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HYPO-q-NORMS ON A CARTESIAN PRODUCT OF ALGEBRAS OF OPERATORS ON BANACH SPACES

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Abstract. In this paper we introduce the hypo-q-operator norm and hypo-q-numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given.

1. Introduction

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. On $\mathbb{K}^n$ endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}(\|\cdot\|_n) := \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n | \, \|\lambda\|_n \le 1\}.$$

As an example of such norms we should mention the usual *p-norms*

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ \left(\sum_{k=1}^n |\lambda_k|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \tag{1.1}$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2\right)^{\frac{1}{2}}.$$

It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following *p-norms*:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max\{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ \left(\sum_{k=1}^n \|x_k\|^p\right)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases} \tag{1.2}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [6], for a given norm $\|\cdot\|_n$ on $\mathbb{K}^n$, we define the functional $\|\cdot\|_{h,n} : E^n \to [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B(\|\cdot\|_n)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.3}$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

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It is easy to see, by the properties of the norm $\|\cdot\|$, that:

(i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
(ii) $\|\mathbf{x}+\mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}$, $\mathbf{y} \in E^n$;
(iii) $\|\alpha\mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n. This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n.

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0)$, $(0, \lambda_2^0, \dots, 0)$, $\dots$, $(0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n.

If by $\mathbb{B}_{n,p}$ with $p \in [1,\infty]$ we denote the balls generated by the p-norms $\|\cdot\|_{n,p}$ on $\mathbb{K}^n$, then we can obtain the following *hypo-q-norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\boldsymbol{\lambda}\in\mathbb{B}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \tag{1.4}$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in $\mathbb{K}^n$, which we denote by $\mathbb{B}_n$, $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \left| \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right. \right\}$ that generates the *hypo-Euclidean norm* on E^n, i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\boldsymbol{\lambda}\in\mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.5}$$

Moreover, if $E = H$, H is a inner product space over $\mathbb{K}$, then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\boldsymbol{\lambda}\in\mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \tag{1.6}$$

Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space over $\mathbb{K}$ and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the n-tuples of vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle\cdot,\cdot\rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \qquad \mathbf{x},\ \mathbf{y} \in H^n, \tag{1.7}$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n, i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \qquad \mathbf{x} \in H^n. \tag{1.8}$$

The following result established in [6] connects the usual Euclidean norm $\|\cdot\|$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [6])**.** *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2, \tag{1.9}$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n.

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [6]). *For any* $\mathbf{x} \in H^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$, *we have*

$$\|\mathbf{x}\|_e = \sup_{\|x\|=1} \left(\sum_{j=1}^{n} |\langle x, x_j \rangle|^2 \right)^{\frac{1}{2}}. \tag{1.10}$$

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$. We denote by E^* its dual space endowed with the norm $\|\cdot\|$ defined by

$$\|f\| := \sup_{\|x\|\le 1} |f(x)| = \sup_{\|u\|=1} |f(u)| < \infty, \text{ where } f \in E^*.$$

We have the following representation result for the *hypo-q-norms* on E^n plays a key role in obtaining different bounds for these norms, see [8]:

Theorem 3 (Dragomir, 2017, [8]). *Let* $(E, \|\cdot\|)$ *be a normed linear space over the real or complex number field* $\mathbb{K}$. *For any* $\mathbf{x} \in E^n$ *with* $\mathbf{x} = (x_1, \ldots, x_n)$, *we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|f\|=1} \left\{ \left(\sum_{j=1}^{n} |f(x_j)|^q \right)^{1/q} \right\} \tag{1.11}$$

where $q \ge 1$, *and*

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j\in\{1,\ldots,n\}} \{\|x_j\|\}. \tag{1.12}$$

We have the following inequalities of interest:

Corollary 1. *With the assumptions of Theorem 3 we have for* $q \ge 1$ *that*

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \le \|\mathbf{x}\|_{h,n,q} \le \|\mathbf{x}\|_{n,q} \tag{1.13}$$

for any any $\mathbf{x} \in E^n$.

We have for $r \ge q \ge 1$ *that*

$$\|\mathbf{x}\|_{h,n,r} \le \|\mathbf{x}\|_{h,n,q} \le n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r} \tag{1.14}$$

for any $\mathbf{x} \in E^n$.

In this paper we introduce the hypo-q-operator norms and hypo-q-numerical radius on a Cartesian product of algebras of bounded linear operators on Banach spaces. A representation of these norms in terms of semi-inner products, the equivalence with the q-norms on a Cartesian product and some reverse inequalities obtained via the scalar reverses of Cauchy-Buniakowski-Schwarz inequality are also given.

2. Semi-Inner Products and Preliminary Results

In what follows, we assume that E is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [11] but the main properties of it were discovered by J. R. Giles [12], P. L. Papini [19], P. M. Miličić [14]–[16], I. Roşca [20], B. Nath [18] and others, see also [2].

In this section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 1. *The mapping $[\cdot,\cdot] : E \times E \to \mathbb{K}$ will be called the semi-inner product in the sense of Lumer-Giles or L-G-s.i.p., for short, if the following properties are satisfied:*

(i) $[x + y, z] = [x, z] + [y, z]$ *for all* $x,\ y,\ z \in E$;
(ii) $[\lambda x, y] = \lambda [x, y]$ *for all* $x,\ y \in E$ *and* λ *a scalar in* $\mathbb{K}$;
(iii) $[x, x] \geq 0$ *for all* $x \in E$ *and* $[x, x] = 0$ *implies that* $x = 0$;
(iv) $|[x, y]|^2 \leq [x, x] [y, y]$ *(Schwarz's inequality) for all* $x,\ y \in E$;
(v) $[x, \lambda y] = \bar{\lambda} [x, y]$ *for all* $x,\ y \in E$ *and* λ *a scalar in* $\mathbb{K}$.

The following results collects some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 1. *Let E be a linear space and $[\cdot,\cdot]$ a L-G-s.i.p on E. Then the following statements are true:*

(i) *The mapping* $E \ni x \overset{\|\cdot\|}{\longrightarrow} [x, x]^{\frac{1}{2}} \in \mathbb{R}_+$ *is a norm on* E;
(ii) *For every* $y \in E$ *the functional* $E \ni x \overset{f_y}{\longrightarrow} [x, y] \in \mathbb{K}$ *is a continuous linear functional on E endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality* $\|f_y\| = \|y\|$.

Definition 2. *The mapping $J : E \to 2^{E^*}$, where E^* is the dual space of E, given by:*

$$J(x) := \{x^* \in E^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \ \ \|x^*\| = \|x\|\}, \ \ x \in E$$

will be called the normalised duality mapping of normed linear space $(E, \|\cdot\|)$.

