\|\leq R$ for any $j\in\{1,...,n\}$.

If we write the inequality (3.2) for $z_j=|\langle T_jx,y\rangle|$, $w_j=y_j=1$, $A=R$ and $a=0$, we get

$$0\leq n\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|^{2}-\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|\right)^{2}\leq\frac{1}{4}n^{2}R^{2}$$

for any $x,\ y\in H$, with $\|x\|=\|y\|=1$.

This implies that

$$\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|^{2}\leq\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|\right)^{2}+\frac{1}{4}nR^{2} \tag{3.13}$$

for any $x,\ y\in H$, with $\|x\|=\|y\|=1$ and, in particular

$$\sum_{j=1}^{n}\left|\left\langle T_{j}x,x\right\rangle\right|^{2}\leq\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,x\right\rangle\right|\right)^{2}+\frac{1}{4}nR^{2} \tag{3.14}$$

for any $x\in H$, with $\|x\|=1$.

Taking the supremum over $\|x\|=\|y\|=1$ in (3.13) and $\|x\|=1$ in (3.14), then we get (3.7) and (3.8).

(ii). Let $(T_1,\dots,T_n)\in B^{(n)}(H)$. If we write the inequality (3.4) for $a_j=|\langle T_jx,y\rangle|$, $b_j=1$, $b=B=1$, $a=0$ and $A=R$, then we get

$$0\leq n\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|^{2}-\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|\right)^{2}\leq nR\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|,$$

for any $x,\ y\in H$, with $\|x\|=\|y\|=1$.

This implies that

$$\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|^{2}\leq\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|\right)^{2}+R\sum_{j=1}^{n}\left|\left\langle T_{j}x,y\right\rangle\right|, \tag{3.15}$$

for any $x,\ y\in H$, with $\|x\|=\|y\|=1$ and, in particular

$$\sum_{j=1}^{n}\left|\left\langle T_{j}x,x\right\rangle\right|^{2}\leq\frac{1}{n}\left(\sum_{j=1}^{n}\left|\left\langle T_{j}x,x\right\rangle\right|\right)^{2}+R\sum_{j=1}^{n}\left|\left\langle T_{j}x,x\right\rangle\right|, \tag{3.16}$$

for any $x\in H$ with $\|x\|=1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.15) and $\|x\| = 1$ in (3.16), then we get (3.9) and (3.10).

(iii). If we write the inequality (3.6) for $a_j = |\langle T_j x, y\rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$ we have

$$0 \le \left(n \sum_{j=1}^{n} |\langle T_j x, y\rangle|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} |\langle T_j x, y\rangle| \le \frac{1}{4} nR,$$

for any x, $y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\left(\sum_{j=1}^{n} |\langle T_j x, y\rangle|^2 \right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |\langle T_j x, y\rangle| + \frac{1}{4}\sqrt{n}R, \tag{3.17}$$

for any x, $y \in H$, with $\|x\| = \|y\| = 1$ and, in particular

$$\left(\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 \right)^{\frac{1}{2}} \le \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |\langle T_j x, x\rangle| + \frac{1}{4}\sqrt{n}R, \tag{3.18}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.17) and $\|x\| = 1$ in (3.18), then we get (3.11) and (3.12). □

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If γ, $\Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$\begin{aligned} 0 &\le \operatorname{Re}\left[(\Gamma - \alpha_j)\left(\overline{\alpha_j} - \bar{\gamma}\right)\right] \\ &= (\operatorname{Re}\Gamma - \operatorname{Re}\alpha_j)(\operatorname{Re}\alpha_j - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}\alpha_j)(\operatorname{Im}\alpha_j - \operatorname{Im}\gamma) \end{aligned} \tag{3.19}$$

or, equivalently,

$$\left| \alpha_j - \frac{\gamma + \Gamma}{2} \right| \le \frac{1}{2} |\Gamma - \gamma| \tag{3.20}$$

for each $j \in \{1, \dots, n\}$, then (see for instance [3, p. 9])

$$n \sum_{j=1}^{n} |\alpha_j|^2 - \left| \sum_{j=1}^{n} \alpha_j \right|^2 \le \frac{1}{4} n^2 |\Gamma - \gamma|^2 . \tag{3.21}$$

