Received 10/03/17

REVERSES OF JENSEN'S INTEGRAL INEQUALITY AND APPLICATIONS: A SURVEY OF RECENT RESULTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Several new reverses of the celebrated Jensen's inequality for convex functions and Lebesgue integral on measurable spaces are surveyed. Applications for weighted discrete means, to Hölder inequality, Cauchy-Bunyakovsky-Schwarz inequality and for f-divergence measures in information theory are also given. Finally, applications for functions of selfadjoint operators in Hilbert spaces with some examples of interest are also provided.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}$. For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. We also assume that $\int_\Omega w d\mu = 1$.

An useful result that is used to provide simpler upper bounds for the difference in Jensen's inequality is the Gruss' inequality. We recall now some facts related to this famous result.

If $f,\ g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(f,g) := \int_\Omega wfg d\mu - \int_\Omega wf d\mu \int_\Omega wg d\mu. \tag{1.1}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for μ -a.e. a. $x \in \Omega$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Note that if $\Omega = \{1, \dots, n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left|\sum_{i=1}^n w_i x_i y_i - \sum_{i=1}^n w_i x_i \cdot \sum_{i=1}^n w_i y_i\right| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.4}$$

1991 *Mathematics Subject Classification.* Primary 26D15, 26D20; Secondary 94A05, 47A63.

Key words and phrases. Jensen's inequality, Reverse of Jensen's inequality, Measurable functions, Lebesgue integral, Divergence measures, f-Divergence measures, Functions of selfadjoint operators, Spectral representation theorem.

RGMIA Res. Rep. Coll. 20 (2017), Art. 26, 60 pp.

provided $\gamma \leq x_i \leq \Gamma$, $\delta \leq y_i \leq \Delta$ for each $i \in \{1, \dots, n\}$ and $w_i \geq 0$ with $W_n := \sum_{i=1}^n w_i = 1$.

With the above assumptions, if $f \in L_w(\Omega, \mu)$ then we may define

$$D_w(f) := D_{w,1}(f) := \int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu. \tag{1.5}$$

In 2002, Cerone & Dragomir [7] obtained the following refinement of the Grüss inequality (1.2):

Theorem 1 (Cerone & Dragomir, 2002 [7])**.** *Let w, f, $g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ-a.e. (almost everywhere) on Ω and $\int_\Omega w d\mu = 1$. If f, g, $fg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for} \quad \mu\text{-a.e. } x \in \Omega, \tag{1.6}$$

then we have the inequality

$$|T_w(f, g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Remark 1. *The inequality (1.7) was obtained for the particular case $\Omega = [a, b]$ and the uniform weight $w(t) = 1$, $t \in [a, b]$ by X. L. Cheng and J. Sun in [6]. However, in that paper the authors did not prove the sharpness of the constant $\frac{1}{2}$.*

For $f \in L_{p,w}(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w |f|^p d\mu < \infty \right\}$, $p \geq 1$ we may also define

$$D_{w,p}(f) := \left[\int_\Omega w \left| f - \int_\Omega w f d\mu \right|^p d\mu \right]^{\frac{1}{p}} = \left\| f - \int_\Omega w f d\mu \right\|_{\Omega,p} \tag{1.8}$$

where $\|\cdot\|_{\Omega,p}$ is the usual p-norm on $L_{p,w}(\Omega, \mathcal{A}, \mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_\Omega w |h|^p d\mu \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Using Hölder's inequality we get

$$D_{w,1}(f) \leq D_{w,p}(f) \quad \text{for } p \geq 1,\ f \in L_{p,w}(\Omega, \mathcal{A}, \mu); \tag{1.9}$$

and, in particular for $p = 2$

$$D_{w,1}(f) \leq D_{w,2}(f) := \left[\int_\Omega w f^2 d\mu - \left(\int_\Omega w f d\mu \right)^2 \right]^{\frac{1}{2}}, \tag{1.10}$$

if $f \in L_{2,w}(\Omega, \mathcal{A}, \mu)$.

For $f \in L_\infty(\Omega, \mathcal{A}, \mu) := \left\{ f : \Omega \to \mathbb{R},\ \|f\|_{\Omega,\infty} := \operatorname{essup}_{x \in \Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \int_\Omega w f d\mu \right\|_{\Omega,\infty}. \tag{1.11}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$
\begin{aligned}
|T_w(f,g)| &\leq \frac{1}{2}(\Delta-\delta) D_w(f) \\
&\leq \frac{1}{2}(\Delta-\delta) D_{w,p}(f) \quad \text{if } f \in L_p(\Omega,\mathcal{A},\mu),\ 1<p<\infty; \\
&\leq \frac{1}{2}(\Delta-\delta) D_{w,\infty}(f) \quad \text{if } f \in L_\infty(\Omega,\mathcal{A},\mu).
\end{aligned} \tag{1.12}
$$

Remark 2. *The inequalities in (1.12) are in order of increasing coarseness. If we assume that $-\infty<\gamma\leq f(x)\leq\Gamma<\infty$ for μ-a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ we have for $p=2$*

$$
\left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{2}(\Gamma-\gamma). \tag{1.13}
$$

By (1.12), we deduce the following sequence of inequalities

$$
\begin{aligned}
|T_w(f,g)| &\leq \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|f-\int_\Omega wf d\mu\right| d\mu \\
&\leq \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma)
\end{aligned} \tag{1.14}
$$

for $f,\ g:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\leq f(x)<\Gamma<\infty$, $-\infty<\delta\leq g(x)\leq\Delta<\infty$ for μ-a.e. $x\in\Omega$. Thus, the inequality (1.14) is a refinement of Grüss' inequality (1.2).

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [14] the following result:

Theorem 2 (Dragomir, 2002 [14])**.** *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f) f\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu=1$. Then we have the inequality:*

$$
\begin{aligned}
0 &\leq \int_\Omega w(\Phi\circ f) d\mu - \Phi\left(\int_\Omega wf d\mu\right) \\
&\leq \int_\Omega w(\Phi'\circ f) f d\mu - \int_\Omega w(\Phi'\circ f) d\mu \int_\Omega wf d\mu \\
&\leq \frac{1}{2}[\Phi'(M)-\Phi'(m)]\int_\Omega w\left|f-\int_\Omega wf d\mu\right| d\mu.
\end{aligned} \tag{1.15}
$$

For a generalization of the first inequality when differentiability is not assumed and the derivative Φ' is replaced with a selection φ from the subdifferential $\partial\Phi$, see the paper [42] by C. P. Niculescu.

Remark 3. *If $\mu(\Omega) < \infty$ and $\Phi \circ f,\ f,\ \Phi' \circ f,\ (\Phi' \circ f) f \in L(\Omega, \mu)$, then we have the inequality:*

$$
\begin{aligned}
(1.16) \qquad 0 &\le \frac{1}{\mu(\Omega)} \int_\Omega (\Phi \circ f)\, d\mu - \Phi\left(\frac{1}{\mu(\Omega)} \int_\Omega f d\mu\right) \\
&\le \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f) f d\mu - \frac{1}{\mu(\Omega)} \int_\Omega (\Phi' \circ f)\, d\mu \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \\
&\le \frac{1}{2} [\Phi'(M) - \Phi'(m)] \frac{1}{\mu(\Omega)} \int_\Omega \left| f - \frac{1}{\mu(\Omega)} \int_\Omega f d\mu \right| d\mu.
\end{aligned}
$$

Remark 4. *On making use of (1.15) and (1.14), one can state the following string of reverse inequalities for the Jensen's difference*

$$
\begin{aligned}
(1.17) \qquad 0 &\le \int_\Omega w (\Phi \circ f)\, d\mu - \Phi\left(\int_\Omega w f d\mu\right) \\
&\le \int_\Omega w (\Phi' \circ f) f d\mu - \int_\Omega w (\Phi' \circ f)\, d\mu \int_\Omega w f d\mu \\
&\le \frac{1}{2} [\Phi'(M) - \Phi'(m)] \int_\Omega w \left| f - \int_\Omega w f d\mu \right| d\mu \\
&\le \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[\int_\Omega w f^2 d\mu - \left(\int_\Omega w f d\mu\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m).
\end{aligned}
$$

We notice that the inequality between the first, second and last term from (1.17) was proved in the general case of positive linear functionals in 2001 by S. S. Dragomir in [13].

The discrete case is as follows. Let $\bar{\mathbf{a}} = (a_1, \dots, a_n)$, $\bar{\mathbf{b}} = (b_1, \dots, b_n)$, $\bar{\mathbf{p}} = (p_1, \dots, p_n)$ be n-tuples of real numbers with $p_i \ge 0$ $(i \in \{1, \dots, n\})$ and $\sum_{i=1}^n p_i = 1$. If $b \le b_i \le B,\ i \in \{1, \dots, n\}$, then one has the inequality

$$
\begin{aligned}
(1.18) \qquad \left| \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right| &\le \frac{1}{2} (B - b) \sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right| \\
&\le \frac{1}{2} (B - b) \left[\sum_{i=1}^n p_i \left| a_i - \sum_{j=1}^n p_j a_j \right|^p\right]^{\frac{1}{p}} \\
&\le \frac{1}{2} (B - b) \max_{i=\overline{1,n}} \left| a_i - \sum_{j=1}^n p_j a_j \right|,
\end{aligned}
$$

where $1 < p < \infty$. The constant $\frac{1}{2}$ is sharp in the first inequality.

If more information about the vector $\mathbf{\bar{a}} = (a_1, \dots, a_n)$ is available, namely, if there exists the constants a and A such that $a \leq a_i \leq A$, $i \in \{1, \dots, n\}$, then

$$\begin{aligned} \left| \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right| &\leq \frac{1}{2} (B - b) \sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right| \\ &\leq \frac{1}{2} (B - b) \left[\sum_{i=1}^{n} p_i \left| a_i - \sum_{j=1}^{n} p_j a_j \right|^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4} (B - b) (A - a), \end{aligned} \tag{1.19}$$

with the constants $\frac{1}{2}$ and $\frac{1}{4}$ best possible.

Corollary 2. *Let $\Phi : [m, M] \to \mathbb{R}$ be a differentiable convex function on (m, M). If $x_i \in [m, M]$ and $w_i \geq 0$ $(i = 1, \dots, n)$ with $W_n := \sum_{i=1}^{n} w_i = 1$, then one has the reverse of Jensen's weighted discrete inequality:*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i \right) \\ &\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right|. \end{aligned} \tag{1.20}$$

Remark 5. *We notice that the inequality between the first and second term in (1.20) was proved in 1994 by Dragomir & Ionescu, see* [26].

On utilizing (1.20) and (1.19) we can state the string of inequalities

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} w_i \Phi(x_i) - \Phi\left(\sum_{i=1}^{n} w_i x_i \right) \\ &\leq \sum_{i=1}^{n} w_i \Phi'(x_i) x_i - \sum_{i=1}^{n} w_i \Phi'(x_i) \sum_{i=1}^{n} w_i x_i \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \sum_{i=1}^{n} w_i \left| x_i - \sum_{j=1}^{n} w_j x_j \right| \\ &\leq \frac{1}{2} [\Phi'(M) - \Phi'(m)] \left[\sum_{i=1}^{n} w_i x_i^2 - \left(\sum_{i=1}^{n} w_i x_i \right)^2 \right]^{1/2} \\ &\leq \frac{1}{4} [\Phi'(M) - \Phi'(m)] (M - m). \end{aligned} \tag{1.21}$$

We notice that the inequality between the first, second and last term in (1.21) was proved in 1999 by S. S. Dragomir in [12].

In this paper we survey several new reverses of the celebrated Jensen's inequality for convex functions and Lebesgue integral on measurable spaces. Applications for weighted discrete means, to Hölder inequality, Cauchy-Bunyakovsky-Schwarz inequality and for f-divergence measures in information theory are also given. Finally, applications for functions of selfadjoint operators in Hilbert spaces with some examples of interest are also provided.

2. A Refinement and a Divided-Difference Reverse

2.1. **General Results.** Following Roberts and Varberg [46, p. 5], we recall that if $f : I \to \mathbb{R}$ is a convex function, then for any $x_0 \in \mathring{I}$ (the interior of the interval I) the limits

$$f'_-(x_0) := \lim_{x\to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \text{ and } f'_+(x_0) := \lim_{x\to x_0+} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and $f'_-(x_0) \le f'_+(x_0)$. The functions f'_- and f'_+ are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [46, p. 7]).

From the monotonicity of the lateral derivatives f'_- and f'_+ we also have *the gradient inequality*

$$f'_-(x)(x - y) \ge f(x) - f(y) \ge f'_+(y)(x - y)$$

for any $x, y \in \mathring{I}$.