Definition 3. *A mapping $\tilde{J} : E \to E^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in E.*

The following theorem due to I. Roşca [20] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 4. *Let $(E, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \quad \text{for all } x, y \text{ in } E,$$

where $\tilde{J}$ is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2. *Let $(E, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *E is smooth;*
(ii) *There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.*

We need the following lemma holding for n-tuples of complex numbers:

Lemma 1. *Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{C}^n$.*

(i) *If* $p,\ q>1$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\sup_{\|\alpha\|_{n,p}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_{n,q}. \tag{2.1}$$

(ii) *We have*

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_{n,1}\ \text{ and }\ \sup_{\|\alpha\|_{n,1}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\|\beta\|_{n,\infty}. \tag{2.2}$$

Proof. (i). Using Hölder's discrete inequality for $p,\ q>1$ and $\frac{1}{p}+\frac{1}{q}=1$ we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\left(\sum_{j=1}^{n}|\alpha_j|^p\right)^{1/p}\left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_{n,p}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_{n,q} \tag{2.3}$$

where $\alpha=(\alpha_1,\ldots,\alpha_n)$ and $\beta=(\beta_1,\ldots,\beta_n)$ are n-tuples of complex numbers.

For $(\beta_1,\ldots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\ldots,\alpha_n)$ with

$$\alpha_j:=\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$

for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We observe that

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}\,|\beta_j|^{q-2}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}\beta_j\right|=\frac{\sum_{j=1}^{n}|\beta_j|^q}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)^{1/p}}$$
$$=\left(\sum_{j=1}^{n}|\beta_j|^q\right)^{1/q}=\|\beta\|_{n,q}$$

and

$$\|\alpha\|_{n,p}^p=\sum_{j=1}^{n}|\alpha_j|^p=\sum_{j=1}^{n}\frac{\left|\overline{\beta_j}\,|\beta_j|^{q-2}\right|^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{\left(|\beta_j|^{q-1}\right)^p}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}$$
$$=\sum_{j=1}^{n}\frac{|\beta_j|^{qp-p}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=\sum_{j=1}^{n}\frac{|\beta_j|^{q}}{\left(\sum_{k=1}^{n}|\beta_k|^q\right)}=1.$$

Therefore, by (2.3) we have the representation (2.1).

(ii). Using the properties of the modulus, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\max_{j\in\{1,\ldots,n\}}|\alpha_j|\sum_{j=1}^{n}|\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_{n,\infty}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_{n,1}\,, \tag{2.4}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_j:=\frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j\neq 0$ and $\alpha_j=0$, for the rest.

We have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\sum_{j=1}^{n}\frac{\overline{\beta_j}}{|\beta_j|}\beta_j\right|=\sum_{j=1}^{n}|\beta_j|=\|\beta\|_{n,1}$$

and

$$\|\alpha\|_{n,\infty}=\max_{j\in\{1,\dots,n\}}|\alpha_j|=\max_{j\in\{1,\dots,n\}}\left|\frac{\overline{\beta_j}}{|\beta_j|}\right|=1$$

and by (2.4) we get the first representation in (2.2).

Moreover, we have

$$\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\sum_{j=1}^{n}|\alpha_j|\max_{j\in\{1,\dots,n\}}|\beta_j|\,,$$

which implies that

$$\sup_{\|\alpha\|_{n,1}\leq 1}\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|\leq\|\beta\|_{n,\infty}\,, \tag{2.5}$$

where $\alpha=(\alpha_1,\dots,\alpha_n)$ and $\beta=(\beta_1,\dots,\beta_n)$.

For $(\beta_1,\dots,\beta_n)\neq 0$, let $j_0\in\{1,...,n\}$ such that $\|\beta\|_\infty=\max_{j\in\{1,\dots,n\}}|\beta_j|=|\beta_{j_0}|$. Consider $\alpha=(\alpha_1,\dots,\alpha_n)$ with $\alpha_{j_0}=\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j=0$ for $j\neq j_0$. For this choice we get

$$\sum_{j=1}^{n}|\alpha_j|=\frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|}=1\text{ and }\left|\sum_{j=1}^{n}\alpha_j\beta_j\right|=\left|\frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}\beta_{j_0}\right|=|\beta_{j_0}|=\|\beta\|_{n,\infty}\,,$$

therefore by (2.5) we obtain the second representation in (2.2). □

Theorem 5. *Let $(E,\|\cdot\|)$ be a normed linear space over the real or complex number field $\mathbb{K}$ and $[\cdot,\cdot]$ a L-G-s.i.p on E that generates the norm $\|\cdot\|$, i.e. $[x,x]^{1/2}=\|x\|$. For any $\mathbf{x}\in E^n$ with $\mathbf{x}=(x_1,\dots,x_n)$, we have*

$$\|\mathbf{x}\|_{h,n,q}=\sup_{\|u\|=1}\left\{\left(\sum_{j=1}^{n}|[x_j,u]|^q\right)^{1/q}\right\}, \tag{2.6}$$

where $q\geq 1$.

Proof. Now if $[\cdot,\cdot]$ is a L-G-s.i.p. that generates the norm $\|\cdot\|$, then

$$\sup_{\|u\|=1}|[x,u]|=\|x\|\text{ for any }x\in X. \tag{2.7}$$

Indeed, if $x=0$ the equality is obvious. If $x\neq 0$, then by Schwarz's inequality we have

$$|[x,u]|\leq \|x\|\,\|u\| \text{ for any } u\in X.$$

By taking the supremum in this inequality we have

$$\sup_{\|u\|=1}|[x,u]|\leq \|x\|\,.$$

On the other hand by taking $u_0:=\frac{x}{\|x\|}$ we have that $\|u_0\|=1$ and since

$$\sup_{\|u\|=1}|[x,u]|\geq |[x,u_0]|=\left|\left[x,\frac{x}{\|x\|}\right]\right|=\frac{\|x\|^2}{\|x\|}=\|x\|\,,$$

then we get the desired equality (3.19).

Assume that $\mathbf{x}\in E^n$ with $\mathbf{x}=(x_1,\dots,x_n)$ and let $p,\,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then by the definition (1.4) and representation (2.7) we have

$$\begin{aligned}\text{(2.8)}\qquad \|\mathbf{x}\|_{h,n,q} &:= \sup_{|\boldsymbol{\alpha}|_p\leq 1}\left\|\sum_{j=1}^n\alpha_j x_j\right\| = \sup_{|\boldsymbol{\alpha}|_p\leq 1}\left(\sup_{\|u\|=1}\left|\left[\left(\sum_{j=1}^n\alpha_j x_j\right),u\right]\right|\right)\\ &= \sup_{\|u\|=1}\left(\sup_{|\boldsymbol{\alpha}|_p\leq 1}\left|\sum_{j=1}^n\alpha_j\,[x_j,u]\right|\right)=\sup_{\|u\|=1}\left(\sum_{j=1}^n|[x_j,u]|^q\right)^{1/q},\end{aligned}$$

where the last equality in (2.8) follows by the representation (2.1) for $\beta_j=[x_j,u]\,,$ $j\in\{1,...,n\}\,.$

For $q=1$, $p=\infty$ the representation (2.6) follows in a similar way by utilising the first equality in (2.2). We omit the details. □

Remark 1. *If $(E,\|\cdot\|)$ is an inner product space with $\langle\cdot,\cdot\rangle$ generating the norm, then we recapture the representation result obtained in the recent paper* [9].