In addition, if $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then (see for example [3, p. 26]):

$$\begin{aligned} n \sum_{j=1}^{n} |\alpha_j|^2 &\le \frac{1}{4} \frac{\left\{ \operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right) \sum_{j=1}^{n} \alpha_j \right] \right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \\ &\le \frac{1}{4} \frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \left| \sum_{j=1}^{n} \alpha_j \right|^2 . \end{aligned} \tag{3.22}$$

Also, if $\Gamma \neq -\gamma$, then (see for instance [3, p. 32]):

$$\left(n\sum_{j=1}^{n}|\alpha_j|^2\right)^{\frac{1}{2}} - \left|\sum_{j=1}^{n}\alpha_j\right| \leq \frac{1}{4}n\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{3.23}$$

Finally, from [5] we can also state that

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \leq n\left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n}\alpha_j\right|, \tag{3.24}$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (3.19) to hold is that

$$\operatorname{Re}\Gamma \geq \operatorname{Re}\alpha_j \geq \operatorname{Re}\gamma \quad \text{and} \quad \operatorname{Im}\Gamma \geq \operatorname{Im}\alpha_j \geq \operatorname{Im}\gamma \tag{3.25}$$

for each $j \in \{1, \ldots, n\}$.

Theorem 4. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(H)$ *and* $\gamma, \Gamma \in \mathbb{C}$ *with* $\Gamma \neq \gamma$. *Assume that*

$$w\left(T_j - \frac{\gamma+\Gamma}{2}I\right) \leq \frac{1}{2}|\Gamma-\gamma| \text{ for any } j \in \{1, \ldots, n\}. \tag{3.26}$$

(i) *We have*

$$w_{h,n,e}^2(T_1, \ldots, T_n) \leq \frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right) + \frac{1}{4}n|\Gamma-\gamma|^2. \tag{3.27}$$

(ii) *If* $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, *then*

$$w_{h,n,e}(T_1, \ldots, T_n) \leq \frac{1}{2\sqrt{n}}\frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}}w\left(\sum_{j=1}^{n}T_j\right) \tag{3.28}$$

and

$$w_{h,n,e}^2(T_1, \ldots, T_n) \leq \left[\frac{1}{n}w\left(\sum_{j=1}^{n}T_j\right) + \left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\right] \times w\left(\sum_{j=1}^{n}T_j\right). \tag{3.29}$$

(iii) *If* $\Gamma \neq -\gamma$, *then*

$$w_{h,n,e}(T_1, \ldots, T_n) \leq \frac{1}{\sqrt{n}}\left(w\left(\sum_{j=1}^{n}T_j\right) + \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right). \tag{3.30}$$

Proof. Let $x \in H$ with $\|x\| = 1$ and $(T_1, \dots, T_n) \in B^{(n)}(H)$ with the property (3.26). By taking $\alpha_j = \langle T_j x, x\rangle$ we have

$$\begin{aligned}\left|\alpha_j - \frac{\gamma+\Gamma}{2}\right| &= \left|\langle T_j x, x\rangle - \frac{\gamma+\Gamma}{2}\langle x, x\rangle\right| = \left|\left\langle \left(T_j - \frac{\gamma+\Gamma}{2} I\right) x, x\right\rangle\right| \\ &\le \sup_{\|x\|=1}\left|\left\langle \left(T_j - \frac{\gamma+\Gamma}{2} I\right) x, x\right\rangle\right| = w\left(T_j - \frac{\gamma+\Gamma}{2}\right) \\ &\le \frac{1}{2}|\Gamma - \gamma|\end{aligned}$$

for any $j \in \{1, \dots, n\}$.