If $I = [a, b]$, then at the end points we also have the inequalities

$$f(x) - f(a) \ge f'_+(a)(x - a)$$

for any $x \in (a, b]$ and

$$f(y) - f(b) \ge f'_-(b)(y - b)$$

for any $y \in [a, b)$.

For a real function $g : [m, M] \to \mathbb{R}$ and two distinct points $\alpha, \beta \in [m, M]$ we recall that the *divided difference* of g in these points is defined by

$$[\alpha, \beta; g] := \frac{g(\beta) - g(\alpha)}{\beta - \alpha}.$$

In what follows, we assume that $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. $x \in \Omega$, is a μ-measurable function with $\int_\Omega w d\mu = 1$.

Theorem 3 (Dragomir, 2011 [20]). *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ the interior of I. If $f : \Omega \to \mathbb{R}$, is μ-measurable, satisfying the bounds*

$$-\infty < m \le f(x) \le M < \infty \quad \textit{for } \mu\textit{-a.e. } x \in \Omega \tag{2.1}$$

and such that $f, \Phi \circ f \in L_w(\Omega, \mu)$, then by denoting

$$\overline{f}_{\Omega,w} := \int_\Omega w f d\mu \in [m, M]$$

and assuming that $\overline{f}_{\Omega,w} \neq m,\ M,$ *we have*

$$\begin{aligned}
(2.2)\qquad & \left| \int_\Omega \left| \Phi(f) - \Phi\left(\overline{f}_{\Omega,w}\right) \right| \operatorname{sgn}\left[f - \overline{f}_{\Omega,w}\right] w d\mu \right| \\
& \leq \int_\Omega (\Phi \circ f)\, w d\mu - \Phi\left(\overline{f}_{\Omega,w}\right) \\
& \leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_w(f) \\
& \leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) D_{w,2}(f) \\
& \leq \frac{1}{4}\left(\left[\overline{f}_{\Omega,w}, M; \Phi\right] - \left[m, \overline{f}_{\Omega,w}; \Phi\right]\right) (M - m),
\end{aligned}$$

where sgn *is the sign function, i.e.* $\operatorname{sgn}(x) = \frac{x}{|x|}$ *for* $x \neq 0$ *and* $\operatorname{sgn}(0) = 0$. *The constant* $\frac{1}{2}$ *in the second inequality from (2.2) is best possible.*

Proof. We recall that if $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $\alpha \in I$ then the divided difference function $\Phi_\alpha : I \setminus \{\alpha\} \to \mathbb{R}$,

$$\Phi_\alpha(t) := [\alpha, t; \Phi] := \frac{\Phi(t) - \Phi(\alpha)}{t - \alpha}$$

is monotonic nondecreasing on $I \setminus \{\alpha\}$.

For f as considered in the statement of the theorem we can assume that that it is not constant μ-almost every where, since for that case the inequality (2.2) is trivially satisfied.

For $\overline{f}_{\Omega,w} \in (m, M)$, we consider now the function defined μ-almost everywhere on Ω by

$$\Phi_{\overline{f}_{\Omega,w}}(x) := \frac{\Phi(f(x)) - \Phi\left(\overline{f}_{\Omega,w}\right)}{f(x) - \overline{f}_{\Omega,w}}.$$

We will show that $\Phi_{\overline{f}_{\Omega,w}}$ and $h := f - \overline{f}_{\Omega,w}$ are synchronous μ-a.e. on Ω.

Let $x,\ y \in \Omega$ with $f(x),\ f(y) \neq \overline{f}_{\Omega,w}$. Assume that $f(x) \geq f(y)$, then

$$(2.3)\qquad \Phi_{\overline{f}_{\Omega,w}}(x) = \frac{\Phi(f(x)) - \Phi\left(\overline{f}_{\Omega,w}\right)}{f(x) - \overline{f}_{\Omega,w}} \geq \frac{\Phi(f(y)) - \Phi\left(\overline{f}_{\Omega,w}\right)}{f(y) - \overline{f}_{\Omega,w}} = \Phi_{\overline{f}_{\Omega,w}}(y)$$

and

$$(2.4)\qquad h(x) \geq h(y),$$

which shows that

$$(2.5)\qquad \left[\Phi_{\overline{f}_{\Omega,w}}(x) - \Phi_{\overline{f}_{\Omega,w}}(y)\right][h(x) - h(y)] \geq 0.$$

If $f(x) < f(y)$, then the inequalities (2.3) and (2.4) reverse but the inequality (2.5) still holds true.

This show that for μ-a.e. $x, y \in \Omega$ we have (2.5) and the claim is proven as stated.

Utilising the continuity property of the modulus we have

$$\begin{aligned}
&\left|\left[\left|\Phi_{\overline{f}_{\Omega,w}}(x)\right|-\left|\Phi_{\overline{f}_{\Omega,w}}(y)\right|\right][h(x)-h(y)]\right| \\
&\leq\left|\left[\Phi_{\overline{f}_{\Omega,w}}(x)-\Phi_{\overline{f}_{\Omega,w}}(y)\right][h(x)-h(y)]\right| \\
&=\left[\Phi_{\overline{f}_{\Omega,w}}(x)-\Phi_{\overline{f}_{\Omega,w}}(y)\right][h(x)-h(y)]
\end{aligned}$$

for μ-a.e. $x,y\in\Omega$.

Multiplying with $w(x)$, $w(y)\geq 0$ and integrating over $\mu(x)$ and $\mu(y)$ we have

$$\begin{aligned}
&\left|\int_\Omega\int_\Omega\left[\left|\Phi_{\overline{f}_{\Omega,w}}(x)\right|-\left|\Phi_{\overline{f}_{\Omega,w}}(y)\right|\right]\right. \\
&\left.\times[h(x)-h(y)]\,w(x)\,w(y)\,d\mu(x)\,d\mu(y)\right| \\
&\leq\int_\Omega\int_\Omega\left[\Phi_{\overline{f}_{\Omega,w}}(x)-\Phi_{\overline{f}_{\Omega,w}}(y)\right] \\
&\times[h(x)-h(y)]\,w(x)\,w(y)\,d\mu(x)\,d\mu(y)\,.
\end{aligned}\tag{2.6}$$

A simple calculation shows that

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega\left[\left|\Phi_{\overline{f}_{\Omega,w}}(x)\right|-\left|\Phi_{\overline{f}_{\Omega,w}}(y)\right|\right] \\
&\times[h(x)-h(y)]\,w(x)\,w(y)\,d\mu(x)\,d\mu(y) \\
&=\int_\Omega\left|\Phi_{\overline{f}_{\Omega,w}}(x)\right|h(x)\,w(x)\,d\mu(x) \\
&-\int_\Omega\left|\Phi_{\overline{f}_{\Omega,w}}(x)\right|w(x)\,d\mu(x)\int_\Omega w(x)\,h(x)\,d\mu(x) \\
&=\int_\Omega\left|\frac{\Phi(f(x))-\Phi\left(\overline{f}_{\Omega,w}\right)}{f(x)-\overline{f}_{\Omega,w}}\right|\left[f(x)-\overline{f}_{\Omega,w}\right]w(x)\,d\mu(x) \\
&=\int_\Omega\left|\Phi(f(x))-\Phi\left(\overline{f}_{\Omega,w}\right)\right|\operatorname{sgn}\left[f(x)-\overline{f}_{\Omega,w}\right]w(x)\,d\mu(x)
\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega\left[\Phi_{\overline{f}_{\Omega,w}}(x)-\Phi_{\overline{f}_{\Omega,w}}(y)\right] \\
&\times[h(x)-h(y)]\,w(x)\,w(y)\,d\mu(x)\,d\mu(y) \\
&=\int_\Omega\Phi_{\overline{f}_{\Omega,w}}(x)\,h(x)\,w(x)\,d\mu(x) \\
&-\int_\Omega\Phi_{\overline{f}_{\Omega,w}}(x)\,w(x)\,d\mu(x)\int_\Omega h(x)\,w(x)\,d\mu(x) \\
&=\int_\Omega\frac{\Phi(f(x))-\Phi\left(\overline{f}_{\Omega,w}\right)}{f(x)-\overline{f}_{\Omega,w}}\left[f(x)-\overline{f}_{\Omega,w}\right]w(x)\,d\mu(x) \\
&=\int_\Omega\left[\Phi(f(x))-\Phi\left(\overline{f}_{\Omega,w}\right)\right]w(x)\,d\mu(x) \\
&=\int_\Omega w(\Phi\circ f)\,d\mu-\Phi\left(\overline{f}_{\Omega,w}\right).
\end{aligned}\tag{2.8}$$

On making use of the identities (2.7) and (2.8) we obtain from (2.6) the first inequality in (2.2).

Now, since f satisfies the condition (2.1) then we have that

$$\begin{aligned} \left[m,\overline{f}_{\Omega,w};\Phi\right] &= \frac{\Phi\left(\overline{f}_{\Omega,w}\right)-\Phi\left(m\right)}{\overline{f}_{\Omega,w}-m} \leq \Phi_{\overline{f}_{\Omega,w}}\left(x\right) \\ &\leq \frac{\Phi\left(M\right)-\Phi\left(\overline{f}_{\Omega,w}\right)}{M-\overline{f}_{\Omega,w}} = \left[\overline{f}_{\Omega,w},M;\Phi\right] \end{aligned} \tag{2.9}$$

for μ-a.e. $x\in\Omega$.

Applying now the Grüss' type inequality (1.7) and taking into account the second part of the equality in (2.7) we have that

$$\begin{aligned} &\int_\Omega w\left(\Phi\circ f\right)d\mu-\Phi\left(\overline{f}_{\Omega,w}\right) \\ &\leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)\int_\Omega w\left|f-\overline{f}_{\Omega,w}\right|d\mu \end{aligned}$$

which proves the second inequality in (2.2).

The other two bounds are obvious from the comments in the introduction.

It is obvious that from (2.2) we get the following reverse of the first Hermite-Hadamard inequality for the convex function $\Phi:[a,b]\to\mathbb{R}$

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \Phi\left(t\right)dt-\Phi\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\left(\left[\frac{a+b}{2},b;\Phi\right]-\left[a,\frac{a+b}{2};\Phi\right]\right)D_w\left(e\right) \end{aligned} \tag{2.10}$$

where $e\left(t\right)=t, t\in\left[a,b\right].$

Since a simple calculation shows that

$$\begin{aligned} &\frac{1}{2}\left(\left[\frac{a+b}{2},b;\Phi\right]-\left[a,\frac{a+b}{2};\Phi\right]\right) \\ &=\frac{2}{b-a}\left[\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\Phi\left(\frac{a+b}{2}\right)\right] \end{aligned}$$

and

$$D_w\left(e\right)=\frac{1}{b-a}\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}\left(b-a\right),$$

and we get from (2.10) that

$$\begin{aligned} 0 &\leq \frac{1}{b-a}\int_a^b\Phi\left(t\right)dt-\Phi\left(\frac{a+b}{2}\right) \\ &\leq \frac{1}{2}\left[\frac{\Phi\left(a\right)+\Phi\left(b\right)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]. \end{aligned} \tag{2.11}$$

To prove the sharpness of the constant $\frac{1}{2}$ in the second inequality from (2.2) we need now only to show that the equality case in (2.11) is realized.

If we take, for instance $\Phi\left(t\right)=\left|t-\frac{a+b}{2}\right|,\ t\in\left[a,b\right],$ then we observe that Φ is convex and we get in both sides of (2.11) the same quantity $\frac{1}{4}\left(b-a\right).$ $\square$

Corollary 3. *With the assumptions in Theorem 3 and if the lateral derivatives $\Phi'_{+}(m)$ and $\Phi'_{-}(M)$ are finite, then we have the inequalities*

$$
\begin{aligned}
0 &\leq \int_{\Omega}(\Phi\circ f)\,w d\mu-\Phi\left(\overline{f}_{\Omega,w}\right) \\
&\leq \frac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)D_{w}(f) \\
&\leq \frac{1}{2}\left(\Phi'_{-}(M)-\Phi'_{+}(m)\right)D_{w}(f) \\
&\leq \frac{1}{2}\left(\Phi'_{-}(M)-\Phi'_{+}(m)\right)D_{w,2}(f) \\
&\leq \frac{1}{4}\left(\Phi'_{-}(M)-\Phi'_{+}(m)\right)(M-m).
\end{aligned}
\tag{2.12}
$$

The constant $\frac{1}{2}$ in the second and third inequality from (2.12) is best possible.

Proof. We need to prove only the third inequality.

By the convexity of Φ we have the gradient inequalities

$$\frac{\Phi(M)-\Phi\left(\overline{f}_{\Omega,w}\right)}{M-\overline{f}_{\Omega,w}}\leq\Phi'_{-}(M)$$

and

$$\frac{\Phi\left(\overline{f}_{\Omega,w}\right)-\Phi(m)}{\overline{f}_{\Omega,w}-m}\geq\Phi'_{+}(m).$$

These imply that

$$\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\leq\Phi'_{-}(M)-\Phi'_{+}(m)$$

and the proof is concluded.

