Received 01/03/17

TRACE INEQUALITIES OF JENSEN TYPE FOR SELFADJOINT OPERATORS IN HILBERT SPACES: A SURVEY OF RECENT RESULTS

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Abstract. Some new trace inequalities for convex functions of selfadjoint operators in Hilbert spaces are surveyed. Reverse Jensen's type trace inequalities and some trace inequalities of Slater's type for convex functions of selfadjoint operators in Hilbert spaces under suitable assumptions for the involved operators are given. Applications for some convex functions of interest and reverses of Hölder and Schwarz trace inequalities are also provided. The superadditivity and monotonicity of some associated functionals are investigated. Some trace inequalities for matrices are also derived. Examples for the operator power and logarithm are presented as well.

1. Introduction

1.1. **Jensen's Inequality.** Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $\mathrm{Sp}(A)$ included in the interval $[m, M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m, M] \to \mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [36, p. 257]):

$$\langle f(A)x, y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x, y\rangle) \tag{1.1}$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2\, d\|E_\lambda x\|^2, \tag{1.2}$$

for any x, $y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y\rangle$ is of *bounded variation* on the interval $[m, M]$ and

$$g_{x,y}(m-0) = 0 \text{ while } g_{x,y}(M) = \langle x, y\rangle$$

for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m, M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality may be found, for instance, in Mond & Pečarić [43] (see also [35, p. 5]):

1991 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Trace class operators, Hilbert-Schmidt operators, Trace, Convex functions, Jensen's inequality, Slater's inequality, Trace inequalities for matrices, Operator power, Operator logarithm, Hölder and Schwarz trace inequalities.

RGMIA Res. Rep. Coll. 20 (2017), Art. 20, 52 pp.

Theorem 1. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$f(\langle Ax, x\rangle) \leq \langle f(A) x, x\rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 1 we have the following Hölder-McCarthy inequality:

Theorem 2 (McCarthy, 1967, [41])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*
(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse for (MP) that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [35, p. 57]:

Theorem 3. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If f is a convex function on $[m, M]$, then*

$$\langle f(A) x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m} f(m) + \frac{\langle Ax, x\rangle - m}{M - m} f(M) \tag{LR}$$

for each $x \in H$ with $\|x\| = 1$.

The following result that provides a reverse of the Jensen inequality has been obtained in [20]:

Theorem 4 (Dragomir, 2008, [20])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$(0 \leq) \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \leq \langle f'(A) Ax, x\rangle - \langle Ax, x\rangle \langle f'(A) x, x\rangle, \tag{1.3}$$

for any $x \in H$ with $\|x\| = 1$.

Perhaps more convenient reverses of (MP) are the following inequalities that have been obtained in the same paper [20]:

Theorem 5 (Dragomir, 2008, [20])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operators on the Hilbert space H with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$, *then*

$$\begin{aligned}
&(0 \leq) \langle f(A) x, x\rangle - f(\langle Ax, x\rangle) \\
&\leq \begin{cases} \frac{1}{2}(M - m)\left[\|f'(A) x\|^2 - \langle f'(A) x, x\rangle^2\right]^{1/2} \\ \\ \frac{1}{2}(f'(M) - f'(m))\left[\|Ax\|^2 - \langle Ax, x\rangle^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}(M - m)(f'(M) - f'(m)),
\end{aligned} \tag{1.4}$$

for any $x \in H$ *with* $\|x\| = 1$.

We also have the inequality

$$
\begin{aligned}
(1.5)\qquad & (0 \le) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\
& \le \frac{1}{4}(M-m)(f'(M) - f'(m)) \\
& - \begin{cases} [\langle Mx - Ax, Ax - mx\rangle \langle f'(M)x - f'(A)x, f'(A)x - f'(m)x\rangle]^{\frac{1}{2}}, \\ \left|\langle Ax, x\rangle - \frac{M+m}{2}\right| \left|\langle f'(A)x, x\rangle - \frac{f'(M)+f'(m)}{2}\right| \end{cases} \\
& \le \frac{1}{4}(M-m)(f'(M) - f'(m)),
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

Moreover, if $m > 0$ *and* $f'(m) > 0$, *then we also have*

$$
\begin{aligned}
(1.6)\qquad & (0 \le) \langle f(A)x, x\rangle - f(\langle Ax, x\rangle) \\
& \le \begin{cases} \frac{1}{4}\frac{(M-m)\left(f'(M)-f'(m)\right)}{\sqrt{Mmf'(M)f'(m)}} \langle Ax, x\rangle \langle f'(A)x, x\rangle, \\ \left(\sqrt{M} - \sqrt{m}\right)\left(\sqrt{f'(M)} - \sqrt{f'(m)}\right)[\langle Ax, x\rangle \langle f'(A)x, x\rangle]^{\frac{1}{2}}, \end{cases}
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

In [21] we obtained the following operator version for Slater's inequality as well as a reverse of it:

Theorem 6 (Dragomir, 2008, [21])**.** *Let* I *be an interval and* $f : I \to \mathbb{R}$ *be a convex and differentiable function on* $\mathring{I}$ *(the interior of* I*) whose derivative* f' *is continuous on* $\mathring{I}$. *If* A *is a selfadjoint operator on the Hilbert space* H *with* $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ *and* $f'(A)$ *is a positive invertible operator on* H *then*

$$
\begin{aligned}
(1.7)\qquad 0 & \le f\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right) - \langle f(A)x, x\rangle \\
& \le f'\left(\frac{\langle Af'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right)\left[\frac{\langle Af'(A)x, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle}{\langle f'(A)x, x\rangle}\right],
\end{aligned}
$$

for any $x \in H$ *with* $\|x\| = 1$.

For other similar results, see [21].

For some inequalities for convex functions see [8]-[13], [33] and [49]. For inequalities for functions of selfadjoint operators, see [15]-[24], [40], [42], [43], [44], [45] and the books [25], [26] and [35].

In order to state our results concerning some trace inequalities for convex functions of selfadjoint operators on Hilbert space $(H, \langle ., .\rangle)$ we need some preparations as follows.

1.2. **Traces for Operators in Hilbert Spaces.** Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\{e_i\}_{i\in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$
(1.8)\qquad \sum_{i\in I} \|Ae_i\|^2 < \infty.
$$

It is well know that, if $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i\in I} \|Ae_i\|^2 = \sum_{j\in I} \|Af_j\|^2 = \sum_{j\in I} \|A^* f_j\|^2 \tag{1.9}$$

showing that the definition (1.8) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i\in I} \|Ae_i\|^2\right)^{1/2} \tag{1.10}$$

for $\{e_i\}_{i\in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm.*

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$. Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (1.9) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 7. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i\in I} \langle Ae_i, Be_i\rangle = \sum_{i\in I} \langle B^*Ae_i, e_i\rangle \tag{1.11}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{1.12}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{1.13}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i\in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{1.14}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 1. *If $A \in \mathcal{B}(H)$, then the following are equivalent:*
(i) $A \in \mathcal{B}_1(H)$;
(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;
(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 8. *With the above notations:*
(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{1.15}$$

for any $A \in \mathcal{B}_1(H)$;
(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{|\langle A, B\rangle_2| \mid B \in \mathcal{B}_2(H), \ \|B\| \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*
(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i \in I} \langle Ae_i, e_i\rangle, \tag{1.16}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (1.16) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 9. *We have*
(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{1.17}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{1.18}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;
(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;
(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [50]. For some classical trace inequalities see [5], [7], [46] and [56], which are continuations of the work of Bellman [2]. For related works the reader can refer to [1], [3], [5], [34], [37], [38], [39], [47] and [53].

2. Jensen's Type Trace Inequalities

2.1. **Some Trace Inequalities for Convex Functions.** Consider the orthonormal basis $\mathcal{E} := \{e_i\}_{i\in I}$ in the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and for a nonzero operator $B \in \mathcal{B}_2(H)$ let introduce the subset of indices from I defined by

$$I_{\mathcal{E},B} := \{i \in I : \ Be_i \neq 0\}.$$

We observe that $I_{\mathcal{E},B}$ is non-empty for any nonzero operator B and if $\ker(B) = 0$, i.e. B is injective, then $I_{\mathcal{E},B} = I$. We also have for $B \in \mathcal{B}_2(H)$ that

$$\operatorname{tr}\left(|B|^2\right) = \operatorname{tr}(B^*B) = \sum_{i\in I}\langle B^*Be_i, e_i\rangle = \sum_{i\in I}\|Be_i\|^2 = \sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2.$$

Theorem 10 (Dragomir, 2014 [29])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $B \in \mathcal{B}_2(H) \setminus \{0\}$, then $\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)} \in [m, M]$ and*

$$\begin{aligned}(2.1)\qquad & f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \\ & \leq J_\varepsilon(f; A, B) \leq \operatorname{tr}\left(|B|^2 f(A)\right) \\ & \leq \frac{1}{M-m}\left(f(m)\operatorname{tr}\left[|B|^2(M1_H - A)\right] + f(M)\operatorname{tr}\left[|B|^2(A - m1_H)\right]\right),\end{aligned}$$

where

$$J_\varepsilon(f; A, B) := \sum_{i\in I_{\mathcal{E},B}} f\left(\frac{\langle B^*ABe_i, e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2. \tag{2.2}$$

Proof. Since $\operatorname{Sp}(A) \subseteq [m, M]$, then $m\|y\|^2 \leq \langle Ay, y\rangle \leq M\|y\|^2$ for any $y \in H$. Therefore

$$m\|Be_i\|^2 \leq \langle ABe_i, Be_i\rangle \leq M\|Be_i\|^2,$$

for any $i \in I$, which implies that

$$m\sum_{i\in I}\|Be_i\|^2 \leq \sum_{i\in I}\langle ABe_i, Be_i\rangle \leq M\sum_{i\in I}\|Be_i\|^2$$

and we conclude that $\frac{\operatorname{tr}(|B|^2A)}{\operatorname{tr}(|B|^2)} \in [m, M]$.

By Jensen's inequality (MP) we have

$$f\left(\frac{\langle Ay, y\rangle}{\|y\|^2}\right) \leq \frac{\langle f(A)y, y\rangle}{\|y\|^2} \tag{2.3}$$

for any $y \in H \setminus \{0\}$.

Let F be a finite part of $I_{\mathcal{E},B}$. Then for any $i \in F$ we have from (2.3) that

$$f\left(\frac{\langle ABe_i, Be_i\rangle}{\|Be_i\|^2}\right) \leq \frac{\langle f(A)Be_i, Be_i\rangle}{\|Be_i\|^2},$$

which is equivalent to

$$f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\le\langle B^*f(A)Be_i,e_i\rangle. \tag{2.4}$$

Summing over $i\in F$ we get

$$\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\le\sum_{i\in F}\langle B^*f(A)Be_i,e_i\rangle. \tag{2.5}$$

Using Jensen's discrete inequality for finite sums and for the positive weights w_i

$$f\left(\frac{\sum_{i\in F}w_iu_i}{\sum_{i\in F}w_i}\right)\le\frac{\sum_{i\in F}w_if(u_i)}{\sum_{i\in F}w_i},$$

we have

$$f\left(\frac{\sum_{i\in F}\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2}\right)\le\frac{\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2}{\sum_{i\in F}\|Be_i\|^2},$$

which is equivalent to

$$f\left(\frac{\sum_{i\in F}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2\le\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2. \tag{2.6}$$

Therefore, for any F a finite part of $I_{\mathcal{E},B}$ we have from (2.5) that

$$\begin{aligned}f\left(\frac{\sum_{i\in F}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in F}\|Be_i\|^2}\right)\sum_{i\in F}\|Be_i\|^2&\le\sum_{i\in F}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\\&\le\sum_{i\in F}\langle B^*f(A)Be_i,e_i\rangle.\end{aligned} \tag{2.7}$$

By the continuity of f we then have from (2.7) that

$$\begin{aligned}&f\left(\frac{\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i,e_i\rangle}{\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2}\right)\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2\\&\le\sum_{i\in I_{\mathcal{E},B}}f\left(\frac{\langle B^*ABe_i,e_i\rangle}{\|Be_i\|^2}\right)\|Be_i\|^2\le\sum_{i\in I_{\mathcal{E},B}}\langle B^*f(A)Be_i,e_i\rangle\end{aligned} \tag{2.8}$$

and since $B\in\mathcal{B}_2(H)\setminus\{0\}$, then also

$$\sum_{i\in I_{\mathcal{E},B}}\|Be_i\|^2=\sum_{i\in I}\|Be_i\|^2=\operatorname{tr}\left(|B|^2\right),$$

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^*ABe_i,e_i\rangle=\sum_{i\in I}\langle B^*ABe_i,e_i\rangle=\operatorname{tr}\left(|B|^2A\right)$$

and

$$\sum_{i\in I_{\mathcal{E},B}}\langle B^*f(A)Be_i,e_i\rangle=\sum_{i\in I}\langle B^*f(A)Be_i,e_i\rangle=\operatorname{tr}\left(|B|^2f(A)\right).$$

From (2.8) we then get the first and the second inequality in (2.1).