(i) By using the inequality (3.21), we have

$$\begin{aligned}\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 &\le \frac{1}{n}\left|\sum_{j=1}^{n}\langle T_j x, x\rangle\right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n}\left|\left\langle \sum_{j=1}^{n} T_j x, x\right\rangle\right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2\end{aligned} \tag{3.31}$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in (3.31) we get

$$\begin{aligned}\sup_{\|x\|=1}\left(\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2\right) &\le \frac{1}{n}\sup_{\|x\|=1}\left|\left\langle \sum_{j=1}^{n} T_j x, x\right\rangle\right|^2 + \frac{1}{4} n |\Gamma - \gamma|^2 \\ &= \frac{1}{n} w^2\left(\sum_{j=1}^{n} T_j\right) + \frac{1}{4} n |\Gamma - \gamma|^2,\end{aligned}$$

which proves (3.27).

(ii) If $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then by (3.22) we have for $\alpha_j = \langle T_j x, x\rangle$, $j \in \{1, \dots, n\}$ that

$$\begin{aligned}\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 &\le \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}\langle T_j x, x\rangle\right|^2 \\ &= \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\left\langle \sum_{j=1}^{n} T_j x, x\right\rangle\right|^2\end{aligned} \tag{3.32}$$

for any $x \in H$ with $\|x\| = 1$.

On taking the supremum over $\|x\| = 1$ in (3.32) we get (3.32).

Also, by (3.24) we get

$$\sum_{j=1}^{n} |\langle T_j x, x\rangle|^2 \le \frac{1}{n}\left|\sum_{j=1}^{n}\langle T_j x, x\rangle\right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n}\langle T_j x, x\rangle\right|,$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\|=1$ in this inequality, we have

$$\begin{aligned}
&\sup_{\|x\|=1}\sum_{j=1}^{n}|\langle T_j x,x\rangle|^2 \\
&\leq \sup_{\|x\|=1}\left[\frac{1}{n}\left|\sum_{j=1}^{n}\langle T_j x,x\rangle\right|^2+\left[|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n}\langle T_j x,x\rangle\right|\right] \\
&\leq \frac{1}{n}\sup_{\|x\|=1}\left|\left\langle\sum_{j=1}^{n}T_j x,x\right\rangle\right|^2+\left[|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\sup_{\|x\|=1}\left|\left\langle\sum_{j=1}^{n}T_j x,x\right\rangle\right| \\
&=\frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right)+\left[|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]w\left(\sum_{j=1}^{n}T_j\right),
\end{aligned}$$

which proves (3.29).

(iii) By the inequality (3.23) we have

$$\begin{aligned}
\left(\sum_{j=1}^{n}|\langle T_j x,x\rangle|^2\right)^{\frac{1}{2}} &\leq \frac{1}{\sqrt{n}}\left(\left|\sum_{j=1}^{n}\langle T_j x,x\rangle\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right) \\
&=\frac{1}{\sqrt{n}}\left(\left|\left\langle\sum_{j=1}^{n}T_j x,x\right\rangle\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\right)
\end{aligned}$$

for any $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\|=1$ in this inequality, we get (3.30). □

Remark 3. *By the use of the elementary inequality $w(T)\leq\|T\|$ that holds for any $T\in B(H)$, a sufficient condition for (3.26) to hold is that*

$$\left\|T_j-\frac{\gamma+\Gamma}{2}\right\|\leq\frac{1}{2}|\Gamma-\gamma| \text{ for any } j\in\{1,\dots,n\}. \tag{3.33}$$

4. Inequalities for δ_p and ϑ_p Norms

For $T\in B(H)$ and $p\geq 1$ we can consider the functionals

$$\delta_p(T):=\sup_{\|x\|=\|y\|=1}\left(|\langle Tx,y\rangle|^p+|\langle T^*x,y\rangle|^p\right)^{1/p}=\|(T,T^*)\|_{h,2,p} \tag{4.1}$$

and

$$\vartheta_p(T):=\sup_{\|x\|=1}\left(\|Tx\|^p+\|T^*x\|^p\right)^{1/p}=\|(T,T^*)\|_{s,2,p}. \tag{4.2}$$

It is easy to see that both δ_p and ϑ_p are *norms* on $B(H)$. The case $p=2$ for the norm $\delta:=\delta_2$ was considered and studied in [4].