Remark 2. *We observe that the representation (2.6) provides a stronger result than the one from Theorem 3 since it makes use of a smaller class of bounded linear functionals, namely the ones generated by a given L-G-s.i.p on E that generates the norm $\|\cdot\|$.*

3. The Case of Operators on Banach Spaces

A fundamental result due to Lumer [11], in the theory of operators on complex Banach spaces X, is that if $T\in\mathcal{B}(X)\,,$ then

$$w(T)\leq\|T\|\leq 4w(T)\,, \tag{3.1}$$

where $w(T):=\sup_{\|x\|=1}|[Tx,x]|$ is the numerical radius of the operator T and $[\cdot,\cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|\,.$

As shown by Glickfeld [13], the second inequality in (3.1) holds with $e=\exp(1)$ instead of 4 and e is the best possible constant. Therefore we have the sharp inequalities

$$\frac{1}{e}\|T\|\leq w(T)\leq\|T\| \tag{3.2}$$

for any $T\in\mathcal{B}(X)\,.$

On the Cartesian product $B^{(n)}(X) := \mathcal{B}(X) \times ... \times \mathcal{B}(X)$ we can define the *hypo-q-operator norms* of $(T_1, \ldots, T_n) \in B^{(n)}(X)$ by

$$\|(T_1, \ldots, T_n)\|_{h,n,q} := \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \text{ where } p,\ q \in [1, \infty], \tag{3.3}$$

with the convention that if $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

If $[\cdot, \cdot]$ is a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $w(T) := \sup_{\|x\|=1} |[Tx, x]|$ is the numerical radius of the operator T we can also define the *hypo-q-numerical radius* of $(T_1, \ldots, T_n) \in B^{(n)}(X)$ by

$$w_{h,n,q}(T_1, \ldots, T_n) := \sup_{\|\lambda\|_{n,p} \leq 1} w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \text{ with } p,\ q \in [1, \infty], \tag{3.4}$$

with the convention that if $p = 1$, $q = \infty$; if $p = \infty$, $q = 1$ and if $p > 1$, then $\frac{1}{p} + \frac{1}{q} = 1$.

Using (3.2) we have

$$\frac{1}{e} \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \leq w\left(\sum_{j=1}^{n} \lambda_j T_j \right) \leq \left\| \sum_{j=1}^{n} \lambda_j T_j \right\|$$

and by taking the supremum over $\|\lambda\|_{n,p} \leq 1$ in this inequality, we get the following fundamental result

$$\frac{1}{e} \|(T_1, \ldots, T_n)\|_{h,n,q} \leq w_{h,n,q}(T_1, \ldots, T_n) \leq \|(T_1, \ldots, T_n)\|_{h,n,q} \tag{3.5}$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$ and $q \geq 1$. The inequalities (3.5) are sharp, which follow by the unidimensional case.

Theorem 6. *Let $(X, \|\cdot\|)$ be a Banach space and $[\cdot, \cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X. Let $(T_1, \ldots, T_n) \in B^{(n)}(X)$ and $x, y \in X$, then for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right] \right| = \left(\sum_{j=1}^{n} |[T_j x, y]|^q \right)^{1/q} \tag{3.6}$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right] \right| = \left(\sum_{j=1}^{n} |[T_j x, y]|^2 \right)^{1/2}. \tag{3.7}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right] \right| = \sum_{j=1}^{n} |[T_j x, y]| \tag{3.8}$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left| \left[\left(\sum_{j=1}^{n} \alpha_j T_j \right) x, y \right] \right| = \max_{j \in \{1, \ldots, n\}} \{ |[T_j x, y]| \}. \tag{3.9}$$

Proof. If we take $\beta = ([T_1 x, y], \dots, [T_n x, y]) \in \mathbb{C}^n$ in (2.1), then we get

$$\left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q} = \|\beta\|_{n,q} = \sup_{\|\alpha\|_p \leq 1} \left|\sum_{j=1}^{n} \alpha_j \beta_j\right|$$

$$= \sup_{\|\alpha\|_{n,p} \leq 1} \left|\sum_{j=1}^{n} \alpha_j [T_j x, y]\right| = \sup_{\|\alpha\|_{n,p} \leq 1} \left|\left[\sum_{j=1}^{n} \alpha_j T_j x, y\right]\right|,$$

which proves (3.6).

The equalities (3.8) and (3.9) follow by (2.2). □

Corollary 2. *With the assumptions of Theorem 6, if* $(T_1, \dots, T_n) \in B^{(n)}(X)$ *and* $x \in X$*, then for* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left|\left[\left(\sum_{j=1}^{n} \alpha_j T_j\right) x, x\right]\right| = \left(\sum_{j=1}^{n} |[T_j x, x]|^q\right)^{1/q} \tag{3.10}$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left|\left[\left(\sum_{j=1}^{n} \alpha_j T_j\right) x, x\right]\right| = \left(\sum_{j=1}^{n} |[T_j x, x]|^2\right)^{1/2}. \tag{3.11}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left|\left[\left(\sum_{j=1}^{n} \alpha_j T_j\right) x, x\right]\right| = \sum_{j=1}^{n} |[T_j x, x]| \tag{3.12}$$

and

$$\sup_{\|\alpha\|_{n,1} \leq 1} \left|\left[\left(\sum_{j=1}^{n} \alpha_j T_j\right) x, x\right]\right| = \max_{j \in \{1,\dots,n\}} \{|[T_j x, x]|\}. \tag{3.13}$$

Corollary 3. *With the assumptions of Theorem 6, if* $(T_1, \dots, T_n) \in B^{(n)}(X)$ *and* $x \in X$*, then for* $p, q > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$ *we have*

$$\sup_{\|\alpha\|_{n,p} \leq 1} \left\|\sum_{j=1}^{n} \alpha_j T_j x\right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q} \tag{3.14}$$

and, in particular

$$\sup_{\|\alpha\|_{n,2} \leq 1} \left\|\sum_{j=1}^{n} \alpha_j T_j x\right\| = \sup_{\|y\|=1} \left(\sum_{j=1}^{n} |[T_j x, y]|^2\right)^{1/2}. \tag{3.15}$$

We also have

$$\sup_{\|\alpha\|_{n,\infty} \leq 1} \left\|\sum_{j=1}^{n} \alpha_j T_j x\right\| = \sup_{\|y\|=1} \sum_{j=1}^{n} |[T_j x, y]| \tag{3.16}$$

and

$$\sup_{\|\alpha\|_{n,1}\leq 1}\left\|\sum_{j=1}^{n}\alpha_j T_j x\right\| = \max_{j\in\{1,\dots,n\}}\{\|T_j x\|\}. \tag{3.17}$$