From (LR) we also have

$$\langle f(A)y,y\rangle\le\frac{1}{M-m}\left[\langle(M1_H-A)y,y\rangle f(m)+\langle(A-m1_H)y,y\rangle f(M)\right] \tag{2.9}$$

for any $y \in H$.

This implies that

$$\langle f(A) Be_i, Be_i \rangle \leq \frac{1}{M-m} \left[\langle (M1_H - A) Be_i, Be_i \rangle f(m) + \langle (A - m1_H) Be_i, Be_i \rangle f(M) \right] \tag{2.10}$$

for any $i \in I$.

By summation we have

$$\sum_{i\in I} \langle f(A) Be_i, Be_i \rangle \leq \frac{1}{M-m} \left[f(m) \sum_{i\in I} \langle (M1_H - A) Be_i, Be_i \rangle + f(M) \sum_{i\in I} \langle (A - m1_H) Be_i, Be_i \rangle \right]$$

and the last part of (2.1) is proved. □

Remark 1. *We observe that the quantities*

$$J_s(f; A, B) = \sup_{\mathcal{E}} J_{\mathcal{E}}(f; A, B) \text{ and } J_i(f; A, B) = \inf_{\mathcal{E}} J_{\mathcal{E}}(f; A, B)$$

are finite and satisfy the bounds

$$f\left(\frac{\operatorname{tr}\left(|B|^2 A \right)}{\operatorname{tr}\left(|B|^2 \right)} \right) \operatorname{tr}\left(|B|^2 \right) \leq J_i(f; A, B) \leq J_s(f; A, B) \leq \operatorname{tr}\left(|B|^2 f(A) \right). \tag{2.11}$$

We have the following version for nonnegative operators $P \geq 0$, i.e. P satisfies the condition $\langle Px, x \rangle \geq 0$ for any $x \in H$.

Corollary 1 (Dragomir, 2014 [29])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m, M]$ and*

$$\begin{aligned} f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \right) \operatorname{tr}(P) &\leq K_{\mathcal{E}}(f; A, P) \leq \operatorname{tr}(Pf(A)) \\ &\leq \frac{1}{M-m} \left(f(m) \operatorname{tr}[P(M1_H - A)] + f(M) \operatorname{tr}[P(A - m1_H)] \right), \end{aligned} \tag{2.12}$$

where

$$K_{\mathcal{E}}(f; A, P) := \sum_{i\in I_{\mathcal{E},P}} f\left(\frac{\langle P^{1/2} A P^{1/2} e_i, e_i \rangle}{\langle Pe_i, e_i \rangle} \right) \langle Pe_i, e_i \rangle$$

and

$$I_{\mathcal{E},P} := \left\{ i \in I : \; P^{1/2} e_i \neq 0 \right\}$$

Moreover, the quantities

$$K_i(f; A, P) := \inf_{\mathcal{E}} K_{\mathcal{E}}(f; A, P) \text{ and } K_s(f; A, P) := \sup_{\mathcal{E}} K_{\mathcal{E}}(f; A, P)$$

are finite and satisfy the bounds

$$f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\operatorname{tr}(P)\leq K_i(f;A,P)\leq K_s(f;A,P)\leq \operatorname{tr}(Pf(A)). \tag{2.13}$$

The finite dimensional case is of interest.

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements.

Corollary 2 (Dragomir, 2014 [29])**.** *Let $A\in\mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $m<M$. If f is a continuous convex function on $[m,M]$, $\mathcal{E}:=\{e_i\}_{i\in\{1,\dots,n\}}$ is an orthonormal basis in $\mathbb{C}^n$, then $\frac{1}{n}\operatorname{tr}(A)\in[m,M]$ and*

$$\begin{aligned} nf\left(\frac{\operatorname{tr}(A)}{n}\right) &\leq J_\varepsilon(f;A)\leq \operatorname{tr}(f(A)) \\ &\leq \frac{1}{M-m}\left[f(m)\operatorname{tr}(MI_n-A)+f(M)\operatorname{tr}(A-mI_n)\right], \end{aligned} \tag{2.14}$$

where

$$J_\varepsilon(f;A):=\sum_{i=1}^n f(\langle Ae_i,e_i\rangle),$$

and I_n is the identity matrix in $\mathcal{M}_n(\mathbb{C})$.

Remark 2. *The second inequality in (2.14), namely*

$$\sum_{i=1}^n f(\langle Ae_i,e_i\rangle)\leq \operatorname{tr}(f(A))$$

for any $\{e_i\}_{i\in\{1,\dots,n\}}$ an orthonormal basis in $\mathbb{C}^n$, is known in literature as Peierls Inequality. For a different proof and some applications, see, for instance [4]*.*

2.2. **Some Functional Properties.** If we denote by $\mathcal{B}_1^+(H)$ the convex cone of nonnegative operators from $\mathcal{B}_1(H)$ we can consider the functional $\sigma_{f,A}:\mathcal{B}_1^+(H)\setminus\{0\}\to[0,\infty)$ defined by

$$\sigma_{f,A}(P):=\operatorname{tr}(Pf(A))-\operatorname{tr}(P)f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\geq 0, \tag{2.15}$$

where A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M $(m<M)$ and f is a continuous convex function on $[m,M]$.

One can easily observe that, if f is a continuous strictly convex function on $[m,M]$, then the inequality is strict in (2.15).

Theorem 11 (Dragomir, 2014 [29])**.** *Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $m<M$ and f is a continuous convex function on $[m,M]$.*

(i) For any P, $Q\in\mathcal{B}_1^+(H)\setminus\{0\}$ we have

$$\sigma_{f,A}(P+Q)\geq\sigma_{f,A}(P)+\sigma_{f,A}(Q)\,(\geq 0), \tag{2.16}$$

i.e. $\sigma_{f,A}(\cdot)$ is a superadditive functional on $\mathcal{B}_1^+(H)\setminus\{0\}$;

(ii) For any P, $Q\in\mathcal{B}_1^+(H)\setminus\{0\}$ with $P\geq Q$ we have

$$\sigma_{f,A}(P)\geq\sigma_{f,A}(Q)\,(\geq 0), \tag{2.17}$$

i.e. $\sigma_{f,A}(\cdot)$ is a monotonic nondecreasing functional on $\mathcal{B}_1^+(H)\setminus\{0\}$;

(iii) If there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma Q \geq P \geq \gamma Q$ with P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\Gamma \sigma_{f,A}(Q) \geq \sigma_{f,A}(P) \geq \gamma \sigma_{f,A}(Q) \, (\geq 0) . \tag{2.18}$$

Proof. (i) Let P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. Then we have

$$\begin{aligned} \sigma_{f,A}(P+Q) &= \operatorname{tr}((P+Q) f(A)) - \operatorname{tr}(P+Q) f\left(\frac{\operatorname{tr}((P+Q)A)}{\operatorname{tr}(P+Q)}\right) \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)] f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right). \end{aligned} \tag{2.19}$$

By the convexity of f we have

$$\begin{aligned} f\left(\frac{\operatorname{tr}(PA) + \operatorname{tr}(QA)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) &= f\left(\frac{\operatorname{tr}(P) \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} + \operatorname{tr}(Q) \frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)}\right) \\ &\leq \frac{\operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)}. \end{aligned} \tag{2.20}$$

Making use of (2.19) and (2.20) we have

$$\begin{aligned} \sigma_{f,A}(P+Q) &\geq \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - [\operatorname{tr}(P) + \operatorname{tr}(Q)] \frac{\operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) + \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \operatorname{tr}(Pf(A)) + \operatorname{tr}(Pf(A)) \\ &\quad - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \operatorname{tr}(Q) f\left(\frac{\operatorname{tr}(QA)}{\operatorname{tr}(Q)}\right) \\ &= \sigma_{f,A}(P) + \sigma_{f,A}(Q) \end{aligned}$$

and the inequality (2.16) is proved.

(ii) Let P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ with $P \geq Q$. Then on applying the superadditivity property of $\sigma_{f,A}$ for $P - Q \geq 0$ and $Q \geq 0$ we have

$$\sigma_{f,A}(P) = \sigma_{f,A}(P - Q + Q) \geq \sigma_{f,A}(P - Q) + \sigma_{f,A}(Q) \geq \sigma_{f,A}(Q)$$

and the inequality (2.17) is proved.

(iii) If $P \geq \gamma Q$, then by the monotonicity property of $\sigma_{f,A}$ we have

$$\sigma_{f,A}(P) \geq \sigma_{f,A}(\gamma Q) = \gamma \sigma_{f,A}(Q)$$

and a similar inequality for Γ. □

We have the following particular case of interest:

Corollary 3. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma I_n \geq P \geq \gamma I_n$ with P*

positive definite, where I_n is the identity matrix, then

$$\Gamma\left[\operatorname{tr}(f(A))-nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right]\geq\operatorname{tr}(Pf(A))-\operatorname{tr}(P)f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)$$
$$\geq\gamma\left[\operatorname{tr}(f(A))-nf\left(\frac{\operatorname{tr}(A)}{n}\right)\right](\geq 0). \tag{2.21}$$

The following result also holds:

Theorem 12 (Dragomir, 2014 [29])**.** *Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $m<M$ and f is a continuous convex function on $[m,M]$. For $p\geq 1$, the functional $\psi_{p,f,A}:\mathcal{B}_1^+(H)\setminus\{0\}\to[0,\infty)$ defined by*

$$\psi_{p,f,A}(P):=[\operatorname{tr}(P)]^{1-\frac{1}{p}}\sigma_{f,A}(P)$$

is superadditive on $\mathcal{B}_1^+(H)\setminus\{0\}$.

Proof. First of all we observe that the following elementary inequality holds:

$$(\alpha+\beta)^p\geq(\leq)\,\alpha^p+\beta^p \tag{2.22}$$

for any $\alpha,\beta\geq 0$ and $p\geq 1$ $(0<p<1)$.

Indeed, if we consider the function $f_p:[0,\infty)\to\mathbb{R}$, $f_p(t)=(t+1)^p-t^p$ we have $f_p'(t)=p\left[(t+1)^{p-1}-t^{p-1}\right]$. Observe that for $p>1$ and $t>0$ we have that $f_p'(t)>0$ showing that f_p is strictly increasing on the interval $[0,\infty)$. Now for $t=\frac{\alpha}{\beta}$ $(\beta>0,\alpha\geq 0)$ we have $f_p(t)>f_p(0)$ giving that $\left(\frac{\alpha}{\beta}+1\right)^p-\left(\frac{\alpha}{\beta}\right)^p>1$, i.e., the desired inequality (2.22).

For $p\in(0,1)$ we have that f_p is strictly decreasing on $[0,\infty)$ which proves the second case in (2.22).