Observe that, for any $T\in B(H)$ and $p\geq 1$, we have

(4.3)

$$\begin{aligned}
w_{h,2,p}((T,T^*)) &= \sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^p+|\langle T^*x,x\rangle|^p\right)^{\frac{1}{p}}=\sup_{\|x\|=1}\left(|\langle Tx,x\rangle|^p+|\langle Tx,x\rangle|^p\right)^{\frac{1}{p}} \\
&=2^{1/p}\sup_{\|x\|=1}|\langle Tx,x\rangle|=2^{1/p}w(T).
\end{aligned}$$

Using the inequality (1.13) we have

$$2^{1/p}w(T) \le \delta_p(T) \le 2^{1+1/p}w(T) \tag{4.4}$$

for any $T \in B(H)$ and $p \ge 1$.

For $p = 2$, we get

$$\sqrt{2}w(T) \le \delta(T) \le \sqrt{8}w(T) \tag{4.5}$$

while for $p = 1$ we get

$$2w(T) \le \delta_1(T) \le 4w(T) \tag{4.6}$$

for any $T \in B(H)$.

We have for any $T \in B(H)$ and $p \ge 1$ that

$$\|(T, T^*)\|_{2,p} = (\|T\|^p + \|T^*\|^p)^{1/p} = 2^{1/p}\|T\|$$

and by (2.25) we get

$$\|T\| \le \delta_p(T) \le 2^{1/p}\|T\| \tag{4.7}$$

for any $T \in B(H)$ and $p \ge 1$.

For $p = 2$, we get

$$\|T\| \le \delta(T) \le \sqrt{2}\|T\| \tag{4.8}$$

while for $p = 1$ we get

$$\|T\| \le \delta_1(T) \le 2\|T\| \tag{4.9}$$

for any $T \in B(H)$.

From (2.32) we get for $r \ge q \ge 1$ that

$$\delta_r(T) \le \delta_q(T) \le 2^{\frac{r-q}{rq}}\delta_r(T) \tag{4.10}$$

for any $T \in B(H)$.

For any $T \in B(H)$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by (2.43) we have

$$\delta_q(T) \ge \frac{1}{2^{1/p}}\|T + T^*\|. \tag{4.11}$$

In particular, for $p = q = 2$ we get

$$\delta(T) \ge \frac{\sqrt{2}}{2}\|T + T^*\|, \tag{4.12}$$

for any $T \in B(H)$.

By using the inequality (2.45) we get

$$\delta_p(T) \le \vartheta_p(T) \le 2^{1/p}\|T\| \tag{4.13}$$

for any $T \in B(H)$ and $p \ge 1$.

For $p = 1$ we get

$$\delta_1(T) \le \vartheta_1(T) \le 2\|T\| \tag{4.14}$$

for any $T \in B(H)$.

For $p \ge 2$, by employing the equality (2.46) we get

$$\vartheta_p(T) = \left[w_{h,2,p/2}\left(|T|^2, |T^*|^2\right)\right]^{1/2} = \left[2^{2/p}w\left(|T|^2\right)\right]^{1/2} = 2^{1/p}\|T\| \tag{4.15}$$

for any $T \in B(H)$.

On utilising (3.7), (3.9) and (3.11) we get

$$0\leq\delta^{2}(T)-\frac{1}{2}\delta_{1}^{2}(T)\leq\frac{1}{2}\|T\|^{2}, \tag{4.16}$$

$$0\leq\delta^{2}(T)-\frac{1}{2}\delta_{1}^{2}(T)\leq\|T\|\,\delta_{1}(T) \tag{4.17}$$

and

$$0\leq\delta(T)-\frac{1}{\sqrt{2}}\delta_{1}(T)\leq\frac{\sqrt{2}}{4}\|T\| \tag{4.18}$$

for any $T\in B(H)$.