Proof. By the properties of semi-inner product, we have for any $u \in X$, $u \neq 0$ (see also (3.19)) that

$$\|u\| = \sup_{\|y\|=1}|[u,y]|. \tag{3.18}$$

Let $x \in X$, then by taking the supremum over $\|y\| = 1$ in (3.6) we get for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}\sup_{\|y\|=1}\left(\sum_{j=1}^{n}|[T_j x,y]|^q\right)^{1/q} &= \sup_{\|y\|=1}\left(\sup_{\|\alpha\|_{n,p}\leq 1}\left|\left[\left(\sum_{j=1}^{n}\alpha_j T_j\right)x,y\right]\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\leq 1}\left(\sup_{\|y\|=1}\left|\left[\left(\sum_{j=1}^{n}\alpha_j T_j\right)x,y\right]\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\leq 1}\left\|\left(\sum_{j=1}^{n}\alpha_j T_j\right)x\right\|,\end{aligned}$$

which proves the equality (3.14). We used for the last equality the property (3.18).

The other equalities can be proved in a similar way by using Theorem 6, however the details are omitted. □

We can state and prove our main representation result.

Theorem 7. *Let $(X,\|\cdot\|)$ be a Banach space, $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $(T_1,\dots,T_n) \in B^{(n)}(X)$.*

(i) *For $q \geq 1$ we have the representation for the hypo-q-operator norm*

$$\|(T_1,\dots,T_n)\|_{h,n,q} = \sup_{\|x\|=\|y\|=1}\left(\sum_{j=1}^{n}|[T_j x,y]|^q\right)^{1/q} \tag{3.19}$$

and

$$\|(T_1,\dots,T_n)\|_{h,n,\infty} = \max_{j\in\{1,\dots,n\}}\{\|T_j\|\}. \tag{3.20}$$

(ii) *For $q \geq 1$ we have the representation for the hypo-q-numerical radius*

$$w_{h,n,q}(T_1,\dots,T_n) = \sup_{\|x\|=1}\left(\sum_{j=1}^{n}|[T_j x,x]|^q\right)^{1/q} \tag{3.21}$$

and

$$w_{h,n,\infty}(T_1,\dots,T_n) = \max_{j\in\{1,\dots,n\}}\{w(T_j)\}. \tag{3.22}$$

Proof. (i) By using the equality (3.14) we have for $(T_1,\dots,T_n)\in B^{(n)}(X)$ that

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\left(\sum_{j=1}^n|[T_jx,y]|^q\right)^{1/q} &= \sup_{\|x\|=1}\left(\sup_{\|y\|=1}\left(\sum_{j=1}^n|[T_jx,y]|^q\right)^{1/q}\right)\\ &= \sup_{\|x\|=1}\left(\sup_{\|\alpha\|_{n,p}\le1}\left\|\sum_{j=1}^n\alpha_jT_jx\right\|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le1}\left(\sup_{\|x\|=1}\left\|\sum_{j=1}^n\alpha_jT_jx\right\|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le1}\left\|\sum_{j=1}^n\alpha_jT_j\right\| = \|(T_1,\dots,T_n)\|_{h,n,q}\,,\end{aligned}$$

which proves (3.19). The rest is obvious.

(ii) By using the equality (3.10) we have for $(T_1,\dots,T_n)\in B^{(n)}(X)$ that

$$\begin{aligned}\sup_{\|x\|=1}\left(\sum_{j=1}^n|[T_jx,x]|^q\right)^{1/q} &= \sup_{\|x\|=1}\left(\sup_{\|\alpha\|_{n,p}\le1}\left|\left[\left(\sum_{j=1}^n\alpha_jT_j\right)x,x\right]\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le1}\left(\sup_{\|x\|=1}\left|\left[\left(\sum_{j=1}^n\alpha_jT_j\right)x,x\right]\right|\right)\\ &= \sup_{\|\alpha\|_{n,p}\le1} w\left(\sum_{j=1}^n\alpha_jT_j\right) = w_{h,n,q}(T_1,\dots,T_n)\,,\end{aligned}$$

which proves (3.21). The rest is obvious. □

We can consider on $B^{(n)}(X)$ the following usual operator and numerical radius q-norms, for $q\ge1$

$$\|(T_1,\dots,T_n)\|_{n,q}:=\left(\sum_{j=1}^n\|T_j\|^q\right)^{1/q}\quad\text{and}\quad w_{n,q}(T_1,\dots,T_n):=\left(\sum_{j=1}^n w^q(T_j)\right)^{1/q}$$

where $(T_1,\dots,T_n)\in B^{(n)}(X)$. For $q=\infty$ we put

$$\|(T_1,\dots,T_n)\|_{n,\infty}:=\max_{j\in\{1,\dots,n\}}\{\|T_j\|\}\ \text{ and }\ w_{n,\infty}(T_1,\dots,T_n):=\max_{j\in\{1,\dots,n\}}\{w(T_j)\}\,.$$

Corollary 4. *With the assumptions of Theorem 7 we have for $q\ge1$ that*

$$\frac{1}{n^{1/q}}\|(T_1,\dots,T_n)\|_{n,q}\le\|(T_1,\dots,T_n)\|_{h,n,q}\le\|(T_1,\dots,T_n)\|_{n,q} \tag{3.23}$$

and

$$\frac{1}{n^{1/q}}w_{n,q}(T_1,\dots,T_n)\le w_{h,n,q}(T_1,\dots,T_n)\le w_{n,q}(T_1,\dots,T_n) \tag{3.24}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(X)$.

Proof. Let $(T_1,\dots,T_n)\in B^{(n)}(X)$ and $x,\ y\in X$ with $\|x\|=\|y\|=1$. Then by Schwarz's inequality we have

$$\left(\sum_{j=1}^{n}|[T_jx,y]|^q\right)^{1/q}\le\left(\sum_{j=1}^{n}\|T_jx\|^q\|y\|^q\right)^{1/q}=\left(\sum_{j=1}^{n}\|T_jx\|^q\right)^{1/q}.$$

By the operator norm inequality we also have

$$\left(\sum_{j=1}^{n}\|T_jx\|^q\right)^{1/q}\le\left(\sum_{j=1}^{n}\|T_j\|^q\|x\|^q\right)^{1/q}=\|(T_1,\dots,T_n)\|_{n,q}.$$

Therefore

$$\left(\sum_{j=1}^{n}|[T_jx,y]|^q\right)^{1/q}\le\|(T_1,\dots,T_n)\|_{n,q}$$

and by taking the supremum over $\|x\|=\|y\|=1$ we get the second inequality in (3.23).

By the properties of complex numbers, we have

$$\max_{j\in\{1,\dots,n\}}\{|[T_jx,y]|\}\le\left(\sum_{j=1}^{n}|[T_jx,y]|^q\right)^{1/q}$$

for any $x,\ y\in X$ with $\|x\|=\|y\|=1$.