Now, since $\sigma_{f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^+(H)\setminus\{0\}$ and $p\geq 1$ then by (2.22) we have

$$\sigma_{f,A}^p(P+Q)\geq[\sigma_{f,A}(P)+\sigma_{f,A}(Q)]^p\geq\sigma_{f,A}^p(P)+\sigma_{f,A}^p(Q) \tag{2.23}$$

for any P, $Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Utilising (2.23) and the additivity property of $\operatorname{tr}(\cdot)$ on $\mathcal{B}_1^+(H)\setminus\{0\}$ we have

$$\begin{aligned}\frac{\sigma_{f,A}^p(P+Q)}{\operatorname{tr}(P+Q)}&\geq\frac{\sigma_{f,A}^p(P)+\sigma_{f,A}^p(Q)}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\\&=\frac{\operatorname{tr}(P)\frac{\sigma_{f,A}^p(P)}{\operatorname{tr}(P)}+\operatorname{tr}(Q)\frac{\sigma_{f,A}^p(Q)}{\operatorname{tr}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\\&=\frac{\operatorname{tr}(P)\left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)}\right)^p+\operatorname{tr}(Q)\left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}\right)^p}{\operatorname{tr}(P)+\operatorname{tr}(Q)}=:I,\end{aligned} \tag{2.24}$$

for any P, $Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Since for $p \geq 1$ the power function $g(t) = t^p$ is convex, then

$$I \geq \left(\frac{\operatorname{tr}(P) \frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/p}(P)} + \operatorname{tr}(Q) \frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/q}(Q)}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \right)^p = \left(\frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P) + \operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)} \right)^p \tag{2.25}$$

for any $P,\ Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

By combining (2.24) with (2.25) we get

$$\frac{\sigma_{f,A}^p(P+Q)}{\operatorname{tr}(P+Q)} \geq \left(\frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P) + \operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)} \right)^p,$$

which is equivalent to

$$\frac{\sigma_{f,A}(P+Q)}{\operatorname{tr}^{1/p}(P+Q)} \geq \frac{\operatorname{tr}^{1-1/p}(P)\,\sigma_{f,A}(P) + \operatorname{tr}^{1-1/q}(Q)\,\sigma_{f,A}(Q)}{\operatorname{tr}(P+Q)}, \tag{2.26}$$

for any $P,\ Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Finally, if we multiply (2.26) by $\operatorname{tr}(P+Q) > 0$ we get

$$\psi_{p,f,A}(P+Q) \geq \psi_{p,f,A}(P) + \psi_{p,f,A}(Q)$$

for any $P,\ Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ and the proof is complete. □

Corollary 4. *With the assumptions of Theorem 12, the two parameters $p,\ q \geq 1$ functional $\psi_{p,q,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0,\infty)$ defined by*

$$\psi_{p,q,f,A}(P) := [\operatorname{tr}(P)]^{q\left(1-\frac{1}{p}\right)} \sigma_{f,A}^q(P)$$

is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. Observe that $\psi_{p,q,f,A}(P) = \left[\psi_{p,f,A}(P)\right]^q$ for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$. Therefore, by Theorem 12 and the inequality (2.22) for $q \geq 1$ we have that

$$\begin{aligned} \psi_{p,q,f,A}(P+Q) &= \left[\psi_{p,f,A}(P+Q)\right]^q \\ &\geq \left[\psi_{p,f,A}(P) + \psi_{p,f,A}(Q)\right]^q \\ &\geq \left[\psi_{p,f,A}(P)\right]^q + \left[\psi_{p,f,A}(Q)\right]^q = \psi_{p,q,f,A}(P) + \psi_{p,q,f,A}(Q) \end{aligned}$$

for any $P,\ Q \in \mathcal{B}_1^+(H) \setminus \{0\}$ and the statement is proved. □

Remark 3. *If we consider the functional*

$$\tilde{\psi}_{p,f,A}(P) := [\operatorname{tr}(P)]^{p-1} \sigma_{f,A}^p(P)$$

then, for $p \geq 1$, $\tilde{\psi}_{p,f,A}(\cdot)$ is superadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Corollary 5. *With the assumptions of Theorem 12 and for parameter $p \geq 1$, if there exists the real numbers $\gamma,\ \Gamma > 0$ such that $\Gamma Q \geq P \geq \gamma Q$ with $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then*

$$\begin{aligned} \Gamma^{2-\frac{1}{p}} [\operatorname{tr}(Q)]^{1-\frac{1}{p}} \sigma_{f,A}(Q) &\geq [\operatorname{tr}(P)]^{1-\frac{1}{p}} \sigma_{f,A}(P) \\ &\geq \gamma^{2-\frac{1}{p}} [\operatorname{tr}(Q)]^{1-\frac{1}{p}} \sigma_{f,A}(Q)\ (\geq 0). \end{aligned} \tag{2.27}$$

The case of finite-dimensional spaces is as follows:

Corollary 6. *Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m, M]$, there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma I_n \geq P \geq \gamma I_n$ with P positive definite, then*

$$\begin{aligned}\Gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} &\left[\operatorname{tr}(f(A)) - n f\left(\frac{\operatorname{tr}(A)}{n}\right)\right] \\ &\geq [\operatorname{tr}(P)]^{1-\frac{1}{p}} \left[\operatorname{tr}(P f(A)) - \operatorname{tr}(P) f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\right] \\ &\geq \gamma^{2-\frac{1}{p}} n^{1-\frac{1}{p}} \left[\operatorname{tr}(f(A)) - n f\left(\frac{\operatorname{tr}(A)}{n}\right)\right] (\geq 0)\end{aligned} \tag{2.28}$$

for any $p \geq 1$.

The following result also holds:

Theorem 13 (Dragomir, 2014 [29])**.** *Let A be a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$ and f is a continuous strictly convex function on $[m, M]$. For $p \in (0, 1)$, the functional $\chi_{p,f,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0, \infty)$ defined by*

$$\chi_{p,f,A}(P) := \frac{[\operatorname{tr}(P)]^{1-\frac{1}{p}}}{\sigma_{f,A}(P)}$$

is subadditive on $\mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. Let $s := -p \in (-1, 0)$. For $s < 0$ we have the following inequality

$$(\alpha + \beta)^s \leq \alpha^s + \beta^s \tag{2.29}$$

for any α, $\beta > 0$.

Indeed, by the convexity of the function $f_s(t) = t^s$ on $(0, \infty)$ with $s < 0$ we have that

$$(\alpha + \beta)^s \leq 2^{s-1}(\alpha^s + \beta^s)$$

for any α, $\beta > 0$ and since, obviously, $2^{s-1}(\alpha^s + \beta^s) \leq \alpha^s + \beta^s$, then (2.29) holds true.

Taking into account that $\sigma_{f,A}(\cdot)$ is superadditive and $s \in (-1, 0)$ we have

$$\sigma_{f,A}^s(P + Q) \leq [\sigma_{f,A}(P) + \sigma_{f,A}(Q)]^s \leq \sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q) \tag{2.30}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Since $\operatorname{tr}(\cdot)$ is additive on $\mathcal{B}_1^+(H) \setminus \{0\}$, then by (2.31) we have

$$\begin{aligned}\frac{\sigma_{f,A}^s(P + Q)}{\operatorname{tr}(P + Q)} &\leq \frac{\sigma_{f,A}^s(P) + \sigma_{f,A}^s(Q)}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P)\left(\frac{\sigma_{f,A}(P)}{\operatorname{tr}^{1/s}(P)}\right)^s + \operatorname{tr}(Q)\left(\frac{\sigma_{f,A}(Q)}{\operatorname{tr}^{1/s}(Q)}\right)^s}{\operatorname{tr}(P) + \operatorname{tr}(Q)} \\ &= \frac{\operatorname{tr}(P)\left(\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}\right)^{-s} + \operatorname{tr}(Q)\left(\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}\right)^{-s}}{\operatorname{tr}(P) + \operatorname{tr}(Q)} =: J\end{aligned} \tag{2.31}$$

for any P, $Q \in \mathcal{B}_1^+(H) \setminus \{0\}$.

By the concavity of the function $g(t)=t^{-s}$ with $s\in(-1,0)$ we also have

$$J \le \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}+\operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\right]^{-s} \tag{2.32}$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Making use of (2.31) and (2.32) we get

$$\frac{\sigma_{f,A}^{s}(P+Q)}{\operatorname{tr}(P+Q)} \le \left[\frac{\operatorname{tr}(P)\frac{\operatorname{tr}^{1/s}(P)}{\sigma_{f,A}(P)}+\operatorname{tr}(Q)\frac{\operatorname{tr}^{1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)}\right]^{-s}$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$, and by taking the power $-1/s>0$ we get

$$\frac{\sigma_{f,A}^{-1}(P+Q)}{\operatorname{tr}^{-1/s}(P+Q)} \le \frac{\frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)}+\frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}}{\operatorname{tr}(P)+\operatorname{tr}(Q)},$$

which is equivalent to

$$\frac{\operatorname{tr}^{1+1/s}(P+Q)}{\sigma_{f,A}(P+Q)} \le \frac{\operatorname{tr}^{1+1/s}(P)}{\sigma_{f,A}(P)}+\frac{\operatorname{tr}^{1+1/s}(Q)}{\sigma_{f,A}(Q)}$$

for any $P,\ Q\in\mathcal{B}_1^+(H)\setminus\{0\}$.

This completes the proof. □

The following result may be stated as well:

Corollary 7. *With the assumptions of Theorem 13, the two parameters* $0<p$, $q<1$ *functional* $\chi_{p,q,f,A}:\mathcal{B}_1^+(H)\setminus\{0\}\to[0,\infty)$ *defined by*

$$\chi_{p,q,f,A}(P)=\frac{\operatorname{tr}^{q\left(1-\frac{1}{p}\right)}(P)}{\sigma_{f,A}^{q}(P)}$$

is subadditive on $\mathcal{B}_1^+(H)\setminus\{0\}$.

Remark 4. *If we consider the functional* $\tilde{\chi}_{p,f,A}(P)=\frac{\operatorname{tr}^{p-1}(P)}{\sigma_{f,A}^{p}(P)}$ *for* $0<p<1$, *then* $\tilde{\chi}_{p,f,A}(\cdot)$ *is also subadditive on* $\mathcal{B}_1^+(H)\setminus\{0\}$.

2.3. **Some Examples.** We consider the power function $f:(0,\infty)\to(0,\infty)$, $f(t)=t^r$ with $t\in\mathbb{R}\setminus\{0\}$. For $r\in(-\infty,0)\cup[1,\infty)$, f is convex while for $r\in(0,1)$, f is concave.

Let $r\ge 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars $m,\ M$ with $0\le m<M$. If $\mathcal{E}:=\{e_i\}_{i\in I}$ is an orthonormal basis in H and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$ then

$$\begin{aligned}&[\operatorname{tr}(PA)]^{r}[\operatorname{tr}(P)]^{1-r}\\ &\le K_{\varepsilon}(r;A,P)\le\operatorname{tr}(PA^{r})\\ &\le\frac{1}{M-m}\left(m^{r}\operatorname{tr}[P(M1_H-A)]+M^{r}\operatorname{tr}[P(A-m1_H)]\right),\end{aligned}\tag{2.33}$$

where

$$K_{\varepsilon}(r;A,P):=\sum_{i\in I_{\mathcal{E},P}}\left\langle P^{1/2}AP^{1/2}e_i,e_i\right\rangle^{r}\left\langle Pe_i,e_i\right\rangle^{1-r}.$$

Moreover, the quantities

$$K_i(r;A,P) := \inf_{\mathcal{E}} K_{\mathcal{E}}(r;A,P) \text{ and } K_s(r;A,P) := \sup_{\mathcal{E}} K_{\mathcal{E}}(r;A,P)$$

are finite and satisfy the bounds

$$[\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \le K_i(r;A,P) \le K_s(r;A,P) \le \operatorname{tr}(PA^r). \tag{2.34}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (2.33) we have

$$\left[\operatorname{tr}\left(P^2\right)\right]^r [\operatorname{tr}(P)]^{1-r} \le K_{\mathcal{E}}(r;P) \le \operatorname{tr}\left(P^{r+1}\right) \tag{2.35}$$

where

$$K_{\mathcal{E}}(r;P) := \sum_{i\in I_{\mathcal{E},P}} \left\langle P^2 e_i, e_i\right\rangle^r \langle Pe_i, e_i\rangle^{1-r}.$$

If we consider the functional $\sigma_{r,A} : \mathcal{B}_1^+(H) \setminus \{0\} \to [0,\infty)$ defined by

$$\sigma_{r,A}(P) := \operatorname{tr}(PA^r) - [\operatorname{tr}(PA)]^r [\operatorname{tr}(P)]^{1-r} \ge 0, \tag{2.36}$$

where A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m,M] \subset [0,\infty)$, then $\sigma_{r,A}(\cdot)$ is superadditive, monotonic nondecreasing and if there exists the real numbers γ, $\Gamma > 0$ such that $\Gamma Q \ge P \ge \gamma Q$ with $P, Q \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$\Gamma\sigma_{r,A}(Q) \ge \sigma_{r,A}(P) \ge \gamma\sigma_{r,A}(Q) \,(\ge 0). \tag{2.37}$$

Consider the convex function $f : (0,\infty) \to (0,\infty)$, $f(t) = -\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\begin{aligned}\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} &\ge L_{\mathcal{E}}(A,P) \ge \exp[\operatorname{tr}(P\ln A)] \\ &\ge m^{\frac{\operatorname{tr}[P(M1_H - A)]}{M-m}} M^{\frac{\operatorname{tr}[P(A-m1_H)]}{M-m}},\end{aligned} \tag{2.38}$$

where

$$L_{\mathcal{E}}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\left\langle P^{1/2}AP^{1/2}e_i, e_i\right\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Moreover, the quantities

$$L_i(A,P) := \inf_{\mathcal{E}} L_{\mathcal{E}}(A,P) \text{ and } L_s(A,P) := \sup_{\mathcal{E}} L_{\mathcal{E}}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \ge L_s(A,P) \ge L_i(A,P) \ge \exp[\operatorname{tr}(P\ln A)]. \tag{2.39}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (2.38) we get

$$\left(\frac{\operatorname{tr}\left(P^2\right)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} \ge L_{\mathcal{E}}(P) \ge \exp[\operatorname{tr}(P\ln P)] \tag{2.40}$$

where

$$L_{\mathcal{E}}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\left\langle P^2 e_i, e_i\right\rangle}{\langle Pe_i, e_i\rangle}\right)^{\langle Pe_i, e_i\rangle}.$$

Consider the functional $\delta_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0,\infty)$ defined by

$$\delta_A(P) := \frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)}}{\exp(\operatorname{tr}(P\ln A))} \geq 1,$$

where A is a selfadjoint operator on the Hilbert space H and such that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$.