We observe that from (2.12) we get the following reverse of the Hermite-Hadamard inequality for the convex function $\Phi:[a,b]\to\mathbb{R}$ having finite lateral derivative $\Phi'_{+}(a)$ and $\Phi'_{-}(b)$

$$
\begin{aligned}
&\frac{1}{b-a}\int_{a}^{b}\Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right) \\
&\leq\frac{1}{2}\left[\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right]\leq\frac{1}{8}\left[\Phi'_{-}(b)-\Phi'_{+}(a)\right](b-a).
\end{aligned}
\tag{2.13}
$$

We observe that the convex function $\Phi(t)=\left|t-\frac{a+b}{2}\right|$ has finite lateral derivatives

$$\Phi'_{-}(b)=1\text{ and }\Phi'_{+}(a)=-1$$

and replacing this function in (2.13) we get in all terms the same quantity $\frac{1}{4}(b-a)$.

This proves that the constant $\frac{1}{2}$ in the second and third inequality from (2.12) is best possible. □

Remark 6. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,\ M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ the interior of I. Let $\bar{\mathbf{a}}=(a_1,\ldots,a_n)$, $\bar{\mathbf{p}}=(p_1,\ldots,p_n)$ be n-tuples of real numbers with $p_i\geq 0$*

$(i \in \{1, \dots, n\})$ and $\sum_{i=1}^{n} p_i = 1$. If $m \le a_i \le M$, $i \in \{1, \dots, n\}$, with $\sum_{i=1}^{n} p_i a_i \ne m, M$, then

$$\left|\sum_{i=1}^{n} p_i \left[\left|\Phi(a_i)\right| - \left|\Phi\left(\sum_{i=1}^{n} p_i a_i\right)\right|\right] \operatorname{sgn}\left|a_i - \sum_{j=1}^{n} p_j a_j\right|\right| \tag{2.14}$$
$$\le \sum_{i=1}^{n} p_i \Phi(a_i) - \Phi\left(\sum_{i=1}^{n} p_i a_i\right)$$
$$\le \frac{1}{2}\left(\left[\sum_{i=1}^{n} p_i a_i, M; \Phi\right] - \left[m, \sum_{i=1}^{n} p_i a_i; \Phi\right]\right) \sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|.$$

If the lateral derivatives $\Phi'_+(m)$ and $\Phi'_-(M)$ are finite, then we also have the inequalities

$$0 \le \sum_{i=1}^{n} p_i \Phi(a_i) - \Phi\left(\sum_{i=1}^{n} p_i a_i\right) \tag{2.15}$$
$$\le \frac{1}{2}\left(\left[\sum_{i=1}^{n} p_i a_i, M; \Phi\right] - \left[m, \sum_{i=1}^{n} p_i a_i; \Phi\right]\right) \sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|$$
$$\le \frac{1}{2}\left(\Phi'_-(M) - \Phi'_+(m)\right) \sum_{i=1}^{n} p_i \left|a_i - \sum_{j=1}^{n} p_j a_j\right|.$$

Remark 7. *Define the weighted arithmetic mean of the positive n-tuple $x = (x_1, ..., x_n)$ with the nonnegative weights $w = (w_1, ..., w_n)$ by*

$$A_n(w, x) := \frac{1}{W_n} \sum_{i=1}^{n} w_i x_i$$

where $W_n := \sum_{i=1}^{n} w_i > 0$ and the weighted geometric mean of the same n-tuple, by

$$G_n(w, x) := \left(\prod_{i=1}^{n} x_i^{w_i}\right)^{1/W_n}.$$

It is well know that the following arithmetic mean-geometric mean inequality holds

$$A_n(w, x) \ge G_n(w, x).$$

Applying the inequality (2.15) for the convex function $\Phi(t) = -\ln t$, $t > 0$ we have the following reverse of the arithmetic mean-geometric mean inequality

$$1 \le \frac{A_n(w, x)}{G_n(w, x)} \tag{2.16}$$
$$\le \left[\frac{\left(\frac{A_n(w,x)}{m}\right)^{A_n(w,x)-m}}{\left(\frac{M}{A_n(w,x)}\right)^{M-A_n(w,x)}}\right]^{\frac{1}{2}A_n(w,|x-A_n(w,x)|)}$$
$$\le \exp\left[\frac{1}{2}\frac{M-m}{mM} A_n(w, |x - A_n(w, x)|)\right],$$

provided that $0 < m \le x_i \le M < \infty$ for $i \in \{1, ..., n\}$.

2.2. **Applications for the Hölder Inequality.** It is well known that if $f \in L_p(\Omega,\mu)$, $p>1$, where the Lebesgue space $L_p(\Omega,\mu)$ is defined by

$$L_p(\Omega,\mu) := \{f:\Omega\to\mathbb{R},\ f \text{ is } \mu\text{-measurable and } \int_\Omega |f(x)|^p d\mu(x) < \infty\}$$

and $g \in L_q(\Omega,\mu)$ with $\frac{1}{p}+\frac{1}{q}=1$ then $fg \in L(\Omega,\mu) := L_1(\Omega,\mu)$ and the *Hölder inequality* holds true

$$\int_\Omega |fg|\, d\mu \le \left(\int_\Omega |f|^p d\mu\right)^{1/p}\left(\int_\Omega |g|^p d\mu\right)^{1/q}.$$

Assume that $p>1$. If $h:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds

$$-\infty < m \le |h(x)| \le M < \infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and is such that h, $|h|^p \in L_w(\Omega,\mu)$, for a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ -a.e. $x\in\Omega$ and $\int_\Omega w d\mu>0$, then from (2.2) we have

$$\begin{aligned}
(2.17)\qquad & \left|\int_\Omega \left||h|^p - \overline{|h|}^p_{\Omega,w}\right| \operatorname{sgn}\left[|h| - \overline{|h|}_{\Omega,w}\right] w d\mu\right| \\
&\le \frac{\int_\Omega |h|^p w d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega |h|\, w d\mu}{\int_\Omega w d\mu}\right)^p \\
&\le \frac{1}{2}\left(\left[\overline{|h|}_{\Omega,w}, M; (\cdot)^p\right] - \left[m, \overline{|h|}_{\Omega,w}; (\cdot)^p\right]\right)\tilde{D}_w(|h|) \\
&\le \frac{1}{2}\left(\left[\overline{|h|}_{\Omega,w}, M; (\cdot)^p\right] - \left[m, \overline{|h|}_{\Omega,w}; (\cdot)^p\right]\right)\tilde{D}_{w,2}(|h|) \\
&\le \frac{1}{4}\left(\left[\overline{|h|}_{\Omega,w}, M; (\cdot)^p\right] - \left[m, \overline{|h|}_{\Omega,w}; (\cdot)^p\right]\right)(M-m),
\end{aligned}$$

where $\overline{|h|}_{\Omega,w} := \frac{\int_\Omega |h| w d\mu}{\int_\Omega w d\mu} \in [m,M]$ and

$$\tilde{D}_w(|h|) := \frac{1}{\int_\Omega w d\mu}\int_\Omega w \left| |h| - \frac{\int_\Omega |h|\, w d\mu}{\int_\Omega w d\mu}\right| d\mu$$

while

$$\tilde{D}_{w,2}(|h|) = \left[\frac{\int_\Omega w\,|h|^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega |h|\, w d\mu}{\int_\Omega w d\mu}\right)^2\right]^{\frac{1}{2}}.$$

The following result related to the Hölder inequality holds:

Proposition 1 (Dragomir, 2011 [20])**.** *If $f \in L_p(\Omega,\mu)$, $g \in L_q(\Omega,\mu)$ with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and there exists the constants γ, $\Gamma>0$ and such that*

$$\gamma \le \frac{|f|}{|g|^{q-1}} \le \Gamma \ \mu\text{-}a.e \text{ on } \Omega,$$

then we have

$$\left|\int_\Omega \left|\frac{|f|^p}{|g|^q} - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)^p\right| \operatorname{sgn}\left[\frac{|f|}{|g|^{q-1}} - \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right] |g|^q\,d\mu\right| \tag{2.18}$$
$$\le \frac{\int_\Omega |f|^p\,d\mu}{\int_\Omega |g|^q\,d\mu} - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)^p$$
$$\le \frac{1}{2}\left(\left[\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}, \Gamma; (\cdot)^p\right] - \left[\gamma, \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}; (\cdot)^p\right]\right) \tilde{D}_{|g|^q}\left(\frac{|f|}{|g|^{q-1}}\right)$$
$$\le \frac{1}{2}\left(\left[\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}, \Gamma; (\cdot)^p\right] - \left[\gamma, \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}; (\cdot)^p\right]\right) \tilde{D}_{|g|^q,2}\left(\frac{|f|}{|g|^{q-1}}\right)$$
$$\le \frac{1}{4}\left(\left[\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}, \Gamma; (\cdot)^p\right] - \left[\gamma, \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}; (\cdot)^p\right]\right) (\Gamma - \gamma),$$

where

$$\tilde{D}_{|g|^q}\left(\frac{|f|}{|g|^{q-1}}\right) = \frac{1}{\int_\Omega |g|^q\,d\mu}\int_\Omega |g|^q \left|\frac{|f|}{|g|^{q-1}} - \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right| d\mu$$

and

$$\tilde{D}_{|g|^q,2}\left(\frac{|f|}{|g|^{q-1}}\right) = \left[\frac{1}{\int_\Omega |g|^q\,d\mu}\int_\Omega \frac{|f|^2}{|g|^{q-2}} d\mu - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)^2\right]^{\frac{1}{2}}.$$

Proof. The inequalities (2.19) follow from (2.17) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q.$$

The details are omitted. □

Remark 8. *We observe that for $p = q = 2$ we have from the first inequality in (2.18) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$\left|\int_\Omega \left|\frac{|f|^2}{|g|^2} - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^2\,d\mu}\right)^2\right| \operatorname{sgn}\left[\frac{|f|}{|g|} - \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^2\,d\mu}\right] |g|^2\,d\mu\right| \tag{2.19}$$
$$\le \frac{\int_\Omega |f|^2\,d\mu}{\int_\Omega |g|^2\,d\mu} - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^2\,d\mu}\right)^2$$
$$\le \frac{1}{2}(\Gamma - \gamma)\frac{1}{\int_\Omega |g|^2\,d\mu}\int_\Omega |g|^2\left|\frac{|f|}{|g|} - \frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^2\,d\mu}\right| d\mu$$
$$\le \frac{1}{2}(\Gamma - \gamma)\left[\frac{1}{\int_\Omega |g|^2\,d\mu}\int_\Omega |f|^2\,d\mu - \left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^2\,d\mu}\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}(\Gamma - \gamma)^2,$$

provided that f, $g \in L_2(\Omega, \mu)$, and there exists the constants $\gamma, \Gamma > 0$ such that

$$\gamma \le \frac{|f|}{|g|} \le \Gamma \ \mu\text{-a.e on } \Omega.$$

2.3. **Applications for f-Divergence.** One of the important issues in many applications of Probability Theory is finding an appropriate measure of *distance* (or *difference* or *discrimination*) between two probability distributions. A number of divergence measures for this purpose have been proposed and extensively studied by Jeffreys [32], Kullback and Leibler [37], Rényi [45], Havrda and Charvat [29], Kapur [35], Sharma and Mittal [48], Burbea and Rao [4], Rao [44], Lin [38], Csiszár [9], Ali and Silvey [1], Vajda [55], Shioya and Da-te [49] and others (see for example [40] and the references therein).

These measures have been applied in a variety of fields such as: anthropology [44], genetics [40], finance, economics, and political science [47], [53], [54], biology [43], the analysis of contingency tables [28], approximation of probability distributions [8], [36], signal processing [33], [34] and pattern recognition [3], [5]. A number of these measures of distance are specific cases of Csiszár f-divergence and so further exploration of this concept will have a flow on effect to other measures of distance and to areas in which they are applied.

Assume that a set Ω and the σ-finite measure μ are given. Consider the set of all probability densities on μ to be $\mathcal{P} := \left\{p | p : \Omega \to \mathbb{R},\, p(x) \geq 0,\ \int_\Omega p(x)\, d\mu(x) = 1\right\}$.

Csiszár f-divergence is defined as follows [10]

$$I_f(p,q) := \int_\Omega p(x) f\left[\frac{q(x)}{p(x)}\right] d\mu(x), \quad p,\ q \in \mathcal{P}, \tag{2.20}$$

where f is convex on $(0,\infty)$. It is assumed that $f(u)$ is zero and strictly convex at $u = 1$. By appropriately defining this convex function, various divergences are derived.