Observe also that, by (3.18) we have for any operator $T\in B(X)$ that

$$\|Tx\|=\sup_{\|y\|=1}|[Tx,y]|\ \text{ for any } x\in X \tag{3.25}$$

and

$$\|T\|=\sup_{\|x\|=1}\|Tx\|=\sup_{\|x\|=1}\left(\sup_{\|y\|=1}|[Tx,y]|\right)=\sup_{\|x\|=\|y\|=1}|[Tx,y]|. \tag{3.26}$$

By taking the supremum over $\|x\|=\|y\|=1$ we get

$$\sup_{\|x\|=\|y\|=1}\left(\max_{j\in\{1,\dots,n\}}\{|[T_jx,y]|\}\right)\le\|(T_1,\dots,T_n)\|_{h,n,q} \tag{3.27}$$

and since

$$\begin{aligned}\sup_{\|x\|=\|y\|=1}\left(\max_{j\in\{1,\dots,n\}}\{|[T_jx,y]|\}\right)&=\max_{j\in\{1,\dots,n\}}\left\{\sup_{\|x\|=\|y\|=1}|[T_jx,y]|\right\}\\&=\max_{j\in\{1,\dots,n\}}\{\|T_j\|\}=\|(T_1,\dots,T_n)\|_{n,\infty},\end{aligned}$$

then by (3.27) we get

$$\|(T_1,\dots,T_n)\|_{n,\infty}\le\|(T_1,\dots,T_n)\|_{h,n,q} \tag{3.28}$$

for any $(T_1,\dots,T_n)\in B^{(n)}(X)$.

Since

$$\|(T_1,\dots,T_n)\|_{n,q} := \left(\sum_{j=1}^{n}\|T_j\|^q\right)^{1/q} \le \left(n\,\|(T_1,\dots,T_n)\|_{n,\infty}^q\right)^{1/q} = n^{1/q}\,\|(T_1,\dots,T_n)\|_{n,\infty}\,, \tag{3.29}$$

then by (3.28) and (3.29) we get

$$\frac{1}{n^{1/q}}\,\|(T_1,\dots,T_n)\|_{n,q} \le \|(T_1,\dots,T_n)\|_{h,n,q}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(X)$.

The inequality (3.24) follows in a similar way and we omit the details. □

Corollary 5. *With the assumptions of Theorem 7 we have for $r \ge q \ge 1$ that*

$$\|(T_1,\dots,T_n)\|_{h,n,r} \le \|(T_1,\dots,T_n)\|_{h,n,q} \le n^{\frac{r-q}{rq}}\,\|(T_1,\dots,T_n)\|_{h,n,r} \tag{3.30}$$

and

$$w_{h,n,r}(T_1,\dots,T_n) \le w_{h,n,q}(T_1,\dots,T_n) \le n^{\frac{r-q}{rq}} w_{h,n,r}(T_1,\dots,T_n) \tag{3.31}$$

for any $(T_1,\dots,T_n) \in B^{(n)}(X)$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j, $j = 1,...,n$ and $r \ge q > 0$ (see for instance [22] and [17])

$$\left(\sum_{j=1}^{n} a_j^r\right)^{1/r} \le \left(\sum_{j=1}^{n} a_j^q\right)^{1/q} \le n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n} a_j^r\right)^{1/r}. \tag{3.32}$$

Let $(T_1,\dots,T_n) \in B^{(n)}(X)$ and x, $y \in X$ with $\|x\| = \|y\| = 1$. Then by (3.32) we get

$$\left(\sum_{j=1}^{n} |[T_j x, y]|^r\right)^{1/r} \le \left(\sum_{j=1}^{n} |[T_j x, y]|^q\right)^{1/q} \le n^{\frac{r-q}{rq}}\left(\sum_{j=1}^{n} |[T_j x, y]|^r\right)^{1/r}.$$

By taking the supremum over $\|x\| = \|y\| = 1$ we get (3.30).

The inequality (3.31) follows in a similar way and we omit the details. □

For $q = 2$, we put

$$\|(T_1,\dots,T_n)\|_{h,n,e} := \|(T_1,\dots,T_n)\|_{h,n,2} \text{ and } w_{h,n,e}(T_1,\dots,T_n) :=_{h,n,2} (T_1,\dots,T_n).$$

Remark 3. *For $q \ge 2$ we have by (3.30) and (3.31) that*

$$\|(T_1,\dots,T_n)\|_{h,n,q} \le \|(T_1,\dots,T_n)\|_{h,n,e} \le n^{\frac{q-2}{2q}}\,\|(T_1,\dots,T_n)\|_{h,n,q} \tag{3.33}$$

and

$$w_{h,n,q}(T_1,\dots,T_n) \le w_{h,n,e}(T_1,\dots,T_n) \le n^{\frac{q-2}{2q}} w_{h,n,q}(T_1,\dots,T_n) \tag{3.34}$$

and for $1 \le q \le 2$ we have

$$\|(T_1,\dots,T_n)\|_{h,n,e} \le \|(T_1,\dots,T_n)\|_{h,n,q} \le n^{\frac{2-q}{2q}}\,\|(T_1,\dots,T_n)\|_{h,n,e} \tag{3.35}$$

and

$$w_{h,n,e}(T_1,\dots,T_n) \le w_{h,n,e}(T_1,\dots,T_n) \le n^{\frac{2-q}{2q}} w_{h,n,e}(T_1,\dots,T_n) \tag{3.36}$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$.

Also, if we take $q = 1$ *and* $r \geq 1$ *in (3.30) and (3.31), then we get*

$$\|(T_1, \ldots, T_n)\|_{h,n,r} \leq \|(T_1, \ldots, T_n)\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|(T_1, \ldots, T_n)\|_{h,n,r} \tag{3.37}$$

and

$$w_{h,n,r}(T_1, \ldots, T_n) \leq w_{h,n,1}(T_1, \ldots, T_n) \leq n^{\frac{r-1}{r}} w_{h,n,r}(T_1, \ldots, T_n) \tag{3.38}$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$.

In particular, for $r = 2$ *we get*

$$\|(T_1, \ldots, T_n)\|_{h,n,e} \leq \|(T_1, \ldots, T_n)\|_{h,n,1} \leq \sqrt{n} \|(T_1, \ldots, T_n)\|_{h,n,e} \tag{3.39}$$

and

$$w_{n,e}(T_1, \ldots, T_n) \leq w_{h,n,1}(T_1, \ldots, T_n) \leq \sqrt{n} w_{n,e}(T_1, \ldots, T_n) \tag{3.40}$$

for any $(T_1, \ldots, T_n) \in B^{(n)}(X)$.