Observe that

$$\sigma_{-\ln,A}(P) := \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P)} - \ln[\exp(\operatorname{tr}(P\ln A))] = \ln[\delta_A(P)]$$

for $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the properties of $\sigma_{-\ln,A}(\cdot)$ we conclude that $\delta_A(\cdot)$ is supermultiplicative, i.e.

$$\delta_A(P+Q) \geq \delta_A(P)\,\delta_A(Q) \geq 1$$

for any $P,Q \in \mathcal{B}_1^+(H) \setminus \{0\}$. The functional $\delta_A(\cdot)$ is also monotonic nondecreasing on $\mathcal{B}_1(H) \setminus \{0\}$.

Consider the convex function $f(t) = t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $0 < m < M$. If $\mathcal{E} := \{e_i\}_{i\in I}$ is an orthonormal basis in H and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ then

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_{\varepsilon}(A,P) \leq \exp[\operatorname{tr}(PA\ln A)] \leq m^{\frac{m\operatorname{tr}[P(M1_H - A)]}{M-m}} M^{\frac{M\operatorname{tr}[P(A-m1_H)]}{M-m}}, \tag{2.41}$$

where

$$I_{\varepsilon}(A,P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^{1/2}AP^{1/2}e_i,e_i\rangle}{\langle Pe_i,e_i\rangle}\right)^{\langle P^{1/2}AP^{1/2}e_i,e_i\rangle}.$$

Moreover, the quantities

$$I_i(A,P) := \inf_{\mathcal{E}} I_{\varepsilon}(A,P) \text{ and } I_s(A,P) := \sup_{\mathcal{E}} I_{\varepsilon}(A,P)$$

are finite and satisfy the bounds

$$\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)} \leq I_i(A,P) \leq I_s(A,P) \leq \exp[\operatorname{tr}(PA\ln A)]. \tag{2.42}$$

Now, if we take $A = P$, $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then by (2.41) we get

$$\left(\frac{\operatorname{tr}(P^2)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(P^2)} \leq I_{\varepsilon}(P) \leq \exp\left[\operatorname{tr}\left(P^2\ln P\right)\right] \tag{2.43}$$

where

$$I_{\varepsilon}(P) := \prod_{i\in I_{\mathcal{E},P}} \left(\frac{\langle P^2e_i,e_i\rangle}{\langle Pe_i,e_i\rangle}\right)^{\langle P^2e_i,e_i\rangle}.$$

Observe that for $f(t) = t\ln t$ we have

$$\begin{aligned}\sigma_{(\cdot)\ln(\cdot),A}(P) &= \operatorname{tr}(PA\ln A) - \operatorname{tr}(PA)\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\\ &= \ln\left[\frac{\exp[\operatorname{tr}(PA\ln A)]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}}\right]\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Consider the functional $\lambda_A : \mathcal{B}_1^+(H) \setminus \{0\} \to (0,\infty)$ defined by

$$\lambda_A(P) := \frac{\exp[\operatorname{tr}(PA\ln A)]}{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{\operatorname{tr}(PA)}} \geq 1.$$

Utilising the properties of $\sigma_{(\cdot)\ln(\cdot),A}(\cdot)$ we can conclude that $\lambda_A(\cdot)$ is supermultiplicative and monotonic nondecreasing on $\mathcal{B}_1^+(H) \setminus \{0\}$.

2.4. **More Inequalities for Convex Functions.** We recall the *gradient inequality* for the convex function $f : [m,M] \to \mathbb{R}$, namely

$$f(\varsigma) - f(\tau) \geq \delta_f(\tau)(\varsigma - \tau) \tag{2.44}$$

for any ς, $\tau \in [m,M]$ where $\delta_f(\tau) \in \left[f'_-(\tau), f'_+(\tau)\right]$, (for $\tau = m$ we take $\delta_f(\tau) = f'_+(m)$ and for $\tau = M$ we take $\delta_f(\tau) = f'_-(M)$). Here $f'_+(m)$ and $f'_-(M)$ are the lateral derivatives of the convex function f.

The following result holds:

Theorem 14 (Dragomir, 2014 [30])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m,M]$ and $B \in \mathcal{B}_2(H) \setminus \{0\}$, then we have $\frac{\operatorname{tr}(|B|^2 A)}{\operatorname{tr}(|B|^2)} \in [m,M]$,*

$$\begin{aligned}&\delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\frac{\operatorname{tr}\left(|B^*|^2 A\right) - \operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\\ &\leq \frac{\operatorname{tr}\left(|B^*|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)} - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right),\end{aligned} \tag{2.45}$$

where

$$\delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \in \left[f'_-\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right), f'_+\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\right]$$

and the Jensen's inequality

$$f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \leq \frac{\operatorname{tr}\left(|B|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}. \tag{2.46}$$

Proof. Let $\mathcal{E} := \{e_i\}_{i\in I}$ be an orthonormal basis in H. Utilising the gradient inequality (2.44) we get

$$f(\varsigma) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\varsigma - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right) \tag{2.47}$$

for any $\varsigma \in [m, M]$, since obviously, by $\operatorname{Sp}(A) \subseteq [m, M]$ we have

$$m\|Be_i\|^2 \leq \langle ABe_i, Be_i\rangle \leq M\|Be_i\|^2,$$

for $i \in I$, which, by summation shows that

$$\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \in [m, M].$$

The inequality (2.47) implies in the operator order of $\mathcal{B}(H)$ that

$$f(A) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)1_H \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right), \tag{2.48}$$

which can be written as

$$\begin{aligned}\langle f(A)y, y\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y, y\rangle \\ \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle Ay, y\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle y, y\rangle\right),\end{aligned} \tag{2.49}$$

for any $y \in H$. This inequality is also of interest in itself.

Taking in (2.49) $y = Be_i$ we get

$$\begin{aligned}\langle f(A)Be_i, Be_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle Be_i, Be_i\rangle \\ \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle ABe_i, Be_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle Be_i, Be_i\rangle\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\langle B^* f(A)Be_i, e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left\langle |B|^2 e_i, e_i\right\rangle \\ \geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\langle B^* ABe_i, e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\left\langle |B|^2 e_i, e_i\right\rangle\right),\end{aligned} \tag{2.50}$$

for any $i \in I$.

Summing in (2.50) we get

$$\sum_{i\in I}\left\langle B^{*}f\left(A\right)Be_{i},e_{i}\right\rangle -f\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\sum_{i\in I}\left\langle \left|B\right|^{2}e_{i},e_{i}\right\rangle \tag{2.51}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left(\sum_{i\in I}\left\langle B^{*}ABe_{i},e_{i}\right\rangle -\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\sum_{i\in I}\left\langle \left|B\right|^{2}e_{i},e_{i}\right\rangle \right).$$

However

$$\sum_{i\in I}\left\langle B^{*}f\left(A\right)Be_{i},e_{i}\right\rangle =\sum_{i\in I}\left\langle BB^{*}f\left(A\right)e_{i},e_{i}\right\rangle$$
$$=\sum_{i\in I}\left\langle \left|B^{*}\right|^{2}f\left(A\right)e_{i},e_{i}\right\rangle =\operatorname{tr}\left(\left|B^{*}\right|^{2}f\left(A\right)\right)$$

and

$$\sum_{i\in I}\left\langle B^{*}ABe_{i},e_{i}\right\rangle =\sum_{i\in I}\left\langle BB^{*}Ae_{i},e_{i}\right\rangle =\operatorname{tr}\left(\left|B^{*}\right|^{2}A\right).$$

By (2.51) we get

$$\operatorname{tr}\left(\left|B^{*}\right|^{2}f\left(A\right)\right)-f\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\operatorname{tr}\left(\left|B\right|^{2}\right) \tag{2.52}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left(\operatorname{tr}\left(\left|B^{*}\right|^{2}A\right)-\operatorname{tr}\left(\left|B\right|^{2}A\right)\right),$$

and the inequality (2.45) is thus proved.

Taking in (2.49) $y=B^{*}e_{i}$ we also get

$$\left\langle f\left(A\right)B^{*}e_{i},B^{*}e_{i}\right\rangle -f\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left\langle B^{*}e_{i},B^{*}e_{i}\right\rangle$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left(\left\langle AB^{*}e_{i},B^{*}e_{i}\right\rangle -\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\left\langle B^{*}e_{i},B^{*}e_{i}\right\rangle \right),$$

which is equivalent to

$$\left\langle Bf\left(A\right)B^{*}e_{i},e_{i}\right\rangle -f\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left\langle BB^{*}e_{i},e_{i}\right\rangle \tag{2.53}$$
$$\geq\delta_{f}\left(\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\right)\left(\left\langle BAB^{*}e_{i},e_{i}\right\rangle -\frac{\operatorname{tr}\left(\left|B\right|^{2}A\right)}{\operatorname{tr}\left(\left|B\right|^{2}\right)}\left\langle BB^{*}e_{i},e_{i}\right\rangle \right),$$

for any $i\in I$.

Summing in (2.53) we get

$$\sum_{i\in I}\langle Bf(A)B^*e_i,e_i\rangle - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle BB^*e_i,e_i\rangle \tag{2.54}$$
$$\geq \delta_f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\left(\sum_{i\in I}\langle BAB^*e_i,e_i\rangle - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle BB^*e_i,e_i\rangle\right).$$

Since

$$\sum_{i\in I}\langle Bf(A)B^*e_i,e_i\rangle = \operatorname{tr}(Bf(A)B^*) = \operatorname{tr}(B^*Bf(A)) = \operatorname{tr}\left(|B|^2 f(A)\right),$$

$$\sum_{i\in I}\langle BB^*e_i,e_i\rangle = \operatorname{tr}(BB^*) = \operatorname{tr}(B^*B) = \operatorname{tr}\left(|B|^2\right)$$

and

$$\sum_{i\in I}\langle BAB^*e_i,e_i\rangle = \operatorname{tr}(BAB^*) = \operatorname{tr}(B^*BA) = \operatorname{tr}\left(|B|^2 A\right),$$

then by (2.54) we get

$$\operatorname{tr}\left(|B|^2 f(A)\right) - f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\operatorname{tr}\left(|B|^2\right) \geq 0$$

and the inequality (2.46) is obtained. □

Remark 5. *The inequality (2.46) is obviously not as good as the first part of (2.1). However it is the natural alternative of Jensen's inequality for trace and provides simple and nice examples for various convex functions of interest. The proof here is also simpler than the one from* [29] *and has some natural reverses as follows.*

Corollary 8. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If f is a continuous convex function on $[m,M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ then $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in [m,M]$ and*

$$f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}. \tag{2.55}$$

The proof follows by either (2.45) or (2.46) on choosing $B = P^{1/2}$, $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

3. Reverses of Jensen's Trace Inequality

3.1. **A Reverse of Jensen's Inequality.**

The following lemma is of interest in itself:

Lemma 1 (Dragomir, 2014 [30])**.** *Let S be a selfadjoint operator such that $\gamma 1_H \leq S \leq \Gamma 1_H$ for some real constants $\Gamma \geq \gamma$. Then for any $B \in \mathcal{B}_2(H) \setminus \{0\}$ we have*

$$
\begin{aligned}
(3.1) \qquad 0 &\leq \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \\
&\leq \frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left|S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right) \\
&\leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \leq \frac{1}{4}(\Gamma-\gamma)^2 .
\end{aligned}
$$

Proof. The first inequality follows by Jensen's inequality (2.46) for the convex function $f(t) = t^2$.

Now, observe that

$$
\begin{aligned}
(3.2) \qquad & \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left(S - \frac{\Gamma+\gamma}{2} 1_H\right)\left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right) \\
&= \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 S \left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right) \\
&\quad - \frac{\Gamma+\gamma}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right) \\
&= \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2
\end{aligned}
$$

since, obviously

$$
\operatorname{tr}\left(|B|^2 \left(S - \frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right) = 0.
$$

Now, since $\gamma 1_H \leq S \leq \Gamma 1_H$ then

$$
\left|S - \frac{\Gamma+\gamma}{2} 1_H\right| \leq \frac{1}{2}(\Gamma-\gamma).
$$

Taking the modulus in (3.2) and using the properties of trace, we have

$$
\begin{aligned}
(3.3)\qquad & \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \\
& =\frac{1}{\operatorname{tr}\left(|B|^2\right)}\left|\operatorname{tr}\left(|B|^2\left(S-\frac{\Gamma+\gamma}{2} 1_H\right)\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right)\right| \\
& \leq \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|\left(S-\frac{\Gamma+\gamma}{2} 1_H\right)\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)\right|\right) \\
& \leq \frac{1}{2}(\Gamma-\gamma) \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right),
\end{aligned}
$$

which proves the first part of (3.1).