The *Kullback-Leibler divergence* [37] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\Omega p(x) \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,\ q \in \mathcal{P}, \tag{2.21}$$

where ln is to base e.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [30], χ^2-*divergence* D_{χ^2}, α-*divergence* D_α, *Bhattacharyya distance* D_B [2], *Harmonic distance* D_{Ha}, *Jeffrey's distance* D_J [32], *triangular discrimination* D_Δ [52], etc... They are defined as follows:

$$D_v(p,q) := \int_\Omega |p(x) - q(x)|\, d\mu(x), \quad p,\ q \in \mathcal{P}; \tag{2.22}$$

$$D_H(p,q) := \int_\Omega \left|\sqrt{p(x)} - \sqrt{q(x)}\right| d\mu(x), \quad p,\ q \in \mathcal{P}; \tag{2.23}$$

$$D_{\chi^2}(p,q) := \int_\Omega p(x) \left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,\ q \in \mathcal{P}; \tag{2.24}$$

$$D_\alpha(p,q) := \frac{4}{1-\alpha^2}\left[1 - \int_\Omega [p(x)]^{\frac{1-\alpha}{2}} [q(x)]^{\frac{1+\alpha}{2}} d\mu(x)\right], \quad p,\ q \in \mathcal{P}; \tag{2.25}$$

$$D_B(p,q) := \int_\Omega \sqrt{p(x)\, q(x)}\, d\mu(x), \quad p,\ q \in \mathcal{P}; \tag{2.26}$$

$$D_{Ha}(p,q) := \int_\Omega \frac{2p(x)\,q(x)}{p(x)+q(x)} d\mu(x)\,, \quad p,\ q \in \mathcal{P}; \tag{2.27}$$

$$D_J(p,q) := \int_\Omega [p(x)-q(x)] \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x)\,, \quad p,\ q \in \mathcal{P}; \tag{2.28}$$

$$D_\Delta(p,q) := \int_\Omega \frac{[p(x)-q(x)]^2}{p(x)+q(x)} d\mu(x)\,, \quad p,\ q \in \mathcal{P}. \tag{2.29}$$

For other divergence measures, see the paper [35] by Kapur or the book on line [51] by Taneja.

Most of the above distances (2.21)-(2.29), are particular instances of Csiszár f-divergence. There are also many others which are not in this class (see for example [51]). For the basic properties of Csiszár f-divergence see [10], [11] and [55].

Before we apply the results obtained in the previous section we observe that, by employing the inequalities from (1.17) we can state the following theorem:

Proposition 2 (Dragomir, 2011 [20])**.** *Let $f:(0,\infty)\to\mathbb{R}$ be a convex function with the property that $f(1)=0$. Assume that $p,\ q\in\mathcal{P}$ and there exists the constants $0<r<1<R<\infty$ such that*

$$r \le \frac{q(x)}{p(x)} \le R \text{ for } \mu\text{-a.e. } x\in\Omega. \tag{2.30}$$

Then we have

$$\begin{aligned} 0 \le I_f(p,q) &\le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] D_v(p,q) \\ &\le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right]\left[D_{\chi^2}(p,q)\right]^{1/2} \\ &\le \frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right]. \end{aligned} \tag{2.31}$$

Proof. From (1.17) we have

$$\begin{aligned} &\int_\Omega p(x) f\left(\frac{q(x)}{p(x)}\right) d\mu(x) - f\left(\int_\Omega q(x)\,d\mu(x)\right) \\ &\le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] \\ &\times \int_\Omega p(x)\left|\frac{q(x)}{p(x)} - \int_\Omega q(y)\,d\mu(y)\right| d\mu(x) \\ &\le \frac{1}{2}\left[f'_-(R)-f'_+(r)\right] \\ &\times \left[\int_\Omega p(x)\left(\frac{q(x)}{p(x)}\right)^2 d\mu - \left(\int_\Omega q(x)\,d\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right], \end{aligned} \tag{2.32}$$

and since

$$\int_\Omega p(x)\left|\frac{q(x)}{p(x)} - \int_\Omega q(y)\,d\mu(y)\right| d\mu(x) = D_v(p,q)$$

and

$$\int_{\Omega} p(x)\left(\frac{q(x)}{p(x)}\right)^2 d\mu - \left(\int_{\Omega} q(x)\, d\mu\right)^2 = D_{\chi^2}(p,q),$$

then we get from (2.32) the desired result (2.31). □

Remark 9. *The inequality*

$$I_f(p,q) \le \frac{1}{4}(R-r)\left[f'_-(R) - f'_+(r)\right] \tag{2.33}$$

was obtained for the discrete divergence measures in 2000 by S. S. Dragomir, see [15].

Proposition 3 (Dragomir, 2011 [20])**.** *With the assumptions in Proposition 2 we have*

$$\begin{aligned}\left|I_{|f|(\operatorname{sgn}(\cdot)-1)}(p,q)\right| &\le I_f(p,q) \\ &\le \frac{1}{2}([1,R;f]-[r,1;f])\, D_v(p,q) \\ &\le \frac{1}{2}([1,R;f]-[r,1;f])\left[D_{\chi^2}(p,q)\right]^{1/2} \\ &\le \frac{1}{4}([1,R;f]-[r,1;f])(R-r),\end{aligned} \tag{2.34}$$

where $I_{|f|(\operatorname{sgn}(\cdot)-1)}(p,q)$ *is the generalized* f*-divergence for the non-necessarily convex function* $|f|(\operatorname{sgn}(\cdot)-1)$ *and is defined by*

$$I_{|f|(\operatorname{sgn}(\cdot)-1)}(p,q) := \int_{\Omega}\left|f\left(\frac{q(x)}{p(x)}\right)\right| \operatorname{sgn}\left[\frac{q(x)}{p(x)} - 1\right] p(x)\, d\mu. \tag{2.35}$$

Proof. From the inequality (2.2) we have

$$\begin{aligned}&\left|\int_{\Omega}\left|f\left(\frac{q(x)}{p(x)}\right)\right| \operatorname{sgn}\left[\frac{q(x)}{p(x)} - 1\right] p(x)\, d\mu.\right| \\ &\le \int_{\Omega} p(x) f\left(\frac{q(x)}{p(x)}\right) d\mu(x) - f\left(\int_{\Omega} q(x)\, d\mu(x)\right) \\ &\le \frac{1}{2}([1,R;f]-[r,1;f]) \\ &\times \int_{\Omega} p(x)\left|\frac{q(x)}{p(x)} - \int_{\Omega} q(y)\, d\mu(y)\right| d\mu(x) \\ &\le \frac{1}{2}([1,R;f]-[r,1;f]) \\ &\times \left[\int_{\Omega} p(x)\left(\frac{q(x)}{p(x)}\right)^2 d\mu - \left(\int_{\Omega} q(x)\, d\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}([1,R;f]-[r,1;f])(R-r),\end{aligned} \tag{2.36}$$

from where we get the desired result (2.34). □

The above results can be utilized to obtain various inequalities for the divergence measures in Information Theory that are particular instances of f-divergence.

Consider the *Kullback-Leibler divergence*

$$D_{KL}(p,q) := \int_\Omega p(x) \ln\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,\ q \in \mathcal{P},$$

which is an f-divergence for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t) = -\ln t$.

If p, $q\in\mathcal{P}$ such that there exists the constants $0<r<1<R<\infty$ with

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x\in\Omega, \tag{2.37}$$

then we get from (2.31) that

$$\begin{aligned} D_{KL}(p,q) &\leq \frac{R-r}{2rR} D_v(p,q) \\ &\leq \frac{R-r}{2rR}\left[D_{\chi^2}(p,q)\right]^{1/2} \leq \frac{(R-r)^2}{4rR} \end{aligned} \tag{2.38}$$

and from (2.34) that

$$\begin{aligned} D_{KL}(p,q) &\leq \frac{1}{2} D_v(p,q) \ln\left(\frac{1}{R^{R-1}r^{1-r}}\right) \\ &\leq \frac{1}{2}\left[D_{\chi^2}(p,q)\right]^{1/2} \ln\left(\frac{1}{R^{R-1}r^{1-r}}\right) \\ &\leq \frac{1}{4}(R-r)\ln\left(\frac{1}{R^{R-1}r^{1-r}}\right). \end{aligned} \tag{2.39}$$

The interested reader can obtain other similar results by considering f-divergence measures generated by other convex functions such as the *Jeffrey's distance* D_J or the *triangular discrimination* D_Δ. The details are omitted.

3. Reverse Inequalities in Terms of First Derivative

3.1. **General Results.** The following reverse of the Jensen's inequality holds:

Theorem 4 (Dragomir, 2011 [19])**.** *Let $\Phi : I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M\in\mathbb{R}$, $m<M$ with $[m,M]\subset \mathring{I}$, where $\mathring{I}$ is the interior of I. If $f:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \leq f(x) \leq M < \infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and such that f, $\Phi\circ f \in L_w(\Omega,\mu)$, then

$$\begin{aligned} 0 &\leq \int_\Omega w(x)\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\ &\leq \left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right], \end{aligned} \tag{3.1}$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(x) f(x)\,d\mu(x) \in [m,M]$, Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

Proof. By the convexity of Φ we have that

$$\begin{aligned}
&\int_{\Omega} w(x)\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\
&= \int_{\Omega} w(x)\Phi\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right] d\mu(x) \\
&- \Phi\left(\int_{\Omega} w(x)\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right] d\mu(x)\right) \\
&\le \int_{\Omega} \frac{(M-f(x))\Phi(m)+(f(x)-m)\Phi(M)}{M-m} w(x)\,d\mu(x) \\
&- \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) \\
&= \frac{\left(M-\bar{f}_{\Omega,w}\right)\Phi(m)+\left(\bar{f}_{\Omega,w}-m\right)\Phi(M)}{M-m} \\
&- \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) := B.
\end{aligned} \tag{3.2}$$

Then, by the convexity of Φ we have the gradient inequality

$$\Phi(t) - \Phi(M) \ge \Phi'_{-}(M)(t-M)$$

for any $t \in [m, M)$. If we multiply this inequality with $t - m \ge 0$, we deduce

$$(t-m)\Phi(t) - (t-m)\Phi(M) \ge \Phi'_{-}(M)(t-M)(t-m), \quad t \in [m, M]. \tag{3.3}$$

Similarly, using the other gradient inequality

$$\Phi(t) - \Phi(m) \ge \Phi'_{+}(m)(t-m)$$

for any $t \in (m, M]$, we also get

$$(M-t)\Phi(t) - (M-t)\Phi(m) \ge \Phi'_{+}(m)(t-m)(M-t), \quad t \in [m, M]. \tag{3.4}$$

Adding (3.3) to (3.4) and dividing by $M - m$, we deduce

$$\Phi(t) - \frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m} \ge \frac{(t-M)(t-m)}{M-m}\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right],$$

for any $t \in (m, M)$.

By denoting

$$\Delta_{\Phi}(t; m, M) := \frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m} - \Phi(t), \quad t \in [m, M]$$

we then get the following inequality of interest

$$\begin{aligned}
0 \le \Delta_{\Phi}(t; m, M) &\le \frac{(M-t)(t-m)}{M-m}\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right] \\
&\le \frac{1}{4}(M-m)\left[\Phi'_{-}(M) - \Phi'_{+}(m)\right]
\end{aligned} \tag{3.5}$$

for any $t \in (m, M)$.

Now, since with the above notations we have $B = \Delta_{\Phi}\left(\bar{f}_{\Omega,w}; m, M\right)$, then by (3.5) we have

$$\begin{aligned} B &\leq \frac{\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right] \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right], \end{aligned}$$

and the proof is completed. □

Corollary 4. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and m, $M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $x_i \in I$ and $p_i \geq 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^{n} p_i = 1$, then we have the inequality*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i \Phi(x_i) - \Phi(\bar{x}_p) \\ &\leq (M-\bar{x}_p)(\bar{x}_p-m)\frac{\Phi'_{-}(M)-\Phi'_{+}(m)}{M-m} \\ &\leq \frac{1}{4}(M-m)\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right], \end{aligned} \tag{3.6}$$

where $\bar{x}_p := \sum_{i=1}^{n} p_i x_i \in I$.

Remark 10. *Consider the positive n-tuple $x = (x_1, ..., x_n)$ with the nonnegative weights $w = (w_1, ..., w_n)$ where $W_n := \sum_{i=1}^{n} w_i > 0$. Applying the inequality (3.6) for the convex function $\Phi(t) = -\ln t$, $t > 0$ we have*

$$\begin{aligned} 1 &\leq \frac{A_n(w,x)}{G_n(w,x)} \leq \exp\left[\frac{1}{Mm}(M-A_n(w,x))(A_n(w,x)-m)\right] \\ &\leq \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right], \end{aligned} \tag{3.7}$$

provided that $0 < m \leq x_i \leq M < \infty$ for $i \in \{1, ..., n\}$.