We have:

Proposition 3. *For any* $(T_1, \ldots, T_n) \in B^{(n)}(X)$ *and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then we have*

$$\|(T_1, \ldots, T_n)\|_{h,n,q} \geq \frac{1}{n^{1/p}} \left\| \sum_{j=1}^{n} T_j \right\| \tag{3.41}$$

and

$$w_{h,n,q}(T_1, \ldots, T_n) \geq \frac{1}{n^{1/p}} w\left(\sum_{j=1}^{n} T_j \right). \tag{3.42}$$

Proof. Let $\lambda_j = \frac{1}{n^{1/p}}$ for $j \in \{1, ..., n\}$, then $\sum_{j=1}^{n} |\lambda_j|^p = 1$. Therefore by (1.8) we get

$$\|(T_1, \ldots, T_n)\|_{h,n,q} = \sup_{\|\lambda\|_{n,p} \leq 1} \left\| \sum_{j=1}^{n} \lambda_j T_j \right\| \geq \left\| \sum_{j=1}^{n} \frac{1}{n^{1/p}} T_j \right\| = \frac{1}{n^{1/p}} \left\| \sum_{j=1}^{n} T_j \right\|.$$

The inequality (3.42) follows in a similar way. □

We can also introduce the following norms for $(T_1, \ldots, T_n) \in B^{(n)}(X)$,

$$\|(T_1, \ldots, T_n)\|_{s,n,p} := \sup_{\|x\|=1} \left(\sum_{j=1}^{n} \|T_j x\|^p \right)^{1/p},$$

where $p \geq 1$ and

$$\|(T_1, \ldots, T_n)\|_{s,n,\infty} := \sup_{\|x\|=1} \left(\max_{j \in \{1,\ldots,n\}} \|T_j x\| \right) = \max_{j \in \{1,\ldots,n\}} \{\|T_j\|\}.$$

The triangle inequality $\|\cdot\|_{s,n,q}$ follows by Minkowski inequality, while the other properties of the norm are obvious.

Proposition 4. *Let* $(T_1, \ldots, T_n) \in B^{(n)}(X)$. *We have for* $p \geq 1$, *that*

$$\|(T_1, \ldots, T_n)\|_{h,n,p} \leq \|(T_1, \ldots, T_n)\|_{s,n,p} \leq \|(T_1, \ldots, T_n)\|_{n,p}. \tag{3.43}$$

Proof. We have for $p \geq 2$ and $x,\, y \in X$ with $\|x\| = \|y\| = 1$, that

$$|[T_j x, y]|^p \leq \|T_j x\|^p \|y\|^p = \|T_j x\|^p \leq \|T_j\|^p \|x\|^p = \|T_j\|^p$$

for $j \in \{1, ..., n\}$.

This implies

$$\sum_{j=1}^{n} |[T_j x, y]|^p \leq \sum_{j=1}^{n} \|T_j x\|^p \leq \sum_{j=1}^{n} \|T_j\|^p ,$$

namely

$$\left(\sum_{j=1}^{n} |[T_j x, y]|^p \right)^{1/p} \leq \left(\sum_{j=1}^{n} \|T_j x\|^p \right)^{1/p} \leq \left(\sum_{j=1}^{n} \|T_j\|^p \right)^{1/p}, \tag{3.44}$$

for any $x,\, y \in X$ with $\|x\| = \|y\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (3.44), we get the desired result (3.43). □

4. Reverse Inequalities

Recall the following reverse of Cauchy-Buniakowski-Schwarz inequality [1] (see also [4, Theorem 5. 14]):

Lemma 2. *Let $a,\, A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:*

$$ay_j \leq z_j \leq Ay_j \quad \text{for each} \quad j \in \{1, \dots, n\}. \tag{4.1}$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^{n} w_j z_j^2 \sum_{j=1}^{n} w_j y_j^2 - \left(\sum_{j=1}^{n} w_j z_j y_j \right)^2 \leq \frac{1}{4} (A-a)^2 \left(\sum_{j=1}^{n} w_j y_j^2 \right)^2. \tag{4.2}$$

The constant $\frac{1}{4}$ is sharp in (4.2).

O. Shisha and B. Mond obtained in 1967 (see [22]) the following counterparts of (CBS)-inequality (see also [4, Theorem 5.20 & 5.21]):

Lemma 3. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists $a,\, A,\, b,\, B$ with the property that:*

$$0 \leq a \leq a_j \leq A \quad \text{and} \quad 0 < b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\}, \tag{4.3}$$

then we have the inequality

$$\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 - \left(\sum_{j=1}^{n} a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^{n} a_j b_j \sum_{j=1}^{n} b_j^2. \tag{4.4}$$

and

Lemma 4. *Assume that $\mathbf{a}$, $\mathbf{b}$ are nonnegative sequences and there exists $\gamma,\, \Gamma$ with the property that*

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \quad \text{for any} \quad j \in \{1, \dots, n\}. \tag{4.5}$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^{n} a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^{n} b_j^2. \tag{4.6}$$

We have:

Theorem 8. *Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot,\cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $(T_1, \ldots, T_n) \in B^{(n)}(X)$.*

(i) *We have*

$$0 \leq \|(T_1, \ldots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \ldots, T_n)\|_{h,n,1}^2 \leq \frac{1}{4} n \|(T_1, \ldots, T_n)\|_{n,\infty}^2 \tag{4.7}$$

and

$$0 \leq w_{n,e}^2 (T_1, \ldots, T_n) - \frac{1}{n} w_{h,n,1}^2 (T_1, \ldots, T_n) \leq \frac{1}{4} n \|(T_1, \ldots, T_n)\|_{n,\infty}^2. \tag{4.8}$$

(ii) *We have*

$$\begin{aligned} 0 &\leq \|(T_1, \ldots, T_n)\|_{h,n,e}^2 - \frac{1}{n} \|(T_1, \ldots, T_n)\|_{h,n,1}^2 \\ &\leq \|(T_1, \ldots, T_n)\|_{n,\infty} \|(T_1, \ldots, T_n)\|_{h,n,1} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} 0 &\leq w_{n,e}^2 (T_1, \ldots, T_n) - \frac{1}{n} w_{h,n,1}^2 (T_1, \ldots, T_n) \\ &\leq \|(T_1, \ldots, T_n)\|_{n,\infty} w_{h,n,1} (T_1, \ldots, T_n). \end{aligned} \tag{4.10}$$

(iii) *We have*

$$0 \leq \|(T_1, \ldots, T_n)\|_{h,n,e} - \frac{1}{\sqrt{n}} \|(T_1, \ldots, T_n)\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|(T_1, \ldots, T_n)\|_{n,\infty} \tag{4.11}$$

and

$$0 \leq w_{n,e} (T_1, \ldots, T_n) - \frac{1}{\sqrt{n}} w_{h,n,1} (T_1, \ldots, T_n) \leq \frac{1}{4} \sqrt{n} \|(T_1, \ldots, T_n)\|_{n,\infty}. \tag{4.12}$$

Proof. (i). Let $(T_1, \ldots, T_n) \in B^{(n)}(H)$ and put

$$R = \max_{j \in \{1,\ldots,n\}} \{\|T_j\|\} = \|(T_1, \ldots, T_n)\|_{n,\infty}.$$

If $x, y \in H$, with $\|x\| = \|y\| = 1$ then

$$|[T_j x, y]| \leq \|T_j x\| \leq \|T_j\| \leq R$$

for any $j \in \{1, ..., n\}$.