By Schwarz inequality for trace we also have

$$
\begin{aligned}
(3.4)\qquad & \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right|\right) \\
& \leq\left[\frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2\left(S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H\right)^2\right)\right]^{1/2} \\
& =\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} .
\end{aligned}
$$

From (3.3) and (3.4) we get

$$
\begin{aligned}
& \frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2 \\
& \leq \frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2},
\end{aligned}
$$

which implies that

$$
\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \leq \frac{1}{2}(\Gamma-\gamma) .
$$

By (3.4) we then obtain

$$\begin{aligned}&\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left|S-\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right)\\&\leq\left[\frac{\operatorname{tr}\left(|B|^2 S^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\frac{\operatorname{tr}\left(|B|^2 S\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2}\leq\frac{1}{2}(\Gamma-\gamma)\end{aligned}$$

that proves the last part of (3.1). □

Remark 6. *Let S be a selfadjoint operator such that $\gamma 1_H \leq S \leq \Gamma 1_H$ for some real constants $\Gamma \geq \gamma$. Then for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ we have*

$$\begin{aligned}0&\leq\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\\&\leq\frac{1}{2}(\Gamma-\gamma)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(P\left|S-\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right|\right)\\&\leq\frac{1}{2}(\Gamma-\gamma)\left[\frac{\operatorname{tr}\left(PS^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\leq\frac{1}{4}(\Gamma-\gamma)^2.\end{aligned}\tag{3.5}$$

The following result provides reverses for the inequalities (2.45) and (2.46) above:

Theorem 15 (Dragomir, 2014 [30])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $B \in \mathcal{B}_2(H) \setminus \{0\}$, then we have*

$$\begin{aligned}&\frac{\operatorname{tr}\left(|B^*|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}-f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\\&\leq\frac{\operatorname{tr}\left(|B^*|^2 f'(A)A\right)}{\operatorname{tr}\left(|B|^2\right)}-\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\frac{\operatorname{tr}\left(|B^*|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}0&\leq\frac{\operatorname{tr}\left(|B|^2 f(A)\right)}{\operatorname{tr}\left(|B|^2\right)}-f\left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\\&\leq\frac{\operatorname{tr}\left(|B|^2 f'(A)A\right)}{\operatorname{tr}\left(|B|^2\right)}-\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)}\frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}=:\mathcal{K}(f',B,A).\end{aligned}\tag{3.7}$$

Moreover, we have

$$\mathcal{K}(f',B,A) \tag{3.8}$$

$$\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\dfrac{\operatorname{tr}\left(|B|^2\left|A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right)}{\operatorname{tr}\left(|B|^2\right)} \\ \\ \frac{1}{2}(M-m)\dfrac{\operatorname{tr}\left(|B|^2\left|f'(A)-\frac{\operatorname{tr}\left(|B|^2f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right|\right)}{\operatorname{tr}\left(|B|^2\right)} \end{cases}$$

$$\leq \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\dfrac{\operatorname{tr}\left(|B|^2A^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\dfrac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\dfrac{\operatorname{tr}\left(|B|^2\left[f'(A)\right]^2\right)}{\operatorname{tr}\left(|B|^2\right)}-\left(\dfrac{\operatorname{tr}\left(|B|^2f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)}\right)^2\right]^{1/2} \end{cases}$$

$$\leq \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).$$

Proof. By the gradient inequality we have

$$f(\tau)-f(\varsigma)\leq f'(\tau)(\tau-\varsigma) \tag{3.9}$$

for any $\tau,\varsigma\in[m,M]$.

This inequality implies in the operator order

$$f(A)-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)1_H\leq f'(A)\left(A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right)$$

that is equivalent to

$$\langle f(A)y,y\rangle-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\langle y,y\rangle \tag{3.10}$$
$$\leq\langle f'(A)Ay,y\rangle-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\langle f'(A)y,y\rangle$$

for any $y\in H$, which is of interest in itself as well.

Let $\mathcal{E}:=\{e_i\}_{i\in I}$ be an orthonormal basis in H. If we take in (3.10) $y=Be_i$ and sum, then we get

$$\sum_{i\in I}\langle f(A)Be_i,Be_i\rangle-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle Be_i,Be_i\rangle$$
$$\leq\sum_{i\in I}\langle f'(A)ABe_i,Be_i\rangle-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle f'(A)Be_i,Be_i\rangle,$$

which is equivalent to

$$\begin{aligned}&\sum_{i\in I}\langle B^{*}f(A)Be_i,e_i\rangle-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^{*}Be_i,e_i\rangle\\&\leq\sum_{i\in I}\langle B^{*}f'(A)ABe_i,e_i\rangle-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle B^{*}f'(A)Be_i,e_i\rangle\end{aligned}$$

and the inequality (3.6) is obtained.

If we take in (3.10) $y=B^{*}e_i$ and sum, then we get

$$\begin{aligned}&\sum_{i\in I}\langle f(A)B^{*}e_i,B^{*}e_i\rangle-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle B^{*}e_i,B^{*}e_i\rangle\\&\leq\sum_{i\in I}\langle f'(A)AB^{*}e_i,B^{*}e_i\rangle-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle f'(A)B^{*}e_i,B^{*}e_i\rangle\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\sum_{i\in I}\langle Bf(A)B^{*}e_i,e_i\rangle-f\left(\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\right)\sum_{i\in I}\langle BB^{*}e_i,e_i\rangle\\&\leq\sum_{i\in I}\langle Bf'(A)AB^{*}e_i,e_i\rangle-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}\sum_{i\in I}\langle Bf'(A)B^{*}e_i,e_i\rangle\end{aligned}\tag{3.11}$$

and the inequality (3.7) is obtained.

Now, since f is continuously convex on $[m,M]$, then f' is monotonic nondecreasing on $[m,M]$ and $f'(m)\leq f'(t)\leq f'(M)$ for any $t\in[m,M]$. We also observe that

$$\begin{aligned}&\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right]\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&=\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2f'(A)\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&-\frac{f'(m)+f'(M)}{2}\frac{1}{\operatorname{tr}\left(|B|^2\right)}\operatorname{tr}\left(|B|^2\left[A-\frac{\operatorname{tr}\left(|B|^2A\right)}{\operatorname{tr}\left(|B|^2\right)}1_H\right]\right)\\&=\mathcal{K}(f',B,A).\end{aligned}\tag{3.12}$$

Since

$$\left|f'(A)-\frac{f'(m)+f'(M)}{2}1_H\right|\leq\frac{1}{2}\left[f'(M)-f'(m)\right]1_H,$$

then by taking the modulus in (3.12) and utilizing the properties of trace we have

$$
\begin{aligned}
(3.13) \qquad & 0 \le \mathcal{K}\left(f', B, A\right) \\
& \le \frac{1}{\operatorname{tr}\left(|B|^2\right)} \\
& \times \operatorname{tr}\left(|B|^2 \left| \left[f'(A) - \frac{f'(m) + f'(M)}{2} 1_H \right] \left[A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right] \right| \right) \\
& \le \frac{1}{2}\left[f'(M) - f'(m)\right] \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left| A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right| \right),
\end{aligned}
$$

and the first inequality in the first branch of (3.8) is proved.

We have $m1_H \le A \le M1_H$ and by applying Lemma 1 we can state that

$$
\begin{aligned}
(3.14) \qquad & \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left| A - \frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right| \right) \\
& \le \left[\frac{\operatorname{tr}\left(|B|^2 A^2\right)}{\operatorname{tr}\left(|B|^2\right)} - \left(\frac{\operatorname{tr}\left(|B|^2 A\right)}{\operatorname{tr}\left(|B|^2\right)} \right)^2 \right]^{1/2} \le \frac{1}{2}(M - m).
\end{aligned}
$$

Making use of (3.13) and (3.14) we deduce the second and the third inequalities in the first branch of (3.8).

We observe that $\mathcal{K}\left(f', B, A\right)$ can be also represented as

$$
\begin{aligned}
& \mathcal{K}\left(f', B, A\right) \\
& = \frac{1}{\operatorname{tr}\left(|B|^2\right)} \operatorname{tr}\left(|B|^2 \left[f'(A) - \frac{\operatorname{tr}\left(|B|^2 f'(A)\right)}{\operatorname{tr}\left(|B|^2\right)} 1_H \right] \left(A - \frac{m+M}{2} 1_H \right) \right).
\end{aligned}
$$

Applying a similar argument as above for this representation, we get the second branch of the inequality (3.8).

The proof is complete. □

Corollary 9. *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$, then we* *have*

$$
\begin{aligned}
(3.15)\qquad 0 &\le \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\cdot\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\[2ex] \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|f'(A)-\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\[2ex] \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(P\left[f'(A)\right]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned}
$$

Remark 7. *Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$, the identity matrix, in (3.15) we get*

$$
\begin{aligned}
(3.16)\qquad 0 &\le \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\
&\le \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n}\frac{\operatorname{tr}(f'(A))}{n} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\frac{\operatorname{tr}\left(\left|A-\frac{\operatorname{tr}(A)}{n}I_n\right|\right)}{n} \\[2ex] \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(\left|f'(A)-\frac{\operatorname{tr}(f'(A))}{n}I_n\right|\right)}{n} \end{cases} \\
&\le \begin{cases} \frac{1}{2}\left[f'(M)-f'(m)\right]\left[\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\[2ex] \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(\left[f'(A)\right]^2\right)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}\left[f'(M)-f'(m)\right](M-m).
\end{aligned}
$$

3.2. **Some Examples.** We consider the power function $f : (0,\infty) \to (0,\infty)$, $f(t) = t^r$ with $t \in \mathbb{R}\setminus\{0\}$. For $r \in (-\infty, 0)\cup[1,\infty)$, f is convex while for $r \in (0, 1)$, f is concave. Denote $\mathcal{B}_1^+(H) := \{P \text{ with } P \in \mathcal{B}_1(H) \text{ and } P \ge 0\}$.

Let $r \ge 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $0 \le m < M$. If $P \in \mathcal{B}_1^+(H)\setminus\{0\}$, then

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \\
&\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right] \\
&\leq \begin{cases} \frac{1}{2}r\left(M^{r-1}-m^{r-1}\right)\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}r(M-m)\frac{\operatorname{tr}\left(P\left|A^{r-1}-\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}r\left(M^{r-1}-m^{r-1}\right)\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}r(M-m)\left[\frac{\operatorname{tr}\left(PA^{2(r-1)}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}r\left(M^{r-1}-m^{r-1}\right)(M-m).
\end{aligned}
\tag{3.17}
$$

Consider the convex function $f:(0,\infty)\to(0,\infty)$, $f(t)=-\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $0<m<M$. If $P\in\mathcal{B}_1^+(H)\setminus\{0\}$, then

$$
\begin{aligned}
0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\
&\leq \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)} - 1 \\
&\leq \begin{cases} \frac{M-m}{2mM}\frac{\operatorname{tr}\left(P\left|A-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m)\frac{\operatorname{tr}\left(P\left|A^{-1}-\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{M-m}{2mM}\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m)\left[\frac{\operatorname{tr}\left(PA^{-2}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{(M-m)^2}{4mM}.
\end{aligned}
\tag{3.18}
$$

Consider the convex function $f(t)=t\ln t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with

$0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(PA\ln A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\operatorname{tr}(PA\ln(eA))}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)} \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right) \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M-m) \frac{\operatorname{tr}\left(P\left|\ln(eA) - \frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\
&\leq \begin{cases} \frac{1}{2}\ln\left(\frac{M}{m}\right) \left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M-m) \left[\frac{\operatorname{tr}\left(P[\ln(eA)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(P\ln(eA))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\leq \frac{1}{4}(M-m)\ln\left(\frac{M}{m}\right).
\end{aligned} \tag{3.19}
$$

3.3. **Further Reverse Inequalities for Convex Functions.** The following reverses of Jensen's trace inequality also hold:

Theorem 16 (Dragomir, 2014 [32])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuos convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ is such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$ then we have*

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right) \\
&\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \sup_{t \in (m,M)} \Psi_f(t; m, M) \\
&\leq \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \frac{f'_-(M) - f'_+(m)}{M-m} \\
&\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right],
\end{aligned} \tag{3.20}
$$

where $\Psi_f(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$
\Psi_f(t; m, M) = \frac{f(M) - f(t)}{M-t} - \frac{f(t) - f(m)}{t-m}.
$$

We also have

$$0 \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{3.21}$$
$$\leq \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m} \Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right)$$
$$\leq \frac{1}{4}(M-m)\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right)$$
$$\leq \frac{1}{4}(M-m)\sup_{t\in(m,M)} \Psi_f(t; m, M)$$
$$\leq \frac{1}{4}(M-m)\left[f'_-(M) - f'_+(m)\right],$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$ *such that* $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \in (m, M)$.