Observe, by (4.3) we have that

$$w_{h,2,e}((T,T^{*}))=\sqrt{2}w(T),$$

for any $T\in B(H)$.

Assume that $T\in B(H)$ and γ, $\Gamma\in\mathbb{C}$ with $\Gamma\neq\gamma$ such that

$$w\left(T-\frac{\gamma+\Gamma}{2}I\right),\ w\left(T^{*}-\frac{\gamma+\Gamma}{2}I\right)\leq\frac{1}{2}|\Gamma-\gamma|, \tag{4.19}$$

then by (3.27) we get

$$w^{2}(T)\leq\|\operatorname{Re}(T)\|^{2}+\frac{1}{4}|\Gamma-\gamma|^{2}, \tag{4.20}$$

where $\operatorname{Re}(T):=\frac{T+T^{*}}{2}$.

If $\operatorname{Re}(\Gamma\bar{\gamma})>0$, then by (3.28) and (3.29)

$$w(T)\leq\frac{1}{2}\frac{|\Gamma+\gamma|}{\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}}\|\operatorname{Re}(T)\| \tag{4.21}$$

and

$$w^{2}(T)\leq\left[\|\operatorname{Re}(T)\|+\left[|\Gamma+\gamma|-2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\right]\|\operatorname{Re}(T)\|. \tag{4.22}$$

If $\Gamma\neq-\gamma$, then by (3.30) we get

$$w(T)\leq\|\operatorname{Re}(T)\|+\frac{1}{8}\frac{|\Gamma-\gamma|^{2}}{|\Gamma+\gamma|}. \tag{4.23}$$

Due to the fact that $w(A)=w(A^{*})$ for any $A\in B(H)$, the condition (4.19) can be simplified as follows.

If m, M are real numbers with $M>m$ and if

$$w\left(T-\frac{m+M}{2}I\right)\leq\frac{1}{2}(M-m),$$

then

$$w^{2}(T)\leq\|\operatorname{Re}(T)\|^{2}+\frac{1}{4}(M-m)^{2}. \tag{4.24}$$

If $m>0$, then

$$w(T)\leq\frac{1}{2}\frac{m+M}{\sqrt{mM}}\|\operatorname{Re}(T)\| \tag{4.25}$$

and

$$w^{2}(T)\leq\left[\|\operatorname{Re}(T)\|+\left(\sqrt{M}-\sqrt{m}\right)^{2}\right]\|\operatorname{Re}(T)\|. \tag{4.26}$$

If $M \neq -m$, then

$$w(T) \leq \|\operatorname{Re}(T)\| + \frac{1}{8}\frac{(M-m)^2}{m+M}. \tag{4.27}$$

5. Inequalities for Real Norms

If X is a complex linear space, then the functional $\|\cdot\|$ is a real norm, if the homogeneity property in the definition of the norms is satisfied only for real numbers, namely we have

$$\|\alpha x\| = |\alpha| \|x\| \text{ for any } \alpha \in \mathbb{R} \text{ and } x \in X.$$

For instance if we consider the complex linear space of complex numbers $\mathbb{C}$ then the functionals

$$|z|_p := \left(|\operatorname{Re}(z)|^p + |\operatorname{Im}(z)|^p\right)^{1/p}, \; p \geq 1$$
and
$$|z|_\infty := \max\left\{|\operatorname{Re}(z)|, |\operatorname{Im}(z)|\right\}, \; p = \infty;$$

are real norms on $\mathbb{C}$.