For the Lebesgue measurable function $g : [\alpha, \beta] \to \mathbb{R}$ we introduce the *Lebesgue p-norms* defined as

$$\|g\|_{[\alpha,\beta],p} := \left(\int_{\alpha}^{\beta} |g(t)|^p \, dt\right)^{1/p} \quad \text{if } g \in L_p[\alpha, \beta],$$

for $p \geq 1$ and

$$\|g\|_{[\alpha,\beta],\infty} := \operatorname*{essup}_{t\in[\alpha,\beta]} |g(t)| \text{ if } g \in L_{\infty}[\alpha, \beta],$$

for $p = \infty$.

The following result also holds:

Theorem 5 (Dragomir, 2011 [19])**.** *With the assumptions in Theorem 4, we have the inequalities*

$$\begin{aligned} 0 &\leq \int_{\Omega} w(x)\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\ &\leq \frac{\left(M-\bar{f}_{\Omega,w}\right)\int_{m}^{\bar{f}_{\Omega,w}}|\Phi'(t)|\,dt + \left(\bar{f}_{\Omega,w}-m\right)\int_{\bar{f}_{\Omega,w}}^{M}|\Phi'(t)|\,dt}{M-m} \\ &:= \Lambda_{\Phi}\left(\bar{f}_{\Omega,w}; m, M\right), \end{aligned} \tag{3.8}$$

where the integral in the second term of the inequality is taken in the Lebesgue sense.

We also have the bounds:

$$\Lambda_{\Phi}\left(\bar{f}_{\Omega,w};m,M\right) \leq \begin{cases} \left[\frac{1}{2}+\frac{\left|\bar{f}_{\Omega,w}-\frac{m+M}{2}\right|}{M-m}\right]\int_{m}^{M}\left|\Phi'(t)\right|dt, \\ \\ \left[\frac{1}{2}\int_{m}^{M}\left|\Phi'(t)\right|dt+\frac{1}{2}\left|\int_{\bar{f}_{\Omega,w}}^{M}\left|\Phi'(t)\right|dt-\int_{m}^{\bar{f}_{\Omega,w}}\left|\Phi'(t)\right|dt\right|\right] \end{cases} \tag{3.9}$$

and

$$\begin{aligned} \Lambda_{\Phi}\left(\bar{f}_{\Omega,w};m,M\right) &\leq \frac{\left(\bar{f}_{\Omega,w}-m\right)\left(M-\bar{f}_{\Omega,w}\right)}{M-m}\left[\left\|\Phi'\right\|_{\left[\bar{f}_{\Omega,w},M\right],\infty}+\left\|\Phi'\right\|_{\left[m,\bar{f}_{\Omega,w}\right],\infty}\right] \\ &\leq \frac{1}{2}(M-m)\frac{\left\|\Phi'\right\|_{\left[\bar{f}_{\Omega,w},M\right],\infty}+\left\|\Phi'\right\|_{\left[m,\bar{f}_{\Omega,w}\right],\infty}}{2}\leq\frac{1}{2}(M-m)\left\|\Phi'\right\|_{[m,M],\infty} \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \Lambda_{\Phi}\left(\bar{f}_{\Omega,w};m,M\right) \leq{}& \frac{1}{M-m}\Big[\left(\bar{f}_{\Omega,w}-m\right)\left(M-\bar{f}_{\Omega,w}\right)^{1/q}\left\|\Phi'\right\|_{\left[\bar{f}_{\Omega,w},M\right],p} \\ &+\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)^{1/q}\left\|\Phi'\right\|_{\left[m,\bar{f}_{\Omega,w}\right],p}\Big] \\ \leq{}& \frac{1}{M-m}\Big[\left(\bar{f}_{\Omega,w}-m\right)^{q}\left(M-\bar{f}_{\Omega,w}\right) \\ &+\left(M-\bar{f}_{\Omega,w}\right)^{q}\left(\bar{f}_{\Omega,w}-m\right)\Big]^{1/q}\left\|\Phi'\right\|_{[m,M],p} \end{aligned} \tag{3.11}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Observe that, with the above notations we have

$$\begin{aligned} \Lambda_{\Phi}(t;m,M) &= \frac{(t-m)\Phi(M)+(M-t)\Phi(m)}{M-m}-\Phi(t) \\ &= \frac{(t-m)\Phi(M)+(M-t)\Phi(m)-(M-m)\Phi(t)}{M-m} \\ &= \frac{(t-m)\Phi(M)+(M-t)\Phi(m)-(M-t+t-m)\Phi(t)}{M-m} \\ &= \frac{(t-m)\left[\Phi(M)-\Phi(t)\right]-(M-t)\left[\Phi(t)-\Phi(m)\right]}{M-m} \end{aligned} \tag{3.12}$$

for any $t\in[m,M]$.

Taking the modulus on (3.12) and noticing that $\Lambda_{\Phi}(t;m,M)\geq 0$ for any $t\in[m,M]$, we have that

$$\begin{aligned} \Lambda_{\Phi}(t;m,M) &\leq \frac{(t-m)\left|\Phi(M)-\Phi(t)\right|+(M-t)\left|\Phi(t)-\Phi(m)\right|}{M-m} \\ &= \frac{(t-m)\left|\int_{t}^{M}\Phi'(s)\,ds\right|+(M-t)\left|\int_{m}^{t}\Phi'(s)\,ds\right|}{M-m} \\ &\leq \frac{(t-m)\int_{t}^{M}\left|\Phi'(s)\right|ds+(M-t)\int_{m}^{t}\left|\Phi'(s)\right|ds}{M-m} \end{aligned} \tag{3.13}$$

for any $t\in[m,M]$.

Finally, if we write the inequality (3.13) for $t = \bar{f}_{\Omega,w} \in [m, M]$ and utilize the inequality (3.2), we deduce the desired result (3.8).

Now, we observe that

$$
\begin{aligned}
&\frac{(t-m)\int_t^M |\Phi'(s)|\,ds + (M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \quad (3.14)\\
&\le \begin{cases} \max\{t-m, M-t\}\int_m^M |\Phi'(t)|\,dt \\ \\ \max\left\{\int_t^M |\Phi'(s)|\,ds, \int_m^t |\Phi'(s)|\,ds\right\}(M-m) \end{cases}\\
&= \begin{cases} \left[\frac{1}{2}(M-m) + \left|t - \frac{m+M}{2}\right|\right]\int_m^M |\Phi'(t)|\,dt \\ \\ \left[\frac{1}{2}\int_m^M |\Phi'(s)|\,ds + \frac{1}{2}\left|\int_t^M |\Phi'(s)|\,ds - \int_m^t |\Phi'(s)|\,ds\right|\right](M-m) \end{cases}
\end{aligned}
$$

for any $t \in [m, M]$. This proves the inequality (3.9).

By the Hölder's inequality we have

$$
\int_t^M |\Phi'(s)|\,ds \le \begin{cases} (M-t)\,\|\Phi'\|_{[t,M],\infty} \\ \\ (M-t)^{1/q}\,\|\Phi'\|_{[t,M],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
$$

and

$$
\int_m^t |\Phi'(s)|\,ds \le \begin{cases} (t-m)\,\|\Phi'\|_{[m,t],\infty} \\ \\ (t-m)^{1/q}\,\|\Phi'\|_{[m,t],p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1 \end{cases}
$$

which give that

$$
\begin{aligned}
&\frac{(t-m)\int_t^M |\Phi'(s)|\,ds + (M-t)\int_m^t |\Phi'(s)|\,ds}{M-m} \quad (3.15)\\
&\le \frac{(t-m)(M-t)\,\|\Phi'\|_{[t,M],\infty} + (M-t)(t-m)\,\|\Phi'\|_{[m,t],\infty}}{M-m}\\
&= \frac{(t-m)(M-t)}{M-m}\left[\|\Phi'\|_{[t,M],\infty} + \|\Phi'\|_{[m,t],\infty}\right]\\
&\le \frac{1}{2}(M-m)\frac{\|\Phi'\|_{[t,M],\infty} + \|\Phi'\|_{[m,t],\infty}}{2}\\
&\le \frac{1}{2}(M-m)\max\left\{\|\Phi'\|_{[t,M],\infty}, \|\Phi'\|_{[m,t],\infty}\right\} = \frac{1}{2}(M-m)\,\|\Phi'\|_{[m,M],\infty}
\end{aligned}
$$

and

$$\begin{aligned}
(3.16)\quad & \frac{(t-m)\int_t^M |\Phi'(s)|\,ds+(M-t)\int_m^t |\Phi'(s)|\,ds}{M-m}\\
&\leq \frac{(t-m)(M-t)^{1/q}\|\Phi'\|_{[t,M],p}+(M-t)(t-m)^{1/q}\|\Phi'\|_{[m,t],p}}{M-m}\\
&\leq \frac{1}{M-m}\left[\left((t-m)(M-t)^{1/q}\right)^q+\left((M-t)(t-m)^{1/q}\right)^q\right]^{1/q}\\
&\times\left[\|\Phi'\|_{[t,M],p}^p+\|\Phi'\|_{[m,t],p}^p\right]^{1/p}\\
&=\frac{1}{M-m}\left[(t-m)^q(M-t)+(M-t)^q(t-m)\right]^{1/q}\|\Phi'\|_{[m,M],p}
\end{aligned}$$

for any $t\in[m,M]$.

These prove the desired inequalities (3.10) and (3.11). □

The discrete case is as follows:

Corollary 5. *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ is the interior of I. If $x_i\in I$ and $p_i\geq 0$ for $i\in\{1,...,n\}$ with $\sum_{i=1}^n p_i=1$, then we have the inequality*

$$\begin{aligned}
(3.17)\quad 0 &\leq \sum_{i=1}^n p_i\Phi(x_i)-\Phi(\bar{x}_p)\\
&\leq \frac{(M-\bar{x}_p)\int_m^{\bar{x}_p}|\Phi'(t)|\,dt+(\bar{x}_p-m)\int_{\bar{x}_p}^M|\Phi'(t)|\,dt}{M-m}\\
&:=\Lambda_\Phi(\bar{x}_p;m,M),
\end{aligned}$$

where $\Lambda_\Phi(\bar{x}_p;m,M)$ satisfies the bounds

$$\begin{aligned}
(3.18)\quad & \Lambda_\Phi(\bar{x}_p;m,M)\\
&\leq\begin{cases}\left[\frac{1}{2}+\frac{\left|\bar{x}_p-\frac{m+M}{2}\right|}{M-m}\right]\int_m^M|\Phi'(t)|\,dt,\\[2ex]
\left[\frac{1}{2}\int_m^M|\Phi'(t)|\,dt+\frac{1}{2}\left|\int_{\bar{x}_p}^M|\Phi'(t)|\,dt-\int_m^{\bar{x}_p}|\Phi'(t)|\,dt\right|\right]\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
(3.19)\quad & \Lambda_\Phi(\bar{x}_p;m,M)\\
&\leq\frac{(\bar{x}_p-m)(M-\bar{x}_p)}{M-m}\left[\|\Phi'\|_{[\bar{x}_p,M],\infty}+\|\Phi'\|_{[m,\bar{x}_p],\infty}\right]\\
&\leq\frac{1}{2}(M-m)\frac{\|\Phi'\|_{[\bar{x}_p,M],\infty}+\|\Phi'\|_{[m,\bar{x}_p],\infty}}{2}\leq\frac{1}{2}(M-m)\|\Phi'\|_{[m,M],\infty}
\end{aligned}$$

and

$$\Lambda_{\Phi}\left(\bar{x}_p; m, M\right) \leq \frac{1}{M-m}\Big[\left(\bar{x}_p - m\right)\left(M-\bar{x}_p\right)^{1/q}\left\|\Phi'\right\|_{[\bar{x}_p, M], p} + \left(M-\bar{x}_p\right)\left(\bar{x}_p - m\right)^{1/q}\left\|\Phi'\right\|_{[m, \bar{x}_p], p}\Big] \leq \frac{1}{M-m}\left[\left(\bar{x}_p - m\right)^q\left(M-\bar{x}_p\right) + \left(M-\bar{x}_p\right)^q\left(\bar{x}_p - m\right)\right]^{1/q}\left\|\Phi'\right\|_{[m, M], p}. \tag{3.20}$$

Remark 11. *Under the assumptions of Remark 10 , on applying the inequality (3.17) for the convex function $\Phi(t) = -\ln t$, we have the following reverse of the arithmetic mean-geometric mean inequality*

$$1 \leq \frac{A_n(w,x)}{G_n(w,x)} \leq \left(\frac{A_n(w,x)}{m}\right)^{M-A_n(w,x)}\left(\frac{M}{A_n(w,x)}\right)^{A_n(w,x)-m}. \tag{3.21}$$

3.2. **Applications for the Hölder Inequality.** Assume that $p>1$. If $h:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds

$$-\infty < m \leq |h(x)| \leq M < \infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and is such that $h, |h|^p \in L_w(\Omega,\mu)$, for a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. $x\in\Omega$ and $\int_\Omega w d\mu > 0$, then from (3.1) we have

$$0 \leq \frac{\int_\Omega |h|^p w d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega |h| w d\mu}{\int_\Omega w d\mu}\right)^p \leq p\frac{M^{p-1}-m^{p-1}}{M-m}\left(M-\overline{|h|}_{\Omega,w}\right)\left(\overline{|h|}_{\Omega,w}-m\right) \leq \frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right), \tag{3.22}$$

where $\overline{|h|}_{\Omega,w} := \frac{\int_\Omega |h| w d\mu}{\int_\Omega w d\mu} \in [m, M]$.