Proof. Since f is convex, then we have

$$f(t) = f\left(\frac{m(M-t) + M(t-m)}{M-m}\right) \leq \frac{(M-t)f(m) + (t-m)f(M)}{M-m}$$

for any $t \in [m, M]$.

This scalar inequality implies, by utilizing the spectral representation of continuous functions of selfadjoint operators, the following inequality

$$f(A) \leq \frac{f(m)(M1_M - A) + f(M)(A - m1_H)}{M-m} \tag{3.22}$$

in the operator order of $\mathcal{B}(H)$.

Utilising the properties of the trace and the inequality (3.22), we have

$$\frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{3.23}$$
$$= \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H - A) + M(A - 1_H m)}{M-m}\right)}{\operatorname{tr}(P)}\right)$$
$$\leq \frac{\operatorname{tr}\left(P\frac{f(m)(M1_M - A) + f(M)(A - m1_H)}{M-m}\right)}{\operatorname{tr}(P)}$$
$$- f\left(\frac{\operatorname{tr}\left(P\frac{m(M1_H - A) + M(A - 1_H m)}{M-m}\right)}{\operatorname{tr}(P)}\right)$$
$$= \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)f(m) + \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)f(M)}{M-m}$$
$$- f\left(\frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)m + \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)M}{M-m}\right)$$
$$=: B(f, P, A, m, M)$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$.

By denoting

$$\Delta_f(t;m,M) := \frac{(t-m)f(M)+(M-t)f(m)}{M-m} - f(t), \quad t\in[m,M]$$

we have

$$\begin{aligned}\Delta_f(t;m,M) &= \frac{(t-m)f(M)+(M-t)f(m)-(M-m)f(t)}{M-m} \\ &= \frac{(t-m)f(M)+(M-t)f(m)-(M-t+t-m)f(t)}{M-m} \\ &= \frac{(t-m)[f(M)-f(t)]-(M-t)[f(t)-f(m)]}{M-m} \\ &= \frac{(M-t)(t-m)}{M-m}\Psi_f(t;m,M)\end{aligned} \tag{3.24}$$

for any $t\in(m,M)$.

Therefore

$$B(f,P,A,m,M) = \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right), \tag{3.25}$$

provided that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$.

If $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$, then

$$\begin{aligned}&\Psi_f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right) \\ &\le \sup_{t\in(m,M)}\Psi_f(t;m,M) \\ &= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}-\frac{f(t)-f(m)}{t-m}\right] \\ &\le \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]+\sup_{t\in(m,M)}\left[-\frac{f(t)-f(m)}{t-m}\right] \\ &= \sup_{t\in(m,M)}\left[\frac{f(M)-f(t)}{M-t}\right]-\inf_{t\in(m,M)}\left[\frac{f(t)-f(m)}{t-m}\right] \\ &= f'_-(M)-f'_+(m),\end{aligned} \tag{3.26}$$

which by (3.23) and (3.25) produces the second, third and fourth inequalities in (3.20).

Since, obviously

$$\frac{1}{M-m}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\le\frac{1}{4}(M-m),$$

then the last part of (3.20) also holds.

The second part of the theorem is clear and the details are omitted. □

The following result also holds:

Theorem 17 (Dragomir, 2014 [32]). *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A)\subseteq[m,M]$ *for some scalars m, M with $m<M$. If*

f is a continuos convex function on $[m,M]$ then for all $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$ we have that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$ and

$$
\begin{aligned}
0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)}-f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\
&\leq 2\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right] \\
&\leq 2\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].
\end{aligned}
\tag{3.27}
$$

Proof. Since $m1_H\leq A\leq M1_H$, it follows that $m\operatorname{tr}(P)\leq\operatorname{tr}(PA)\leq M\operatorname{tr}(P)$ for any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\geq 0$, which shows that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in[m,M]$.

Further on, we recall the following result (see for instance [12]) that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$
\begin{aligned}
& n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \\
&\leq \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i)-f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \\
&\leq n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n f(x_i)-f\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right],
\end{aligned}
\tag{3.28}
$$

where $f:C\to\mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}\subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n:=\sum_{i=1}^n p_i>0$.

For $n=2$ we deduce from (3.28) that

$$
\begin{aligned}
& 2\min\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right] \\
&\leq tf(x)+(1-t)f(y)-f(tx+(1-t)y) \\
&\leq 2\max\{t,1-t\}\left[\frac{f(x)+f(y)}{2}-f\left(\frac{x+y}{2}\right)\right]
\end{aligned}
\tag{3.29}
$$

for any $x,y\in C$ and $t\in[0,1]$.

If we use the second inequality in (3.29) for the convex function $f:I\to\mathbb{R}$ where m, $M\in\mathbb{R}$, $m<M$ with $[m,M]=I$, we have for $x=m$, $y=M$ and $t=\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}$

that

$$\begin{aligned}B(f,P,A,m,M)&=\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)f(m)+\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)f(M)}{M-m}\\&\quad-f\left(\frac{m\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)+M\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\right)\\&\le 2\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\\&\quad\times\left[\frac{f(m)+f(M)}{2}-f\left(\frac{m+M}{2}\right)\right].\end{aligned}$$

Making use of (3.23) we deduce the first inequality in (3.27).

Since

$$\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\le 1,$$

the last part of (3.27) is also proved. □

3.4. **Some Examples.** For $p>1$ and $0<m<M<\infty$ consider the convex function $f(t)=t^p$ defined on $[m,M]$. Then $\Psi_p(\cdot;m,M):(m,M)\to\mathbb{R}$ is defined by

$$\begin{aligned}\Psi_p(t;m,M)&=\frac{M^p-t^p}{M-t}-\frac{t^p-m^p}{t-m}\\&=\frac{t(M^p-m^p)-t^p(M-m)-mM\left(M^{p-1}-m^{p-1}\right)}{(M-t)(t-m)}.\end{aligned}$$

Let A be a nonnegative selfadjoint operator on the Hilbert space H and assume that $\operatorname{Sp}(A)\subseteq[m,M]$ for some scalars m, M with $0\le m<M$. If $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge 0$ such that $\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\in(m,M)$, then we have from (3.20) that

$$\begin{aligned}0&\le\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\\&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)};m,M\right)\\&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_p(t;m,M)\\&\le p\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\frac{M^{p-1}-m^{p-1}}{M-m}\\&\le\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\end{aligned}\tag{3.30}$$

and from (3.21) that

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \tag{3.31}$$
$$\le \frac{\left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right)}{M-m}\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right)$$
$$\le \frac{1}{4}(M-m)\Psi_p\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}; m, M\right)$$
$$\le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_p(t; m, M)$$
$$\le \frac{1}{4}p(M-m)\left(M^{p-1} - m^{p-1}\right).$$

For $p = 2$, we have

$$\Psi_2(t; m, M) = \frac{M^2 - t^2}{M-t} - \frac{t^2 - m^2}{t-m} = M - m$$

and by (3.30) we get

$$0 \le \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2 \le \left(M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right) \tag{3.32}$$
$$\le \frac{1}{4}(M-m)^2$$

for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

Making use of the inequality (3.27) we have

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \tag{3.33}$$
$$\le 2\max\left\{\frac{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m}{M-m}\right\}\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right]$$
$$\le 2\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right],$$

for any positive operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and for any $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

In particular, for $p = 2$ we get

$$0 \le \frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2 \tag{3.34}$$
$$\le \frac{1}{2}(M-m)\max\left\{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right\}$$
$$\le \frac{1}{2}(M-m)^2.$$

Since

$$\max\left\{M - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}, \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - m\right\} = \frac{1}{2}(M-m) + \left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{1}{2}(m+M)\right|,$$

then the second inequality in (3.34) is not as good as the second inequality in (3.32).

For $p=-1$ and $0<m<M<\infty$ consider the convex function $f(t)=t^{-1}$ defined on $[m,M]$. Then $\Psi_{-1}(\cdot;m,M):(m,M)\to\mathbb{R}$ is defined by

$$\Psi_{-1}(t;m,M)=\frac{M^{-1}-t^{-1}}{M-t}-\frac{t^{-1}-m^{-1}}{t-m}=\frac{M-m}{mMt}.$$

The definition of $\Psi_{-1}(\cdot;m,M)$ can be extended to the closed interval $[m,M]$. We also have that

$$\sup_{t\in(m,M)}\Psi_{-1}(t;m,M)=\frac{M-m}{m^2M}.$$

From the inequality (3.20) we get

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\\&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\\&\le\frac{1}{m^2M}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)\\&\le\frac{1}{4}\frac{(M-m)^2(M+m)}{m^2M^2},\end{aligned}\tag{3.35}$$

while from (3.21) we get

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\\&\le\frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\\&\le\frac{1}{4}\frac{(M-m)^2}{mM}\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\le\frac{1}{4}\frac{(M-m)^2}{m^2M}\end{aligned}\tag{3.36}$$

for any positive definite operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge0$. Since $m>0$, then $\operatorname{tr}(PA)\ge m\operatorname{tr}(P)>0$.

From the inequality (3.27) we have

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA)}\\&\le\frac{(M-m)^2}{mM(m+M)}\max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m},\frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\\&\le\frac{(M-m)^2}{mM(m+M)},\end{aligned}\tag{3.37}$$

for any positive definite operator A with $\operatorname{Sp}(A)\subseteq[m,M]$ and any $P\in\mathcal{B}_1(H)\setminus\{0\}$, $P\ge0$.

In order to compare the upper bounds provided by (3.36) and (3.37) consider the difference

$$\begin{aligned}\Delta(m,M) &:= \frac{1}{4}\frac{(M-m)^2}{m^2M} - \frac{(M-m)^2}{mM(m+M)} \\ &= \frac{(M-m)^2}{mM}\left(\frac{1}{4m} - \frac{1}{m+M}\right) = \frac{(M-m)^2(M-3m)}{4m^2M(m+M)},\end{aligned}$$

where $0 < m < M$.

We observe that if $M < 3m$, then the upper bound provided by (3.36) is better than the bound provided by (3.37). The conclusion is the other way around if $M \geq 3m$.

If we consider the convex function $f(t) = -\ln t$ defined on $[m,M] \subset (0,\infty)$, then $\Psi_{-\ln}(\cdot;m,M) : (m,M) \to \mathbb{R}$ is defined by

$$\begin{aligned}\Psi_{-\ln}(t;m,M) &= \frac{-\ln M + \ln t}{M-t} - \frac{-\ln t + \ln m}{t-m} \\ &= \frac{(M-m)\ln t - (M-t)\ln m - (t-m)\ln M}{(M-t)(t-m)} \\ &= \ln\left(\frac{t^{M-m}}{m^{M-t}M^{t-m}}\right)^{\frac{1}{(M-t)(t-m)}}.\end{aligned}$$

Utilising the inequality (3.20) we have

$$\begin{aligned}0 &\leq \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \\ &\leq \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right) \\ &\leq \frac{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}{M-m}\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M) \\ &\leq \frac{1}{Mm}\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right) \leq \frac{(M-m)^2}{4mM},\end{aligned} \tag{3.38}$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m,M]$ and $P \in \mathcal{B}_1(H)\setminus\{0\}$, $P \geq 0$.

From (3.21) we have

$$0 \le \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \tag{3.39}$$
$$\le \frac{1}{M-m}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right)$$
$$\le \frac{1}{4}\frac{(M-m)}{\left(M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m\right)}\ln\left(\frac{\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^{M-m}}{m^{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}M^{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}}\right)$$
$$\le \frac{1}{4}(M-m)\sup_{t\in(m,M)}\Psi_{-\ln}(t;m,M)$$
$$\le \frac{(M-m)^2}{4mM},$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

From the inequality (3.27) we get

$$0 \le \ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)} \tag{3.40}$$
$$\le \max\left\{\frac{M-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{M-m}, \frac{\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}-m}{M-m}\right\}\ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right)$$
$$\le \ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right),$$

for any positive definite operator A with $\operatorname{Sp}(A) \subseteq [m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \ge 0$.

We observe that, since $\ln x \le x - 1$ for any $x > 0$, then

$$\ln\left(\frac{\left(\frac{m+M}{2}\right)^2}{mM}\right) \le \frac{\left(\frac{m+M}{2}\right)^2}{mM} - 1 = \frac{(M-m)^2}{4mM},$$

which shows that the absolute upper bound for

$$\ln\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) - \frac{\operatorname{tr}(P\ln A)}{\operatorname{tr}(P)}$$

provided by the inequality (3.40) is better than the one provided by (3.39).