For $T \in B(H)$ we consider the Cartesian decomposition

$$T = \operatorname{Re}(T) + i \operatorname{Im}(T)$$

where the selfadoint operators $\operatorname{Re}(T)$ and $\operatorname{Im}(T)$ are uniquely defined by

$$\operatorname{Re}(T) = \frac{T+T^*}{2} \text{ and } \operatorname{Im}(T) = \frac{T-T^*}{2i}.$$

We can introduce the following functionals

$$\|T\|_{r,p} := \left(\|\operatorname{Re}(T)\|^p + \|\operatorname{Im}(T)\|^p\right)^{1/p}, \; p \geq 1$$

and

$$\|T\|_{r,\infty} := \max\left\{\|\operatorname{Re}(T)\|, \|\operatorname{Im}(T)\|\right\}, \; p = \infty$$

where $\|\cdot\|$ is the usual operator norm on $B(H)$. *The definition can be extended for any other norms on $B(H)$ or its subspaces.*

Using the properties of the norm $\|\cdot\|$ and the Minkowski's inequality

$$\left(|a+b|^p + |c+d|^p\right)^{1/p} \leq \left(|a|^p + |c|^p\right)^{1/p} + \left(|b|^p + |d|^p\right)^{1/p}$$

for $p \geq 1$ and $a, b, c, d \in \mathbb{C}$, we observe that $\|\cdot\|_{r,p}$, $p \in [1, \infty]$ is a real norm on $B(H)$.

For $p \geq 1$ and $T \in B$ we can introduce the following functionals

$$\begin{aligned}\eta_{r,p}(T) &:= \sup_{\|x\|=\|y\|=1}\left(|\operatorname{Re}\langle Tx, y\rangle|^p + |\operatorname{Im}\langle Tx, y\rangle|^p\right)^{1/p} \\ &= \sup_{\|x\|=\|y\|=1}\left(|\langle \operatorname{Re} Tx, y\rangle|^p + |\langle \operatorname{Im} Tx, y\rangle|^p\right)^{1/p} = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,p},\end{aligned}$$

$$\begin{aligned}\theta_{r,p}(T) &:= \sup_{\|x\|=1}\left(|\operatorname{Re}\langle Tx, x\rangle|^p + |\operatorname{Im}\langle Tx, x\rangle|^p\right)^{1/p} \\ &= \sup_{\|x\|=1}\left(|\langle \operatorname{Re} Tx, x\rangle|^p + |\langle \operatorname{Im} Tx, x\rangle|^p\right)^{1/p} = w_{h,2,p}(\operatorname{Re} T, \operatorname{Im} T)\end{aligned}$$

and

$$\kappa_{r,p}(T) := \sup_{\|x\|=1}\left(\|\operatorname{Re} Tx\|^p + \|\operatorname{Im} Tx\|^p\right)^{1/p} = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,p}.$$

The case $p=2$ is of interest since for $T\in B(H)$ we have

$$\eta_{r,2}(T):=\sup_{\|x\|=\|y\|=1}\left(\left|\operatorname{Re}\langle Tx,y\rangle\right|^2+\left|\operatorname{Im}\langle Tx,y\rangle\right|^2\right)^{1/2}=\sup_{\|x\|=\|y\|=1}\left|\langle Tx,y\rangle\right|=\|T\|,$$

$$\theta_{r,2}(T):=\sup_{\|x\|=1}\left(\left|\operatorname{Re}\langle Tx,x\rangle\right|^2+\left|\operatorname{Im}\langle Tx,x\rangle\right|^2\right)^{1/2}=\sup_{\|x\|=1}\left|\langle Tx,x\rangle\right|=w(T)$$

and

$$\begin{aligned}\kappa_{r,2}(T)&:=\sup_{\|x\|=1}\left(\|\operatorname{Re}Tx\|^2+\|\operatorname{Im}Tx\|^2\right)^{1/2}\\&=\sup_{\|x\|=1}\left(\left\langle(\operatorname{Re}T)^2x,x\right\rangle+\left\langle(\operatorname{Im}T)^2x,x\right\rangle\right)^{1/2}\\&=\sup_{\|x\|=1}\left(\left\langle\left[(\operatorname{Re}T)^2+(\operatorname{Im}T)^2\right]x,x\right\rangle\right)^{1/2}\\&=\left\|(\operatorname{Re}T)^2+(\operatorname{Im}T)^2\right\|^{1/2}=\left\|\frac{|T|^2+|T^*|^2}{2}\right\|^{1/2}.\end{aligned}$$