Proposition 4 (Dragomir, 2011 [19]). *If $f\in L_p(\Omega,\mu)$, $g\in L_q(\Omega,\mu)$ with $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and there exists the constants γ, $\Gamma>0$ and such that*

$$\gamma \leq \frac{|f|}{|g|^{q-1}} \leq \Gamma \ \mu\text{-a.e on } \Omega$$

then we have

$$0 \leq \frac{\int_\Omega |f|^p d\mu}{\int_\Omega |g|^q d\mu} - \left(\frac{\int_\Omega |fg| d\mu}{\int_\Omega |g|^q d\mu}\right)^p \leq p\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma}\left(\Gamma - \frac{\int_\Omega |fg| d\mu}{\int_\Omega |g|^q d\mu}\right)\left(\frac{\int_\Omega |fg| d\mu}{\int_\Omega |g|^q d\mu} - \gamma\right) \leq \frac{1}{4}p(\Gamma-\gamma)\left(\Gamma^{p-1}-\gamma^{p-1}\right). \tag{3.23}$$

Proof. The inequalities (3.23) follow from (3.22) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q.$$

The details are omitted. □

Remark 12. *We observe that for $p=q=2$ we have from the first inequality in (3.23) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$0 \leq \int_\Omega |g|^2 \, d\mu \int_\Omega |f|^2 \, d\mu - \left(\int_\Omega |fg| \, d\mu\right)^2 \\ \leq \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu} - \gamma\right)\left(\int_\Omega |g|^2 \, d\mu\right)^2 \\ \leq \frac{1}{4}(\Gamma - \gamma)^2 \left(\int_\Omega |g|^2 \, d\mu\right)^2, \tag{3.24}$$

provided that $f,\ g \in L_2(\Omega, \mu)$ and there exists the constants $\gamma,\ \Gamma > 0$ such that

$$\gamma \leq \frac{|f|}{|g|} \leq \Gamma \ \mu\text{-}a.e \text{ on } \Omega.$$

Corollary 6. *With the assumptions of Proposition 4 we have the following additive reverses of the Hölder inequality:*

(3.25)

$$0 \leq \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^q \, d\mu\right)^{1/q} - \int_\Omega |fg| \, d\mu \\ \leq p^{1/p} \left(\frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma}\right)^{\frac{1}{p}} \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^{\frac{1}{p}} \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)^{\frac{1}{p}} \int_\Omega |g|^q \, d\mu \\ \leq \frac{1}{4^{1/p}} p^{1/p} (\Gamma - \gamma)^{1/p} \left(\Gamma^{p-1} - \gamma^{p-1}\right)^{1/p} \int_\Omega |g|^q \, d\mu$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By multiplying in (3.23) with $\left(\int_\Omega |g|^q \, d\mu\right)^p$ we have

$$\int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{p-1} - \left(\int_\Omega |fg| \, d\mu\right)^p \\ \leq p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)\left(\int_\Omega |g|^q \, d\mu\right)^p \\ \leq \frac{1}{4} p (\Gamma - \gamma)\left(\Gamma^{p-1} - \gamma^{p-1}\right)\left(\int_\Omega |g|^q \, d\mu\right)^p,$$

which is equivalent with

$$\int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{p-1} \\ \leq \left(\int_\Omega |fg| \, d\mu\right)^p + p\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \\ \times \left(\int_\Omega |g|^q \, d\mu\right)^p \frac{\Gamma^{p-1} - \gamma^{p-1}}{\Gamma - \gamma} \\ \leq \left(\int_\Omega |fg| \, d\mu\right)^p + \frac{1}{4} p (\Gamma - \gamma)\left(\Gamma^{p-1} - \gamma^{p-1}\right)\left(\int_\Omega |g|^q \, d\mu\right)^p. \tag{3.26}$$

Taking the power $1/p$ with $p>1$ and employing the following elementary inequality that state that for $p>1$ and $\alpha,\ \beta>0$,

$$(\alpha+\beta)^{1/p}\le\alpha^{1/p}+\beta^{1/p}$$

we have from the first part of (3.26) that

$$\begin{aligned}(3.27)\quad &\int_\Omega |f|^p\,d\mu\left(\int_\Omega |g|^q\,d\mu\right)^{1-\frac{1}{p}}\\ &\le\int_\Omega |fg|\,d\mu\\ &+\left[p\left(\Gamma-\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)\left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\gamma\right)\left(\int_\Omega |g|^q\,d\mu\right)^p\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma}\right]^{1/p}.\end{aligned}$$

Since $1-\frac{1}{p}=\frac{1}{q}$, we get from (3.27) the first inequality in (3.25). The rest is obvious. □

If $h:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds

$$-\infty<m\le|h(x)|\le M<\infty\text{ for }\mu\text{-a.e. }x\in\Omega$$

and is such that $h,\ |h|^p\in L_w(\Omega,\mu)$, for a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge0$ for μ-a.e. $x\in\Omega$ and $\int_\Omega wd\mu>0$, then from Theorem 5 we have amongst other the following inequality

$$\begin{aligned}(3.28)\qquad 0&\le\frac{\int_\Omega |h|^p\,wd\mu}{\int_\Omega wd\mu}-\left(\frac{\int_\Omega |h|\,wd\mu}{\int_\Omega wd\mu}\right)^p\\ &\le(M^p-m^p)\left[\frac{1}{2}+\frac{1}{M-m}\left|\frac{\int_\Omega |h|\,wd\mu}{\int_\Omega wd\mu}-\frac{m+M}{2}\right|\right].\end{aligned}$$

From this inequality we can state that:

Proposition 5 (Dragomir, 2011 [19])**.** *With the assumptions of Proposition 4 we have*

$$\begin{aligned}(3.29)\qquad 0&\le\frac{\int_\Omega |f|^p\,d\mu}{\int_\Omega |g|^q\,d\mu}-\left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)^p\\ &\le(\Gamma^p-\gamma^p)\left[\frac{1}{2}+\frac{1}{\Gamma-\gamma}\left|\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\frac{\gamma+\Gamma}{2}\right|\right].\end{aligned}$$

Finally, the following additive reverse of the Hölder inequality can be stated as well:

Corollary 7. *With the assumptions of Proposition 4 we have*

$$\begin{aligned}(3.30)\qquad &\left(\int_\Omega |f|^p\,d\mu\right)^{1/p}\left(\int_\Omega |g|^q\,d\mu\right)^{1/q}-\int_\Omega |fg|\,d\mu\\ &\le(\Gamma^p-\gamma^p)^{1/p}\left[\frac{1}{2}+\frac{1}{\Gamma-\gamma}\left|\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\frac{\gamma+\Gamma}{2}\right|\right]^{1/p}\int_\Omega |g|^q\,d\mu.\end{aligned}$$

Remark 13. *We observe that for $p = q = 2$ we have from the first inequality in (3.29) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$\begin{aligned}(3.31)\qquad &\int_\Omega |g|^2 d\mu \int_\Omega |f|^2 d\mu - \left(\int_\Omega |fg|\, d\mu\right)^2 \\ &\le \left(\Gamma^2 - \gamma^2\right)\left[\frac{1}{2} + \frac{1}{\Gamma - \gamma}\left|\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^2 d\mu} - \frac{\gamma + \Gamma}{2}\right|\right]\left(\int_\Omega |g|^2 d\mu\right)^2\end{aligned}$$

provided that f, $g \in L_2(\Omega, \mu)$ and there exists the constants γ, $\Gamma > 0$ such that

$$\gamma \le \frac{|f|}{|g|} \le \Gamma \ \mu\text{-a.e on } \Omega.$$

One can easily observe that the bound provided by (3.31) is not as good as the one given by (3.24). The details are omitted.

3.3. **Applications for f-Divergence.** The following result holds:

Proposition 6 (Dragomir, 2011 [19])**.** *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that p, $q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that*

$$(3.32)\qquad r \le \frac{q(x)}{p(x)} \le R \text{ for } \mu\text{-a.e. } x \in \Omega.$$

Then we have the inequalities

$$\begin{aligned}(3.33)\qquad 0 \le I_f(p, q) &\le (R - 1)(1 - r)\frac{f'_-(R) - f'_+(r)}{R - r} \\ &\le \frac{1}{4}(R - r)\left[f'_-(R) - f'_+(r)\right].\end{aligned}$$

Proof. Utilising Theorem 4 we can write that

$$\begin{aligned}(3.34)\qquad &\int_\Omega p(x) f\left(\frac{q(x)}{p(x)}\right) d\mu(x) - f\left(\int_\Omega q(x)\, d\mu(x)\right) \\ &\le \left(R - \int_\Omega q(x)\, d\mu(x)\right)\left(\int_\Omega q(x)\, d\mu(x) - r\right)\frac{f'_-(R) - f'_+(r)}{R - r} \\ &\le \frac{1}{4}(R - r)\left[f'_-(R) - f'_+(r)\right],\end{aligned}$$

for p, $q \in \mathcal{P}$ satisfying (3.32) and since $f\left(\int_\Omega q(x)\, d\mu(x)\right) = f(1) = 0$ we get from (3.34) the desired result (3.33). □

By the use of Theorem 5 we can also state the following result:

Proposition 7 (Dragomir, 2011 [19])**.** *With the assumptions in Proposition 6, we have the inequalities*

$$(3.35)\qquad 0 \le I_f(p, q) \le B_f(r, R)$$

where

$$(3.36)\qquad B_f(r, R) := \frac{(R - 1)\int_r^1 |f'(t)|\, dt + (1 - r)\int_1^R |f'(t)|\, dt}{R - r}.$$

Moreover, we have the following bounds for $B_f(r,R)$,

$$B_f(r,R) \leq \begin{cases} \left[\frac{1}{2}+\frac{\left|1-\frac{r+R}{2}\right|}{R-r}\right]\int_r^R |f'(t)|\,dt, \\ \\ \left[\frac{1}{2}\int_r^R |f'(t)|\,dt+\frac{1}{2}\left|\int_1^R |f'(t)|\,dt-\int_r^1 |f'(t)|\,dt\right|\right] \end{cases} \tag{3.37}$$

and

$$\begin{aligned} B_f(r,R) &\leq \frac{(1-r)(R-1)}{R-r}\left[\|f'\|_{[1,R],\infty}+\|f'\|_{[r,1],\infty}\right] \\ &\leq \frac{1}{2}(R-r)\frac{\|f'\|_{[1,R],\infty}+\|f'\|_{[r,1],\infty}}{2} \leq \frac{1}{2}(R-r)\|f'\|_{[r,R],\infty} \end{aligned} \tag{3.38}$$

and

$$\begin{aligned} B_f(r,R) &\leq \frac{1}{R-r}\left[(1-r)(R-1)^{1/q}\|f'\|_{[1,R],p}+(R-1)(1-r)^{1/q}\|f'\|_{[r,1],p}\right] \\ &\leq \frac{1}{R-r}\left[(1-r)^q(R-1)+(R-1)^q(1-r)\right]^{1/q}\|f'\|_{[r,R],p} \end{aligned} \tag{3.39}$$

where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$.

The above results can be utilized to obtain various inequalities for the divergence measures in information theory that are particular instances of f-divergences.

Consider, for example, the Kullback-Leibler divergence measure

$$D_{KL}(p,q) := \int_\Omega p(x)\ln\left[\frac{p(x)}{q(x)}\right]d\mu(x), \quad p,\ q\in\mathcal{P},$$

which is an f-divergence for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=-\ln t$.