3.5. **Reverses of Hölder's Inequality.** We have the following result:

Theorem 18 (Dragomir, 2014 [32])**.** *Assume that p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let S be a positive operator that commutes with Q, a positive invertible operator and such that there exists the constants k, $K > 0$ with*

$$k1_H \le SQ^{1-q} \le K1_H. \tag{3.41}$$

If S^p, $Q^q \in \mathcal{B}_1(H)$, then we have

$$0 \le [\operatorname{tr}(S^p)]^{1/p}[\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \le B_p(k, K)\operatorname{tr}(Q^q), \tag{3.42}$$

where

$$B_p(k,K)=\begin{cases}\frac{1}{4^{1/p}}p^{1/p}(K-k)^{1/p}\left(K^{p-1}-k^{p-1}\right)^{1/p},\\ \\ 2^{1/p}\left[\frac{k^p+K^p}{2}-\left(\frac{k+K}{2}\right)^p\right]^{1/p}.\end{cases}\tag{3.43}$$

Proof. If we write the inequality

$$0\le\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p\le\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right)\tag{3.44}$$

for the operators $P=Q^q$ and $A=SQ^{1-q}$ then we get

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(Q^q\left(SQ^{1-q}\right)^p\right)}{\operatorname{tr}(Q^q)}-\left(\frac{\operatorname{tr}\left(Q^qSQ^{1-q}\right)}{\operatorname{tr}(Q^q)}\right)^p\\&\le\frac{1}{4}p(K-k)\left(K^{p-1}-k^{p-1}\right).\end{aligned}\tag{3.45}$$

Observe that, by the properties of trace we have

$$\operatorname{tr}\left(Q^qSQ^{1-q}\right)=\operatorname{tr}\left(SQ^{1-q}Q^q\right)=\operatorname{tr}(SQ).$$

It is known, see for instance [52, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator T on H and two bounded functions φ and ψ such that $A=\varphi(T)$ and $B=\psi(T)$. Moreover, if $\{E_\lambda\}$ is the spectral family over the closed interval $[0,1]$ for the selfadjoint operator T, then $T=\int_{0-}^1\lambda dE_\lambda$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_\lambda\}$ on $[0,1]$ and

$$A=\varphi(T)=\int_{0-}^1\varphi(\lambda)\,dE_\lambda\text{ and }B=\psi(T)=\int_{0-}^1\psi(\lambda)\,dE_\lambda.$$

Now, if A and B are as above with $\operatorname{Sp}(A),\operatorname{Sp}(B)\subseteq J$ an interval of real numbers, then for any continuous functions $f,\,g:J\to\mathbb{C}$ we have the representations

$$f(A)=\int_{0-}^1(f\circ\varphi)(\lambda)\,dE_\lambda\text{ and }g(B)=\int_{0-}^1(g\circ\psi)(\lambda)\,dE_\lambda.$$

If we apply the above property to the commuting selfadjoint operators S and Q, then we have two positive functions φ and ψ such that $S=\varphi(T)$ and $Q=\psi(T)$. Moreover, using the integral representation for functions of selfadjoint operators, we have

$$\begin{aligned}Q^q\left(SQ^{1-q}\right)^p&=[\psi(T)]^q\left(\varphi(T)[\psi(T)]^{1-q}\right)^p\\&=\int_{0-}^1[\psi(\lambda)]^q\left(\varphi(\lambda)[\psi(\lambda)]^{1-q}\right)^pdE_\lambda\\&=\int_{0-}^1[\psi(\lambda)]^q[\varphi(\lambda)]^p[\psi(\lambda)]^{(1-q)p}dE_\lambda\\&=\int_{0-}^1[\varphi(\lambda)]^p[\psi(\lambda)]^{q+p-qp}dE_\lambda=\int_{0-}^1[\varphi(\lambda)]^pdE_\lambda=S^p.\end{aligned}$$

Therefore, the inequality (3.45) is equivalent to

$$0 \le \frac{\operatorname{tr}(S^p)}{\operatorname{tr}(Q^q)} - \left(\frac{\operatorname{tr}(SQ)}{\operatorname{tr}(Q^q)}\right)^p \le \frac{1}{4} p (K-k) \left(K^{p-1} - k^{p-1}\right), \tag{3.46}$$

which is of interest in itself.

From this inequality we have

$$\operatorname{tr}(S^p) [\operatorname{tr}(Q^q)]^{p-1} \le (\operatorname{tr}(SQ))^p + \frac{1}{4} p (K-k) \left(K^{p-1} - k^{p-1}\right) [\operatorname{tr}(Q^q)]^p .$$

Taking the power $1/p \in (0,1)$ and using the property that

$$(\alpha+\beta)^r \le \alpha^r + \beta^r, \text{ where } \alpha, \beta \ge 0 \text{ and } r \in (0,1),$$

we get

$$\begin{aligned} &[\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{(p-1)/p} \\ &\le \left[(\operatorname{tr}(SQ))^p + \frac{1}{4} p (K-k) \left(K^{p-1} - k^{p-1}\right) [\operatorname{tr}(Q^q)]^p\right]^{1/p} \\ &\le \operatorname{tr}(SQ) + \frac{1}{4^{1/p}} p^{1/p} (K-k)^{1/p} \left(K^{p-1} - k^{p-1}\right)^{1/p} [\operatorname{tr}(Q^q)], \end{aligned}$$

i.e.

$$\begin{aligned} &[\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \\ &\le \frac{1}{4^{1/p}} p^{1/p} (K-k)^{1/p} \left(K^{p-1} - k^{p-1}\right)^{1/p} [\operatorname{tr}(Q^q)] \end{aligned}$$

The second part follows from the inequality

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p \le 2\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right],$$

and the details are omitted. □

Remark 8. *We observe that under the previous assumptions, from any upper bound for the difference*

$$0 \le \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^p$$

we can deduce in a similar way an upper bound for the Hölder's difference

$$0 \le [\operatorname{tr}(S^p)]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ).$$

Also, if the commutativity property of the operators S and Q is dropped, then we can prove that

$$0 \le \left[\operatorname{tr}\left(Q^q \left(SQ^{1-q}\right)^p\right)\right]^{1/p} [\operatorname{tr}(Q^q)]^{1/q} - \operatorname{tr}(SQ) \le B_p(k,K) \operatorname{tr}(Q^q) \tag{3.47}$$

with the same $B_p(k,K)$. However, the noncommutative case of the second inequality in (3.42) is an open question for the author.

The following reverse of Schwarz inequality holds:

Corollary 10. *Let S be a positive operator that commutes with Q, a positive invertible operator and such that there exists the constants k, $K > 0$ with*

$$k 1_H \le SQ^{-1} \le K 1_H. \tag{3.48}$$

If S^2, $Q^2 \in \mathcal{B}_1(H)$, *then we have*

$$0 \leq \left[\operatorname{tr}\left(S^2\right)\right]^{1/2}\left[\operatorname{tr}\left(Q^2\right)\right]^{1/2} - \operatorname{tr}(SQ) \leq \frac{\sqrt{2}}{2}(K-k)\operatorname{tr}\left(Q^2\right). \tag{3.49}$$

Remark 9. *If we take* $p = q = 2$ *in (3.47) and drop the commutativity assumption, then we get*

$$0 \leq \left[\operatorname{tr}\left(QSQ^{-1}S\right)\right]^{1/2}\left[\operatorname{tr}\left(Q^2\right)\right]^{1/2} - \operatorname{tr}(SQ) \leq \frac{\sqrt{2}}{2}(K-k)\operatorname{tr}\left(Q^2\right),$$

provided that (3.48) holds true.

Also, if we use the inequality (3.32), then we have

$$\begin{aligned} 0 &\leq \operatorname{tr}\left(QSQ^{-1}S\right)\operatorname{tr}\left(Q^2\right) - \left[\operatorname{tr}(SQ)\right]^2 \\ &\leq \left(K\operatorname{tr}\left(Q^2\right) - \operatorname{tr}(SQ)\right)\left(\operatorname{tr}(SQ) - k\operatorname{tr}\left(Q^2\right)\right) \leq \frac{1}{4}(K-k)^2\left[\operatorname{tr}\left(Q^2\right)\right]^2 \end{aligned} \tag{3.50}$$

provided that (3.42) holds true.

4. Slater's Type Trace Inequalities

4.1. **Slater's Type Inequalities.** Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if x, $y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I.$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function. If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The following result is well known in the literature as *Slater inequality:*

Theorem 19 (Slater, 1981, [51])**.** *If* $f : I \to \mathbb{R}$ *is a nonincreasing (nondecreasing) convex function,* $x_i \in I$, $p_i \geq 0$ *with* $P_n := \sum_{i=1}^n p_i > 0$ *and* $\sum_{i=1}^n p_i\varphi(x_i) \neq 0$, *where* $\varphi \in \partial f$, *then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leq f\left(\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)}\right). \tag{4.1}$$

As pointed out in [11, p. 208], the monotonicity assumption for the derivative φ can be replaced with the condition

$$\frac{\sum_{i=1}^n p_i x_i \varphi(x_i)}{\sum_{i=1}^n p_i \varphi(x_i)} \in I, \tag{4.2}$$

which is more general and can hold for suitable points in I and for not necessarily monotonic functions.

The following result holds:

Theorem 20 (Dragomir, 2014 [31])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ (the interior of I) whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, then*

$$0 \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \le f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right), \tag{4.3}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. Since f is convex and differentiable on $\mathring{I}$, then we have

$$f'(s)(t-s) \le f(t) - f(s) \le f'(t)(t-s) \tag{4.4}$$

for any $t, s \in [m, M]$.

Now, if we fix $t \in [m, M]$ and use the continuous functional calculus for the operator A, then we have

$$tf'(A) - Af'(A) \le f(t) \cdot 1_H - f(A) \le f'(t)\, t \cdot 1_H - f'(t) A \tag{4.5}$$

for any $t \in [m, M]$.

If we apply the property of the trace to the inequality (4.5) then we have

$$t \operatorname{tr}[Pf'(A)] - \operatorname{tr}[PAf'(A)] \le f(t)\operatorname{tr}(P) - \operatorname{tr}[Pf(A)] \le f'(t)\, t \operatorname{tr}(P) - f'(t)\operatorname{tr}(PA) \tag{4.6}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Now, since A is selfadjoint with $m1_H \le A \le M1_H$ and $f'(A)$ is positive, then

$$mf'(A) \le Af'(A) \le Mf'(A).$$

If we apply again the property of the trace, then we get

$$m \operatorname{tr}[Pf'(A)] \le \operatorname{tr}[PAf'(A)] \le M \operatorname{tr}[Pf'(A)],$$

which shows that

$$t_0 := \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in [m, M].$$

Observe that since $f'(A)$ is a positive invertible operator on H, then $\operatorname{tr}[Pf'(A)] > 0$ for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Finally, if we put $t = t_0$ in the equation (4.6), then we get

$$\begin{aligned} &\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\operatorname{tr}[Pf'(A)] - \operatorname{tr}[PAf'(A)] \\ &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\operatorname{tr}(P) - \operatorname{tr}[Pf(A)] \\ &\le f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\operatorname{tr}(P) \\ &- f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\operatorname{tr}(PA), \end{aligned} \tag{4.7}$$

which is equivalent to the desired result (4.3). □

Remark 10. *It is important to observe that, the condition that $f'(A)$ is a positive invertible operator on H can be replaced with the more general assumption that*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I} \text{ and } \operatorname{tr}[Pf'(A)] \neq 0 \tag{4.8}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, which may be easily verified for particular convex functions f in various examples as follows.

Also, as pointed out by the referee, if $\langle f'(A)x, x\rangle > 0$ for any $x \in H$, $x \neq 0$, then $\operatorname{tr}[Pf'(A)] > 0$ for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and the inequality (4.3) is valid as well.

Remark 11. *Now, if the function is concave on $\mathring{I}$ and the condition (4.8) holds, then we have the inequalities*

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\ &\leq f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right), \end{aligned} \tag{4.9}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the inequality (4.9) for the concave function $f : (0, \infty) \to \mathbb{R}$, $f(t) = \ln t$, then we can state that

$$0 \leq \frac{\operatorname{tr}(P \ln A)}{\operatorname{tr}(P)} - \ln\left(\frac{\operatorname{tr}(P)}{\operatorname{tr}(PA^{-1})}\right) \leq \frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} - 1 \tag{4.10}$$

for any positive invertible operator A and P with $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Utilising the inequality (4.3) for the convex function $f : (0, \infty) \to \mathbb{R}$, $f(t) = t^{-1}$, then we can state that

$$0 \leq \frac{\operatorname{tr}\left(PA^{-2}\right)}{\operatorname{tr}(PA^{-1})} - \frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PA^{-2}\right)}{\operatorname{tr}(PA^{-1})} - \frac{\operatorname{tr}\left(PA^{-1}\right)}{\operatorname{tr}(PA^{-2})}, \tag{4.11}$$

for any positive invertible operator $A$and P with $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

If we take $B = A^{-1}$ in (4.11), then we get the equivalent inequality

$$0 \leq \frac{\operatorname{tr}\left(PB^2\right)}{\operatorname{tr}(PB)} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)} \leq \frac{\operatorname{tr}\left(PB^2\right)}{\operatorname{tr}(PB)} \frac{\operatorname{tr}\left(PB^{-1}\right)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PB)}{\operatorname{tr}(PB^2)}, \tag{4.12}$$

for any positive invertible operator B and P with $P \in \mathcal{B}_1(H) \setminus \{0\}$.