If we write the inequality (3.2) for $z_j = |[T_j x, y]|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^{n} |[T_j x, y]|^2 - \left(\sum_{j=1}^{n} |[T_j x, y]| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $x, y \in H$, with $\|x\| = \|y\| = 1$.

This implies that

$$\sum_{j=1}^{n}|[T_jx,y]|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|[T_jx,y]|\right)^2+\frac{1}{4}nR^2 \tag{4.13}$$

for any x, $y\in H$, with $\|x\|=\|y\|=1$ and, in particular

$$\sum_{j=1}^{n}|[T_jx,x]|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|[T_jx,x]|\right)^2+\frac{1}{4}nR^2 \tag{4.14}$$

for any $x\in H$, with $\|x\|=1$.

Taking the supremum over $\|x\|=\|y\|=1$ in (4.13) and $\|x\|=1$ in (4.14), then we get (4.7) and (4.8).

(ii). Let $(T_1,\dots,T_n)\in B^{(n)}(H)$. If we write the inequality (3.4) for $a_j=|[T_jx,y]|$, $b_j=1$, $b=B=1$, $a=0$ and $A=R$, then we get

$$0\le n\sum_{j=1}^{n}|[T_jx,y]|^2-\left(\sum_{j=1}^{n}|[T_jx,y]|\right)^2\le nR\sum_{j=1}^{n}|[T_jx,y]|,$$

for any x, $y\in H$, with $\|x\|=\|y\|=1$.

This implies that

$$\sum_{j=1}^{n}|[T_jx,y]|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|[T_jx,y]|\right)^2+R\sum_{j=1}^{n}|[T_jx,y]|, \tag{4.15}$$

for any x, $y\in H$, with $\|x\|=\|y\|=1$ and, in particular

$$\sum_{j=1}^{n}|[T_jx,x]|^2 \le \frac{1}{n}\left(\sum_{j=1}^{n}|[T_jx,x]|\right)^2+R\sum_{j=1}^{n}|[T_jx,x]|, \tag{4.16}$$

for any $x\in H$ with $\|x\|=1$.

Taking the supremum over $\|x\|=\|y\|=1$ in (4.15) and $\|x\|=1$ in (4.16), then we get (4.9) and (4.10).

(iii). If we write the inequality (4.6) for $a_j=|[T_jx,y]|$, $b_j=1$, $b=B=1$, $\gamma=0$ and $\Gamma=R$ we have

$$0\le\left(n\sum_{j=1}^{n}|[T_jx,y]|^2\right)^{\frac{1}{2}}-\sum_{j=1}^{n}|[T_jx,y]|\le\frac{1}{4}nR,$$

for any x, $y\in H$, with $\|x\|=\|y\|=1$.

This implies that

$$\left(\sum_{j=1}^{n}|[T_jx,y]|^2\right)^{\frac{1}{2}}\le\frac{1}{\sqrt{n}}\sum_{j=1}^{n}|[T_jx,y]|+\frac{1}{4}\sqrt{n}R, \tag{4.17}$$

for any x, $y\in H$, with $\|x\|=\|y\|=1$ and, in particular

$$\left(\sum_{j=1}^{n}|[T_jx,x]|^2\right)^{\frac{1}{2}}\le\frac{1}{\sqrt{n}}\sum_{j=1}^{n}|[T_jx,x]|+\frac{1}{4}\sqrt{n}R, \tag{4.18}$$

for any $x \in H$ with $\|x\| = 1$.

Taking the supremum over $\|x\| = \|y\| = 1$ in (4.17) and $\|x\| = 1$ in (4.18), then we get (4.11) and (4.12). □

Before we proceed with establishing some reverse inequalities for the hypo-Euclidean numerical radius, we recall some reverse results of the Cauchy-Bunyakovsky-Schwarz inequality for complex numbers as follows:

If $\gamma, \Gamma \in \mathbb{C}$ and $\alpha_j \in \mathbb{C}$, $j \in \{1, \dots, n\}$ with the property that

$$0 \le \operatorname{Re}\left[(\Gamma - \alpha_j)\left(\overline{\alpha_j} - \bar{\gamma}\right)\right] = (\operatorname{Re}\Gamma - \operatorname{Re}\alpha_j)(\operatorname{Re}\alpha_j - \operatorname{Re}\gamma) + (\operatorname{Im}\Gamma - \operatorname{Im}\alpha_j)(\operatorname{Im}\alpha_j - \operatorname{Im}\gamma) \tag{4.19}$$

or, equivalently,

$$\left|\alpha_j - \frac{\gamma + \Gamma}{2}\right| \le \frac{1}{2}\left|\Gamma - \gamma\right| \tag{4.20}$$

for each $j \in \{1, \dots, n\}$, then (see for instance [5, p. 9])

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \le \frac{1}{4}n^2|\Gamma - \gamma|^2. \tag{4.21}$$

In addition, if $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$, then (see for example [5, p. 26]):

$$n\sum_{j=1}^{n}|\alpha_j|^2 \le \frac{1}{4}\frac{\left\{\operatorname{Re}\left[\left(\bar{\Gamma} + \bar{\gamma}\right)\sum_{j=1}^{n}\alpha_j\right]\right\}^2}{\operatorname{Re}(\Gamma\bar{\gamma})} \le \frac{1}{4}\frac{|\Gamma + \gamma|^2}{\operatorname{Re}(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}\alpha_j\right|^2. \tag{4.22}$$

Also, if $\Gamma \ne -\gamma$, then (see for instance [5, p. 32]):

$$\left(n\sum_{j=1}^{n}|\alpha_j|^2\right)^{\frac{1}{2}} - \left|\sum_{j=1}^{n}\alpha_j\right| \le \frac{1}{4}n\frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|}. \tag{4.23}$$

Finally, from [7] we can also state that

$$n\sum_{j=1}^{n}|\alpha_j|^2 - \left|\sum_{j=1}^{n}\alpha_j\right|^2 \le n\left[|\Gamma + \gamma| - 2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}\right]\left|\sum_{j=1}^{n}\alpha_j\right|, \tag{4.24}$$

provided $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$.

We notice that a simple sufficient condition for (4.19) to hold is that

$$\operatorname{Re}\Gamma \ge \operatorname{Re}\alpha_j \ge \operatorname{Re}\gamma \quad \text{and} \quad \operatorname{Im}\Gamma \ge \operatorname{Im}\alpha_j \ge \operatorname{Im}\gamma \tag{4.25}$$

for each $j \in \{1, \dots, n\}$.