If $p,\ q\in\mathcal{P}$ such that there exists the constants $0<r<1<R<\infty$ with

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x\in\Omega, \tag{3.40}$$

then we get from (3.33) that

$$D_{KL}(p,q) \leq \frac{(R-1)(1-r)}{rR} \tag{3.41}$$

and from (3.35) that

$$D_{KL}(p,q) \leq \ln\left(\frac{R^{1-r}}{r^{R-1}}\right)^{\frac{1}{R-r}}.$$

The interested reader can obtain similar results for other divergence measures as listed above. However, the details are omitted.

4. More Reverse Inequalities

4.1. General Results.

The following reverse of the Jensen's inequality that provides a refinement and an alternative for the inequality in Theorem 4 holds:

Theorem 6 (Dragomir, 2011 [18]). *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, $\mathring{I}$ is the interior of I. If $f : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds*

$$-\infty < m \le f(x) \le M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and such that f, $\Phi \circ f \in L_w(\Omega, \mu)$, where $w \ge 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$, then

$$\begin{aligned} 0 &\le \int_\Omega w(\Phi \circ f)\, d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \\ &\le \frac{\left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right)}{M - m} \sup_{t \in (m,M)} \Psi_\Phi(t; m, M) \\ &\le \left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\ &\le \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{4.1}$$

where $\bar{f}_{\Omega,w} := \int_\Omega w(x) f(x)\, d\mu(x) \in [m, M]$ and $\Psi_\Phi(\cdot; m, M) : (m, M) \to \mathbb{R}$ is defined by

$$\Psi_\Phi(t; m, M) = \frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}.$$

We also have the inequality

$$\begin{aligned} 0 &\le \int_\Omega w(\Phi \circ f)\, d\mu - \Phi\left(\bar{f}_{\Omega,w}\right) \le \frac{1}{4}(M - m)\Psi_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) \\ &\le \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right], \end{aligned} \tag{4.2}$$

provided that $\bar{f}_{\Omega,w} \in (m, M)$.

Proof. By the convexity of Φ we have that

$$\begin{aligned}
(4.3)\qquad & \int_\Omega w(x)\,\Phi(f(x))\,d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\
&= \int_\Omega w(x)\,\Phi\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right]d\mu(x) \\
&\quad - \Phi\left(\int_\Omega w(x)\left[\frac{m(M-f(x))+M(f(x)-m)}{M-m}\right]d\mu(x)\right) \\
&\le \int_\Omega \frac{(M-f(x))\,\Phi(m)+(f(x)-m)\,\Phi(M)}{M-m}\,w(x)\,d\mu(x) \\
&\quad - \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) \\
&= \frac{\left(M-\bar{f}_{\Omega,w}\right)\Phi(m)+\left(\bar{f}_{\Omega,w}-m\right)\Phi(M)}{M-m} \\
&\quad - \Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) := B.
\end{aligned}$$

By denoting

$$\Delta_\Phi(t;m,M) := \frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)}{M-m} - \Phi(t), \quad t\in[m,M]$$

we have

$$\begin{aligned}
(4.4)\qquad \Delta_\Phi(t;m,M) &= \frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)-(M-m)\,\Phi(t)}{M-m} \\
&= \frac{(t-m)\,\Phi(M)+(M-t)\,\Phi(m)-(M-t+t-m)\,\Phi(t)}{M-m} \\
&= \frac{(t-m)\,[\Phi(M)-\Phi(t)]-(M-t)\,[\Phi(t)-\Phi(m)]}{M-m} \\
&= \frac{(M-t)(t-m)}{M-m}\Psi_\Phi(t;m,M)
\end{aligned}$$

for any $t\in(m,M)$.

Therefore we have the equality

$$(4.5)\qquad B = \frac{\left(M-\bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\Psi_\Phi\left(\bar{f}_{\Omega,w};m,M\right)$$

provided that $\bar{f}_{\Omega,w}\in(m,M)$.

For $\bar{f}_{\Omega,w} = m$ or $\bar{f}_{\Omega,w} = M$ the inequality (4.1) is obvious. If $\bar{f}_{\Omega,w} \in (m, M)$, then

$$\begin{aligned}
\Psi_\Phi\left(\bar{f}_{\Omega,w}; m, M\right) &\le \sup_{t\in(m,M)} \Psi_\Phi(t; m, M) \\
&= \sup_{t\in(m,M)} \left[\frac{\Phi(M) - \Phi(t)}{M - t} - \frac{\Phi(t) - \Phi(m)}{t - m}\right] \\
&\le \sup_{t\in(m,M)} \left[\frac{\Phi(M) - \Phi(t)}{M - t}\right] + \sup_{t\in(m,M)} \left[-\frac{\Phi(t) - \Phi(m)}{t - m}\right] \\
&= \sup_{t\in(m,M)} \left[\frac{\Phi(M) - \Phi(t)}{M - t}\right] - \inf_{t\in(m,M)} \left[\frac{\Phi(t) - \Phi(m)}{t - m}\right] \\
&= \Phi'_-(M) - \Phi'_+(m),
\end{aligned}$$

which by (4.3) and (4.5) produces the desired result (4.1).

Since, obviously

$$\frac{\left(M - \bar{f}_{\Omega,w}\right)\left(\bar{f}_{\Omega,w} - m\right)}{M - m} \le \frac{1}{4}(M - m),$$

then by (4.3) and (4.5) we deduce the first inequality (4.2). The second part is clear. □

Corollary 8. *Let $\Phi : I \to \mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$. If $x_i \in [m, M]$ and $p_i \ge 0$ for $i \in \{1, ..., n\}$ with $\sum_{i=1}^n p_i = 1$, then we have the inequalities*

$$\begin{aligned}
0 &\le \sum_{i=1}^n p_i \Phi(x_i) - \Phi(\bar{x}_p) \\
&\le \frac{(M - \bar{x}_p)(\bar{x}_p - m)}{M - m} \sup_{t\in(m,M)} \Psi_\Phi(t; m, M) \\
&\le (M - \bar{x}_p)(\bar{x}_p - m) \frac{\Phi'_-(M) - \Phi'_+(m)}{M - m} \\
&\le \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right],
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
0 &\le \sum_{i=1}^n p_i \Phi(x_i) - \Phi(\bar{x}_p) \le \frac{1}{4}(M - m)\Psi_\Phi(\bar{x}_p; m, M) \\
&\le \frac{1}{4}(M - m)\left[\Phi'_-(M) - \Phi'_+(m)\right],
\end{aligned} \tag{4.7}$$

where $\bar{x}_p := \sum_{i=1}^n p_i x_i \in (m, M)$.

Remark 14. *Consider the positive n-tuple $x = (x_1, ..., x_n)$ with the nonnegative weights $w = (w_1, ..., w_n)$ where $W_n := \sum_{i=1}^n w_i > 0$. Applying the inequality between the first and third term in (4.6) for the convex function $\Phi(t) = -\ln t$, $t > 0$ we* *have*

$$1 \le \frac{A_n(w,x)}{G_n(w,x)} \le \exp\left[\frac{1}{Mm}\left(M - A_n(w,x)\right)\left(A_n(w,x) - m\right)\right] \le \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right], \tag{4.8}$$

provided that $0 < m \le x_i \le M < \infty$ *for* $i \in \{1, ..., n\}$.

Also, if we apply the inequality (4.7) for the same function Φ *we get that*

$$\begin{aligned} 1 &\le \frac{A_n(w,x)}{G_n(w,x)} \\ &\le \left[\left(\frac{M}{A_n(w,x)}\right)^{M - A_n(w,x)}\left(\frac{m}{A_n(w,x)}\right)^{A_n(w,x) - m}\right]^{-\frac{1}{4}(M-m)} \\ &\le \exp\left[\frac{1}{4}\frac{(M-m)^2}{mM}\right]. \end{aligned} \tag{4.9}$$

The following result also holds:

Theorem 7 (Dragomir, 2011 [18])**.** *With the assumptions of Theorem 6, we have the inequalities*

$$\begin{aligned} 0 &\le \int_\Omega w\left(\Phi \circ f\right) d\mu(x) - \Phi\left(\bar{f}_{\Omega,w}\right) \\ &\le 2\max\left\{\frac{M - \bar{f}_{\Omega,w}}{M-m}, \frac{\bar{f}_{\Omega,w} - m}{M-m}\right\}\left[\frac{\Phi(m) + \Phi(M)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \\ &\le \frac{1}{2}\max\left\{M - \bar{f}_{\Omega,w}, \bar{f}_{\Omega,w} - m\right\}\left[\Phi'_-(M) - \Phi'_+(m)\right]. \end{aligned} \tag{4.10}$$

Proof. First of all, we recall the following result obtained by the author in [16] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} & n\min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n \Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right] \\ &\le \frac{1}{P_n}\sum_{i=1}^n p_i\Phi(x_i) - \Phi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \\ & n\max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^n \Phi(x_i) - \Phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right)\right], \end{aligned} \tag{4.11}$$

where $\Phi : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}} \subset C$ are vectors and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^n p_i > 0$.

For $n = 2$ we deduce from (4.11) that

$$\begin{aligned} & 2\min\{t, 1-t\}\left[\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \\ &\le t\Phi(x) + (1-t)\Phi(y) - \Phi(tx + (1-t)y) \\ &\le 2\max\{t, 1-t\}\left[\frac{\Phi(x) + \Phi(y)}{2} - \Phi\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{4.12}$$

for any $x, y \in C$ and $t \in [0,1]$.

If we use the second inequality in (4.12) for the convex function $\Phi : I \to \mathbb{R}$ and $m, M \in \mathbb{R}$, $m < M$ with $[m, M] \subset \mathring{I}$, we have for $t = \frac{M - \bar{f}_{\Omega,w}}{M-m}$ that

$$\begin{aligned} &\frac{\left(M-\bar{f}_{\Omega,w}\right)\Phi(m)+\left(\bar{f}_{\Omega,w}-m\right)\Phi(M)}{M-m} \\ &-\Phi\left(\frac{m\left(M-\bar{f}_{\Omega,w}\right)+M\left(\bar{f}_{\Omega,w}-m\right)}{M-m}\right) \\ &\leq 2\max\left\{\frac{M-\bar{f}_{\Omega,w}}{M-m},\frac{\bar{f}_{\Omega,w}-m}{M-m}\right\} \\ &\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]. \end{aligned} \tag{4.13}$$

Utilizing the inequality (4.3) and (4.13) we deduce the first inequality in (4.10).

Since

$$\begin{aligned} &\frac{\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)}{M-m} \\ &=\frac{1}{4}\left[\frac{\Phi(M)-\Phi\left(\frac{m+M}{2}\right)}{M-\frac{m+M}{2}}-\frac{\Phi\left(\frac{m+M}{2}\right)-\Phi(m)}{\frac{m+M}{2}-m}\right] \end{aligned}$$

and, by the gradient inequality, we have that

$$\frac{\Phi(M)-\Phi\left(\frac{m+M}{2}\right)}{M-\frac{m+M}{2}} \leq \Phi'_{-}(M)$$

and

$$\frac{\Phi\left(\frac{m+M}{2}\right)-\Phi(m)}{\frac{m+M}{2}-m} \geq \Phi'_{+}(m),$$

then we get

$$\frac{\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)}{M-m} \leq \frac{1}{4}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]. \tag{4.14}$$

On making use of (4.13) and (4.14) we deduce the last part of (4.10). □

Corollary 9. *With the assumptions in Corollary 8, we have the inequalities*

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} p_i\Phi(x_i)-\Phi(\bar{x}_p) \\ &\leq 2\max\left\{\frac{M-\bar{x}_p}{M-m},\frac{\bar{x}_p-m}{M-m}\right\}\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right] \\ &\leq \frac{1}{2}\max\left\{M-\bar{x}_p,\bar{x}_p-m\right\}\left[\Phi'_{-}(M)-\Phi'_{+}(m)\right]. \end{aligned} \tag{4.15}$$

Remark 15. *Since, obviously,*

$$\frac{M-\bar{f}_{\Omega,w}}{M-m},\frac{\bar{f}_{\Omega,w}-m}{M-m} \leq 1$$

then we obtain from the first inequality in (4.10) the simpler, however coarser inequality

$$0 \leq \int_{\Omega} w\left(\Phi \circ f\right) d\mu\left(x\right) - \Phi\left(\bar{f}_{\Omega,w}\right) \leq 2\left[\frac{\Phi\left(m\right)+\Phi\left(M\right)}{2} - \Phi\left(\frac{m+M}{2}\right)\right]. \tag{4.16}$$

We notice that the discrete version of this result, namely

$$0 \leq \sum_{i=1}^{n} p_i \Phi\left(x_i\right) - \Phi\left(\bar{x}_p\right) \leq 2\left[\frac{\Phi\left(m\right)+\Phi\left(M\right)}{2} - \Phi\left(\frac{m+M}{2}\right)\right] \tag{4.17}$$

was obtained in 2008 by S. Simić in [50].