If we write the inequality (4.3) for the convex function $f(t) = \exp(\alpha t)$ with $\alpha \in \mathbb{R} \setminus \{0\}$, then we get

$$\begin{aligned} 0 &\leq \exp\left(\alpha \frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right) - \frac{\operatorname{tr}[P\exp(\alpha A)]}{\operatorname{tr}(P)} \\ &\leq \alpha \exp\left(\alpha \frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]}\right)\left(\frac{\operatorname{tr}[PA\exp(\alpha A)]}{\operatorname{tr}[P\exp(\alpha A)]} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right), \end{aligned} \tag{4.13}$$

for any selfadjoint operator A and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

4.2. **Further Reverses.** We use the following Grüss' type inequalities [28]:

Lemma 2 (Dragomir, 2014 [28])**.** *Let S be a selfadjoint operator with $m1_H \leq S \leq M1_H$ and $f : [m, M] \to \mathbb{C}$ a continuous function of bounded variation on $[m, M]$. For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leq \frac{1}{2}\bigvee_m^M(f)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq \frac{1}{2}\bigvee_m^M(f)\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}, \tag{4.14}$$

where $\bigvee_m^M(f)$ is the total variation of f on the interval.

If the function $f : [m, M] \to \mathbb{C}$ is Lipschitzian with the constant $L > 0$ on $[m, M]$, i.e.

$$|f(t) - f(s)| \leq L|t - s|$$

for any $t, s \in [m, M]$, then

$$\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \leq L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \leq L\left\|S - \frac{\operatorname{tr}(PS)}{\operatorname{tr}(P)}1_H\right\|\left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \tag{4.15}$$

for any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. For the sake of completeness we give here a simple proof.

We observe that, for any $\lambda \in \mathbb{C}$ we have

$$\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[P(A - \lambda 1_H)\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] = \frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left[PA\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] - \frac{\lambda}{\operatorname{tr}(P)}\operatorname{tr}\left[P\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right] = \frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}. \tag{4.16}$$

Taking the modulus in (4.16) and utilising the properties of the trace, we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[P\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)\right]\right| \\
&=\frac{1}{\operatorname{tr}(P)}\left|\operatorname{tr}\left[\left(A-\lambda 1_H\right)\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right]\right| \\
&\leq\left\|A-\lambda 1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)
\end{aligned}\tag{4.17}$$

for any $\lambda\in\mathbb{C}$.

From the inequality (4.17) we have

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\leq\left\|f(S)-\lambda 1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)
\end{aligned}\tag{4.18}$$

for any $\lambda\in\mathbb{C}$.

From (4.18) we get

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\leq\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right).
\end{aligned}\tag{4.19}$$

Since f is of bounded variation on $[m,M]$, then we have

$$\begin{aligned}
\left|f(t)-\frac{f(m)+f(M)}{2}\right|&=\left|\frac{f(t)-f(m)+f(t)-f(M)}{2}\right| \\
&\leq\frac{1}{2}\left[\left|f(t)-f(m)\right|+\left|f(M)-f(t)\right|\right]\leq\frac{1}{2}\bigvee_m^M(f)
\end{aligned}\tag{4.20}$$

for any $t\in[m,M]$.

From (4.20) we get in the order $\mathcal{B}(H)$ that

$$\left|f(S)-\frac{f(m)+f(M)}{2}1_H\right|\leq\frac{1}{2}\bigvee_m^M(f)\,1_H,$$

which implies that

$$\left\|f(S)-\frac{f(m)+f(M)}{2}1_H\right\|\leq\frac{1}{2}\bigvee_m^M(f).\tag{4.21}$$

Making use of (4.20) and (4.21) we get the first inequality in (4.14).

The second part is obvious by the Schwarz inequality for traces

$$\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)}{\operatorname{tr}(P)}\leq\left(\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)}{\operatorname{tr}(P)}\right)^{1/2},$$

and by noticing that

$$\frac{\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P^{1/2}\right|^2\right)}{\operatorname{tr}(P)}=\frac{\operatorname{tr}\left(P\left|C\right|^2\right)}{\operatorname{tr}(P)}-\left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \tag{4.22}$$

for any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

From (4.18) we also have

$$\begin{aligned}&\left|\frac{\operatorname{tr}(Pf(S)C)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(Pf(S))}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|\\&\le\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\|\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\end{aligned} \tag{4.23}$$

any $C\in\mathcal{B}(H)$ and $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Since

$$|f(t)-f(s)|\le L|t-s|$$

for any $t,\ s\in[m,M]$, then we have in the order $\mathcal{B}(H)$ that

$$|f(S)-f(s)1_H|\le L|S-s1_H|$$

for any $s\in[m,M]$. In particular, we have

$$\left|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right|\le L\left|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right|,$$

which implies that

$$\left\|f(S)-f\left(\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}\right)1_H\right\|\le L\left\|S-\frac{\operatorname{tr}(SP)}{\operatorname{tr}(P)}1_H\right\|$$

and by (4.23) we get the first inequality in (4.15).

The second part is obvious. □

We also have the following reverse of Schwarz inequality [28]:

Lemma 3 (Dragomir, 2014 [28])**.** *If C is a selfadjoint operator with $k1_H\le C\le K1_H$ for some real numbers $k<K$, then*

$$\begin{aligned}0&\le\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\\&\le\frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_H\right)P\right|\right)\\&\le\frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}\le\frac{1}{4}(K-k)^2,\end{aligned} \tag{4.24}$$

for any $P\in\mathcal{B}_1^+(H)\setminus\{0\}$.

Proof. If we take in (4.14) $f(t)=t$ and $S=C$ we get

$$\begin{aligned}\left|\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\right| &\leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_{H}\right)P\right|\right)\\ &\leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}.\end{aligned}\tag{4.25}$$

Since by (4.22) we have

$$\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\geq 0,$$

then by (4.25) we get

$$\begin{aligned}0 &\leq \frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\\ &\leq \frac{1}{2}(K-k)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(C-\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}1_{H}\right)P\right|\right)\\ &\leq \frac{1}{2}(K-k)\left[\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}.\end{aligned}\tag{4.26}$$

Utilising the inequality between the first and last term in (4.26) we also have

$$\left[\frac{\operatorname{tr}\left(PC^{2}\right)}{\operatorname{tr}(P)}-\left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^{2}\right]^{1/2}\leq\frac{1}{2}(K-k),$$

which proves the last part of (4.24). □

Theorem 21 (Dragomir, 2014 [31]). *Let I be an interval and $f: I\to\mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A)\subseteq[m,M]\subset\mathring{I}$ and $f'(A)$ is a positive invertible operator on H, or*

$$\frac{\operatorname{tr}\left[PAf'(A)\right]}{\operatorname{tr}\left[Pf'(A)\right]}\in\mathring{I},\quad \operatorname{tr}\left[Pf'(A)\right]\neq 0$$

for any $P\in\mathcal{B}_{1}^{+}(H)\setminus\{0\}$, then

$$\begin{aligned}0 &\leq f\left(\frac{\operatorname{tr}\left[PAf'(A)\right]}{\operatorname{tr}\left[Pf'(A)\right]}\right)-\frac{\operatorname{tr}\left[Pf(A)\right]}{\operatorname{tr}(P)}\\ &\leq \frac{\operatorname{tr}(P)}{\operatorname{tr}\left[Pf'(A)\right]}f'\left(\frac{\operatorname{tr}\left[PAf'(A)\right]}{\operatorname{tr}\left[Pf'(A)\right]}\right)L\left(P,A,f'(A)\right),\end{aligned}\tag{4.27}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *where*

$$
\begin{aligned}
L(P, A, f'(A)) &:= \frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \frac{\operatorname{tr}[Pf'(A)]}{\operatorname{tr}(P)} \\
&\le \begin{cases} \frac{1}{2}(f'(M) - f'(m)) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\ \\ \frac{1}{2}(M - m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \end{cases} \\
&\le \begin{cases} \frac{1}{2}(f'(M) - f'(m)) \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\
&\le \frac{1}{4}(f'(M) - f'(m))(M - m).
\end{aligned}
$$

Proof. Utilising Lemma 2 and Lemma 3 we have

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\
&\le \frac{1}{2}(f'(M) - f'(m)) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \frac{1}{2}(f'(M) - f'(m)) \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\le \frac{1}{4}(f'(M) - f'(m))(M - m)
\end{aligned} \tag{4.28}
$$

and

$$
\begin{aligned}
0 &\le \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \\
&\le \frac{1}{2}(M - m) \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}\left(P[f'(A)]^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\le \frac{1}{4}(f'(M) - f'(m))(M - m)
\end{aligned} \tag{4.29}
$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The positivity of

$$
\frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}
$$

follows by Čebyšev's trace inequality for synchronous functions of selfadjoint operators, see [27]. □

The case of convex and monotonic functions is as follows:

Corollary 11. *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and differentiable function on $\mathring{I}$ whose derivative f' is continuous on $\mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(m) > 0$, then*

$$0 \le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \le \frac{f'(M)}{f'(m)} L(P, A, f'(A)), \tag{4.30}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

The proof follows by (4.27) observing that

$$0 \le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \le \frac{f'(M)}{f'(m)}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

If we consider the monotonic nondecreasing convex function $f(t) = t^p$ with $p \ge 1$ and $t \ge 0$, then by (4.30) we have the sequence of inequalities

$$\begin{aligned} 0 &\le \left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(PA^{p-1})}\right)^p - \frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} \\ &\le p\left(\frac{M}{m}\right)^{p-1}\left(\frac{\operatorname{tr}(PA^p)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}\right) \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\quad\times \begin{cases} \left(M^{p-1} - m^{p-1}\right)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\ \\ (M - m)\frac{1}{\operatorname{tr}(P)}\operatorname{tr}\left(\left|\left(A^{p-1} - \frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \end{cases} \\ &\le \frac{1}{2}p^2\left(\frac{M}{m}\right)^{p-1} \\ &\quad\times \begin{cases} \left(M^{p-1} - m^{p-1}\right)\left[\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ (M - m)\left[\frac{\operatorname{tr}\left(PA^{2(p-1)}\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}\left(PA^{p-1}\right)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\le \frac{1}{4}p^2\left(\frac{M}{m}\right)^{p-1}\left(M^{p-1} - m^{p-1}\right)(M - m) \end{aligned} \tag{4.31}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ and A with $\operatorname{Sp}(A) \subseteq [m, M] \subset (0, \infty)$.

Theorem 22 (Dragomir, 2014 [31])**.** *Let I be an interval and $f : I \to \mathbb{R}$ be a convex and twice differentiable function on $\mathring{I}$ whose second derivative f'' is bounded on $\mathring{I}$, i.e. there is a positive constant K such that $0 \le f''(t) \le K$ for any $t \in \mathring{I}$. If A is a selfadjoint operator on the Hilbert space H with $\operatorname{Sp}(A) \subseteq [m, M] \subset \mathring{I}$ and $f'(A)$ is a positive invertible operator on H, or*

$$\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]} \in \mathring{I}, \ \ \operatorname{tr}[Pf'(A)] \neq 0$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, *then*

$$\begin{aligned}
(4.32)\qquad 0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le \frac{1}{2}(M-m)K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Proof. From (4.27) we have

$$\begin{aligned}
(4.33)\qquad 0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) L(P, A, f'(A)),
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

From (4.15) we also have

$$\begin{aligned}
(4.34)\qquad & (0 \le) L(P, A, f'(A)) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2}
\end{aligned}$$

for any $P \in \mathcal{B}_1^+(H) \setminus \{0\}$.

Therefore, by (4.33) and (4.34) we get

$$\begin{aligned}
0 &\le f\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) - \frac{\operatorname{tr}[Pf(A)]}{\operatorname{tr}(P)} \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right)P\right|\right) \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right) \\
&\le K\left\|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}1_H\right\| \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\
&\times \frac{\operatorname{tr}(P)}{\operatorname{tr}[Pf'(A)]} f'\left(\frac{\operatorname{tr}[PAf'(A)]}{\operatorname{tr}[Pf'(A)]}\right)
\end{aligned}$$

that proves the second and third inequalities in (4.32).

The last part follows by Lemma 3. □

The inequality (4.32) can be also written for the convex function $f(t) = t^p$ with $p \geq 1$ and $t \geq 0$, however the details are not presented here.

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