For $p=\infty$ we have

$$\begin{aligned}\eta_{r,\infty}(T)&:=\sup_{\|x\|=\|y\|=1}\left(\max\left\{\left|\operatorname{Re}\langle Tx,y\rangle\right|,\left|\operatorname{Im}\langle Tx,y\rangle\right|\right\}\right)\\&=\max\left\{\sup_{\|x\|=\|y\|=1}\left|\langle\operatorname{Re}Tx,y\rangle\right|,\sup_{\|x\|=\|y\|=1}\left|\langle\operatorname{Im}Tx,y\rangle\right|\right\}\\&=\max\left\{\|\operatorname{Re}T\|,\|\operatorname{Im}T\|\right\},\end{aligned}$$

and in a similar way

$$\theta_{r,\infty}(T)=\kappa_{r,\infty}(T)=\max\left\{\|\operatorname{Re}T\|,\|\operatorname{Im}T\|\right\}=\|T\|_{r,\infty}.$$

The functionals $\eta_{r,p}$, $\theta_{r,p}$ and $\kappa_{r,p}$ with $p\in[1,\infty]$ are real norms on $B(H)$.

We have

$$\begin{aligned}\eta_{r,p}(T)&=\sup_{\|x\|=\|y\|=1}\left(\left|\operatorname{Re}\langle Tx,y\rangle\right|^p+\left|\operatorname{Im}\langle Tx,y\rangle\right|^p\right)^{1/p}\\&\le\left(\sup_{\|x\|=\|y\|=1}\left|\operatorname{Re}\langle Tx,y\rangle\right|^p+\sup_{\|x\|=\|y\|=1}\left|\operatorname{Im}\langle Tx,y\rangle\right|^p\right)^{1/p}\\&=\left(\|\operatorname{Re}(T)\|^p+\|\operatorname{Im}(T)\|^p\right)^{1/p}=\|T\|_{r,p}\end{aligned}$$

and

$$\begin{aligned}\|T\|_{r,\infty}&=\sup_{\|x\|=\|y\|=1}\left(\max\left\{\left|\operatorname{Re}\langle Tx,y\rangle\right|,\left|\operatorname{Im}\langle Tx,y\rangle\right|\right\}\right)\\&\le\sup_{\|x\|=\|y\|=1}\left(\left|\operatorname{Re}\langle Tx,y\rangle\right|^p+\left|\operatorname{Im}\langle Tx,y\rangle\right|^p\right)^{1/p}=\eta_{r,p}(T)\end{aligned}$$

for any $p\ge1$ and $T\in B(H)$.

In a similar way we have

$$\|T\|_{r,\infty}\le\theta_{r,p}(T)\le\|T\|_{r,p}$$

and

$$\|T\|_{r,\infty}\le\kappa_{r,p}(T)\le\|T\|_{r,p}$$

for any $p \geq 1$ and $T \in B(H)$.

If we write the inequality (1.13) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get

$$\theta_{r,p}(T) \leq \eta_{r,p}(T) \leq 2\theta_{r,p}(T) \tag{5.1}$$

for any $p \geq 1$ and $T \in B(H)$.

Using the inequlities (2.25) and (2.26) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \eta_{r,p}(T) \leq \|T\|_{r,p} \tag{5.2}$$

and

$$\frac{1}{2^{1/p}} \|T\|_{r,p} \leq \theta_{r,p}(T) \leq \|T\|_{r,p} \tag{5.3}$$

for any $p \geq 1$ and $T \in B(H)$.

If we use the inequalities (2.32) and (2.33) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$ then we get for $t \geq p \geq 1$ that

$$\eta_{r,t}(T) \leq \eta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \eta_{r,t}(T) \tag{5.4}$$

and

$$\theta_{r,t}(T) \leq \theta_{r,p}(T) \leq 2^{\frac{t-p}{tp}} \theta_{r,t}(T) \tag{5.5}$$

for any $T \in B(H)$.