Theorem 9. *Let $(X, \|\cdot\|)$ be a Banach space, $[\cdot, \cdot]$ a s-L-G-s.i.p. that generates the norm $\|\cdot\|$ of X and $\gamma, \Gamma \in \mathbb{C}$ with $\Gamma \ne \gamma$. Assume that*

$$w\left(T_j - \frac{\gamma + \Gamma}{2}I\right) \le \frac{1}{2}|\Gamma - \gamma| \quad \text{for any } j \in \{1, \dots, n\}. \tag{4.26}$$

(i) *We have*

$$w_{h,n,e}^{2}\left(T_{1},\ldots,T_{n}\right)\leq\frac{1}{n}w^{2}\left(\sum_{j=1}^{n}T_{j}\right)+\frac{1}{4}n\left|\Gamma-\gamma\right|^{2}. \tag{4.27}$$

(ii) *If* $\operatorname{Re}(\Gamma\bar{\gamma})>0$, *then*

$$w_{h,n,e}\left(T_{1},\ldots,T_{n}\right)\leq\frac{1}{2\sqrt{n}}\frac{\left|\Gamma+\gamma\right|}{\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}}w\left(\sum_{j=1}^{n}T_{j}\right) \tag{4.28}$$

and

$$\begin{aligned}w_{h,n,e}^{2}\left(T_{1},\ldots,T_{n}\right)\leq&\left[\frac{1}{n}w\left(\sum_{j=1}^{n}T_{j}\right)+\left[\left|\Gamma+\gamma\right|-2\sqrt{\operatorname{Re}\left(\Gamma\bar{\gamma}\right)}\right]\right]\\&\times w\left(\sum_{j=1}^{n}T_{j}\right).\end{aligned} \tag{4.29}$$

(iii) *If* $\Gamma\neq-\gamma$, *then*

$$w_{h,n,e}\left(T_{1},\ldots,T_{n}\right)\leq\frac{1}{\sqrt{n}}\left(w\left(\sum_{j=1}^{n}T_{j}\right)+\frac{1}{4}\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\right). \tag{4.30}$$

Proof. Let $x\in H$ with $\|x\|=1$ and $(T_1,\ldots,T_n)\in B^{(n)}(H)$ with the property (4.26). By taking $\alpha_j=[T_jx,x]$ we have

$$\begin{aligned}\left|\alpha_{j}-\frac{\gamma+\Gamma}{2}\right|&=\left|[T_{j}x,x]-\frac{\gamma+\Gamma}{2}[x,x]\right|=\left|\left[\left(T_{j}-\frac{\gamma+\Gamma}{2}I\right)x,x\right]\right|\\&\leq\sup_{\|x\|=1}\left|\left[\left(T_{j}-\frac{\gamma+\Gamma}{2}I\right)x,x\right]\right|=w\left(T_{j}-\frac{\gamma+\Gamma}{2}\right)\\&\leq\frac{1}{2}\left|\Gamma-\gamma\right|\end{aligned}$$

for any $j\in\{1,\ldots,n\}$.

(i) By using the inequality (4.21), we have

$$\begin{aligned}\sum_{j=1}^{n}\left|[T_{j}x,x]\right|^{2}&\leq\frac{1}{n}\left|\sum_{j=1}^{n}[T_{j}x,x]\right|^{2}+\frac{1}{4}n\left|\Gamma-\gamma\right|^{2}\\&=\frac{1}{n}\left|\left[\sum_{j=1}^{n}T_{j}x,x\right]\right|^{2}+\frac{1}{4}n\left|\Gamma-\gamma\right|^{2}\end{aligned} \tag{4.31}$$

for any $x\in H$ with $\|x\|=1$.

By taking the supremum over $\|x\| = 1$ in (4.31) we get

$$\sup_{\|x\|=1}\left(\sum_{j=1}^{n}\left|[T_jx,x]\right|^2\right) \le \frac{1}{n}\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|^2 + \frac{1}{4}n\left|\Gamma-\gamma\right|^2$$
$$= \frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right) + \frac{1}{4}n\left|\Gamma-\gamma\right|^2,$$

which proves (4.27).

(ii) If $\mathrm{Re}\,(\Gamma\bar{\gamma}) > 0$, then by (4.22) we have for $\alpha_j = [T_jx,x]$, $j \in \{1,\dots,n\}$ that

$$\sum_{j=1}^{n}\left|[T_jx,x]\right|^2 \le \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\mathrm{Re}\,(\Gamma\bar{\gamma})}\left|\sum_{j=1}^{n}[T_jx,x]\right|^2 = \frac{1}{4n}\frac{|\Gamma+\gamma|^2}{\mathrm{Re}\,(\Gamma\bar{\gamma})}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|^2 \tag{4.32}$$

for any $x \in H$ with $\|x\| = 1$.

On taking the supremum over $\|x\| = 1$ in (4.32) we get (4.32).

Also, by (4.24) we get

$$\sum_{j=1}^{n}\left|[T_jx,x]\right|^2 \le \frac{1}{n}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\right]\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|,$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we have

$$\sup_{\|x\|=1}\sum_{j=1}^{n}\left|[T_jx,x]\right|^2$$
$$\le \sup_{\|x\|=1}\left[\frac{1}{n}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\right]\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|\right]$$
$$\le \frac{1}{n}\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|^2 + \left[|\Gamma+\gamma| - 2\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\right]\sup_{\|x\|=1}\left|\left[\sum_{j=1}^{n}T_jx,x\right]\right|$$
$$= \frac{1}{n}w^2\left(\sum_{j=1}^{n}T_j\right) + \left[|\Gamma+\gamma| - 2\sqrt{\mathrm{Re}\,(\Gamma\bar{\gamma})}\right]w\left(\sum_{j=1}^{n}T_j\right),$$

which proves (4.29).

(iii) By the inequality (4.23) we have

$$\left(\sum_{j=1}^{n}\left|[T_j x,x]\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}}\left(\left|\sum_{j=1}^{n}[T_j x,x]\right| + \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\Gamma+\gamma\right|}\right) = \frac{1}{\sqrt{n}}\left(\left|\left[\sum_{j=1}^{n}T_j x,x\right]\right| + \frac{1}{4}\frac{\left|\Gamma-\gamma\right|^2}{\left|\Gamma+\gamma\right|}\right)$$

for any $x \in H$ with $\|x\| = 1$.

By taking the supremum over $\|x\| = 1$ in this inequality, we get (4.30). □

Remark 4. *By the use of the elementary inequality $w(T) \leq \|T\|$ that holds for any $T \in B(X)$, a sufficient condition for (4.26) to hold is that*

$$\left\|T_j - \frac{\gamma+\Gamma}{2}\right\| \leq \frac{1}{2}\left|\Gamma-\gamma\right| \text{ for any } j \in \{1,\ldots,n\}. \tag{4.33}$$

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