For $p = 1$ we have the functionals

$$\eta_{r,1}(T) = \sup_{\|x\|=\|y\|=1} \left(|\langle \operatorname{Re} Tx, y\rangle| + |\langle \operatorname{Im} Tx, y\rangle| \right) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{h,2,1},$$

$$\theta_{r,1}(T) := \sup_{\|x\|=1} \left(|\langle \operatorname{Re} Tx, x\rangle| + |\langle \operatorname{Im} Tx, x\rangle| \right) = w_{h,2,1}(\operatorname{Re} T, \operatorname{Im} T)$$

and

$$\kappa_{r,1}(T) := \sup_{\|x\|=1} \left(\|\operatorname{Re} Tx\| + \|\operatorname{Im} Tx\| \right) = \|(\operatorname{Re} T, \operatorname{Im} T)\|_{s,2,1}.$$

By utilising the inequalities (3.7), (3.9) and (3.11) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$, then

$$0 \leq \|T\|^2 - \frac{1}{2}\eta_{r,1}^2(T) \leq \frac{1}{2}\left(\max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \right)^2, \tag{5.6}$$

$$0 \leq \|T\|^2 - \frac{1}{2}\eta_{r,1}^2(T) \leq \max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \eta_{r,1}(T) \tag{5.7}$$

and

$$0 \leq \|T\| - \frac{\sqrt{2}}{2}\eta_{r,1}(T) \leq \frac{\sqrt{2}}{4} \max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \tag{5.8}$$

for any $T \in B(H)$.

Also, by utilising the inequalities (3.8), (3.10) and (3.12) for $n = 2$, $T_1 = \operatorname{Re} T$ and $T_2 = \operatorname{Im} T$, then

$$0 \leq w^2(T) - \frac{1}{2}\theta_{r,1}^2(T) \leq \frac{1}{2}\left(\max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \right)^2, \tag{5.9}$$

$$0 \leq w^2(T) - \frac{1}{2}\theta_{r,1}^2(T) \leq \max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \theta_{r,1}(T) \tag{5.10}$$

and

$$0 \le w(T) - \frac{\sqrt{2}}{2}\theta_{r,1}(T) \le \frac{\sqrt{2}}{4}\max\{\|\operatorname{Re} T\|, \|\operatorname{Im} T\|\} \tag{5.11}$$

for any $T \in B(H)$.

If m, M are real numbers with $M > m$ and if

$$\left\|\operatorname{Re} T - \frac{m+M}{2}I\right\|, \ \left\|\operatorname{Im} T - \frac{m+M}{2}I\right\| \le \frac{1}{2}(M-m), \tag{5.12}$$

then by (3.27) we get

$$w^2(T) \le \frac{1}{2}\|\operatorname{Re} T + \operatorname{Im} T\|^2 + \frac{1}{2}(M-m)^2. \tag{5.13}$$

If $m > 0$, then (3.28) and (3.29) we have

$$w(T) \le \frac{1}{2\sqrt{2}}\frac{m+M}{\sqrt{mM}}\|\operatorname{Re} T + \operatorname{Im} T\| \tag{5.14}$$

and

$$w^2(T) \le \left[\frac{1}{2}\|\operatorname{Re} T + \operatorname{Im} T\| + \left(\sqrt{M} - \sqrt{m}\right)^2\right]\|\operatorname{Re} T + \operatorname{Im} T\|. \tag{5.15}$$

If $M \ne -m$, then by (3.30) we get

$$w(T) \le \frac{1}{\sqrt{2}}\left(\|\operatorname{Re} T + \operatorname{Im} T\| + \frac{1}{4}\frac{(M-m)^2}{M+m}\right). \tag{5.16}$$

Finally, we observe that a simple sufficient condition for (5.12) to hold, is that

$$mI \le \operatorname{Re} T, \ \ \operatorname{Im} T \le MI$$

in the operator order of $B(H)$.

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