Remark 16. *With the assumptions in Remark 14 we have the following reverse of the arithmetic mean-geometric mean inequality*

$$1 \leq \frac{A_n\left(w,x\right)}{G_n\left(w,x\right)} \leq \left(\frac{A\left(m,M\right)}{G\left(m,M\right)}\right)^{2\max\left\{\frac{M-A_n(w,x)}{M-m}, \frac{A_n(w,x)-m}{M-m}\right\}}, \tag{4.18}$$

where $A\left(m,M\right)$ is the arithmetic mean while $G\left(m,M\right)$ is the geometric mean of the positive numbers m and M.

4.2. **Applications for the Hölder Inequality.** Assume that $p > 1$. If $h : \Omega \to \mathbb{R}$ is μ-measurable, satisfies the bounds

$$0 < m \leq \left|h\left(x\right)\right| \leq M < \infty \text{ for } \mu\text{-a.e. } x \in \Omega$$

and is such that $h, \left|h\right|^p \in L_w\left(\Omega,\mu\right)$, for a $\mu-$measurable function $w : \Omega \to \mathbb{R}$, with $w\left(x\right) \geq 0$ for μ-a.e. $x \in \Omega$ and $\int_{\Omega} w d\mu > 0$, then from (4.1) we have

$$\begin{aligned} 0 &\leq \frac{\int_{\Omega}\left|h\right|^p w d\mu}{\int_{\Omega} w d\mu} - \left(\frac{\int_{\Omega}\left|h\right| w d\mu}{\int_{\Omega} w d\mu}\right)^p \\ &\leq \frac{\left(M - \overline{\left|h\right|}_{\Omega,w}\right)\left(\overline{\left|h\right|}_{\Omega,w} - m\right)}{M-m} B_p\left(m,M\right) \\ &\leq p\frac{M^{p-1}-m^{p-1}}{M-m}\left(M - \overline{\left|h\right|}_{\Omega,w}\right)\left(\overline{\left|h\right|}_{\Omega,w} - m\right) \\ &\leq \frac{1}{4}p\left(M-m\right)\left(M^{p-1}-m^{p-1}\right), \end{aligned} \tag{4.19}$$

where $\overline{\left|h\right|}_{\Omega,w} := \frac{\int_{\Omega}\left|h\right| w d\mu}{\int_{\Omega} w d\mu} \in \left[m,M\right]$ and $\Psi_p\left(\cdot;m,M\right) : \left(m,M\right) \to \mathbb{R}$ is defined by

$$\Psi_p\left(t;m,M\right) = \frac{M^p - t^p}{M-t} - \frac{t^p - m^p}{t-m}$$

while

$$B_p\left(m,M\right) := \sup_{t \in \left(m,M\right)} \Psi_p\left(t;m,M\right). \tag{4.20}$$

From (4.2) we also have the inequality

$$\begin{aligned} 0 &\leq \frac{\int_{\Omega}\left|h\right|^p w d\mu}{\int_{\Omega} w d\mu} - \left(\frac{\int_{\Omega}\left|h\right| w d\mu}{\int_{\Omega} w d\mu}\right)^p \leq \frac{1}{4}\left(M-m\right)\Psi_p\left(\overline{\left|h\right|}_{\Omega,w};m,M\right) \\ &\leq \frac{1}{4}p\left(M-m\right)\left(M^{p-1}-m^{p-1}\right). \end{aligned} \tag{4.21}$$

Proposition 8 (Dragomir, 2011 [18])**.** *If $f \in L_p(\Omega,\mu)$, $g \in L_q(\Omega,\mu)$ with $p > 1, \frac{1}{p}+\frac{1}{q}=1$ and there exists the constants γ, $\Gamma > 0$ and such that*

$$\gamma \leq \frac{|f|}{|g|^{q-1}} \leq \Gamma \ \mu\text{-}a.e \text{ on } \Omega,$$

then we have

$$\begin{aligned}
(4.22)\qquad 0 &\leq \frac{\int_\Omega |f|^p \, d\mu}{\int_\Omega |g|^q \, d\mu} - \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^p \\
&\leq \frac{B_p(\gamma,\Gamma)}{\Gamma-\gamma}\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \\
&\leq p\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma}\left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \\
&\leq \frac{1}{4}p(\Gamma-\gamma)\left(\Gamma^{p-1}-\gamma^{p-1}\right),
\end{aligned}$$

and

$$\begin{aligned}
(4.23)\qquad 0 &\leq \frac{\int_\Omega |f|^p \, d\mu}{\int_\Omega |g|^q \, d\mu} - \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^p \\
&\leq \frac{1}{4}(\Gamma-\gamma)\Psi_p\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu};\gamma,\Gamma\right) \leq \frac{1}{4}p(\Gamma-\gamma)\left(\Gamma^{p-1}-\gamma^{p-1}\right),
\end{aligned}$$

where $B_p(\cdot,\cdot)$ and $\Psi_p(\cdot;\cdot,\cdot)$ are defined above.

Proof. The inequalities (4.22) and (4.23) follow from (4.19) and (4.21) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q.$$

The details are omitted. □

Remark 17. *We observe that for $p = q = 2$ we have $\Psi_2(t;\gamma,\Gamma) = \Gamma-\gamma = B_2(\gamma,\Gamma)$ and then from the first inequality in (4.22) we get the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality:*

$$\begin{aligned}
(4.24)\qquad &\int_\Omega |g|^2 \, d\mu \int_\Omega |f|^2 \, d\mu - \left(\int_\Omega |fg| \, d\mu\right)^2 \\
&\leq \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu}\right)\left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^2 \, d\mu} - \gamma\right)\left(\int_\Omega |g|^2 \, d\mu\right)^2
\end{aligned}$$

provided that f, $g \in L_2(\Omega,\mu)$, and there exists the constants γ, $\Gamma > 0$ such that

$$\gamma \leq \frac{|f|}{|g|} \leq \Gamma \ \mu\text{-}a.e \text{ on } \Omega.$$

Corollary 10. *With the assumptions of Proposition 8 we have the following additive reverses of the Hölder inequality*

$$
\begin{aligned}
(4.25)\qquad 0 &\le \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^q \, d\mu\right)^{1/q} - \int_\Omega |fg| \, d\mu \\
&\le \left[\frac{B_p(\gamma,\Gamma)}{\Gamma-\gamma}\right]^{1/p} \left(\Gamma - \frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^{1/p} \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)^{1/p} \\
&\times \int_\Omega |g|^q \, d\mu \\
&\le p^{1/p} \left(\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma}\right)^{1/p} \left(\Gamma - \frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu}\right)^{1/p} \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right)^{1/p} \\
&\times \int_\Omega |g|^q \, d\mu \\
&\le \frac{1}{4^{1/p}} p^{1/p} (\Gamma-\gamma)^{1/p} \left(\Gamma^{p-1}-\gamma^{p-1}\right)^{1/p} \int_\Omega |g|^q \, d\mu
\end{aligned}
$$

and

$$
\begin{aligned}
(4.26)\qquad 0 &\le \left(\int_\Omega |f|^p \, d\mu\right)^{1/p} \left(\int_\Omega |g|^q \, d\mu\right)^{1/q} - \int_\Omega |fg| \, d\mu \\
&\le \frac{1}{4^{1/p}} (\Gamma-\gamma)^{1/p} \, \Psi_p^{1/p} \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu}; m, M\right) \int_\Omega |g|^q \, d\mu \\
&\le \frac{1}{4^{1/p}} p^{1/p} (\Gamma-\gamma)^{1/p} \left(\Gamma^{p-1}-\gamma^{p-1}\right)^{1/p} \int_\Omega |g|^q \, d\mu
\end{aligned}
$$

where $p > 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By multiplying in (4.22) with $\left(\int_\Omega |g|^q \, d\mu\right)^p$ we have

$$
\begin{aligned}
&\int_\Omega |f|^p \, d\mu \left(\int_\Omega |g|^q \, d\mu\right)^{p-1} - \left(\int_\Omega |fg| \, d\mu\right)^p \\
&\le \frac{B_p(\gamma,\Gamma)}{\Gamma-\gamma} \left(\Gamma - \frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu}\right) \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \left(\int_\Omega |g|^q \, d\mu\right)^p \\
&\le p\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma} \left(\Gamma - \frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu}\right) \left(\frac{\int_\Omega |fg|\, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma\right) \left(\int_\Omega |g|^q \, d\mu\right)^p \\
&\le \frac{1}{4} p (\Gamma-\gamma) \left(\Gamma^{p-1}-\gamma^{p-1}\right) \left(\int_\Omega |g|^q \, d\mu\right)^p,
\end{aligned}
$$

which is equivalent with

$$\begin{aligned}(4.27)\quad &\int_\Omega |f|^p\,d\mu\left(\int_\Omega |g|^q\,d\mu\right)^{p-1}\\ &\leq \left(\int_\Omega |fg|\,d\mu\right)^p+\frac{B_p(\gamma,\Gamma)}{\Gamma-\gamma}\left(\Gamma-\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)\left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\gamma\right)\\ &\times\left(\int_\Omega |g|^q\,d\mu\right)^p\\ &\leq \left(\int_\Omega |fg|\,d\mu\right)^p+p\left(\Gamma-\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)\left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\gamma\right)\\ &\times\left(\int_\Omega |g|^q\,d\mu\right)^p\frac{\Gamma^{p-1}-\gamma^{p-1}}{\Gamma-\gamma}\\ &\leq \left(\int_\Omega |fg|\,d\mu\right)^p+\frac{1}{4}p(\Gamma-\gamma)\left(\Gamma^{p-1}-\gamma^{p-1}\right)\left(\int_\Omega |g|^q\,d\mu\right)^p.\end{aligned}$$

Taking the power $1/p$ with $p>1$ and employing the following elementary inequality that state that for $p>1$ and α, $\beta>0$,

$$(\alpha+\beta)^{1/p}\leq \alpha^{1/p}+\beta^{1/p}$$

we have from the first part of (4.27) that

$$\begin{aligned}(4.28)\quad &\left(\int_\Omega |f|^p\right)^{1/p}d\mu\left(\int_\Omega |g|^q\,d\mu\right)^{1-\frac{1}{p}}\\ &\leq \int_\Omega |fg|\,d\mu+\left[\frac{B_p(\gamma,\Gamma)}{\Gamma-\gamma}\right]^{1/p}\left(\Gamma-\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}\right)^{1/p}\left(\frac{\int_\Omega |fg|\,d\mu}{\int_\Omega |g|^q\,d\mu}-\gamma\right)^{1/p}\\ &\times\int_\Omega |g|^q\,d\mu\end{aligned}$$

and since $1-\frac{1}{p}=\frac{1}{q}$ we get from (4.28) the first inequality in (4.25). The rest is obvious.

The inequality (4.26) can be proved in a similar manner, however the details are omitted. □

If $h:\Omega\to\mathbb{R}$ is μ-measurable, satisfies the bounds

$$0<m\leq |h(x)|\leq M<\infty \text{ for } \mu\text{-a.e. } x\in\Omega$$

and is such that $h,\ |h|^p\in L_w(\Omega,\mu)$, for a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. $x\in\Omega$ and $\int_\Omega w d\mu>0$, then from (4.10) we also have the inequality

$$\begin{aligned}(4.29)\quad 0&\leq \frac{\int_\Omega |h|^p\,wd\mu}{\int_\Omega wd\mu}-\left(\frac{\int_\Omega |h|\,wd\mu}{\int_\Omega wd\mu}\right)^p\\ &\leq 2\left[\frac{m^p+M^p}{2}-\left(\frac{m+M}{2}\right)^p\right]\max\left\{\frac{M-\overline{|h|}_{\Omega,w}}{M-m},\frac{\overline{|h|}_{\Omega,w}-m}{M-m}\right\}\\ &\leq \frac{1}{2}p\left(M^{p-1}-m^{p-1}\right)\max\left\{M-\overline{|h|}_{\Omega,w},\overline{|h|}_{\Omega,w}-m\right\}.\end{aligned}$$

where, as above, $\overline{|h|}_{\Omega,w}:=\frac{\int_\Omega |h|wd\mu}{\int_\Omega wd\mu}\in[m,M]$.

From the inequality (4.29) we can state:

Proposition 9 (Dragomir, 2011 [18])**.** *With the assumptions of Proposition 8 we have*

$$
\begin{aligned}
0 &\le \frac{\int_\Omega |f|^p \, d\mu}{\int_\Omega |g|^q \, d\mu} - \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} \right)^p \\
&\le 2 \cdot \frac{\frac{\gamma^p + \Gamma^p}{2} - \left(\frac{\gamma+\Gamma}{2} \right)^p}{\Gamma - \gamma} \max \left\{ \Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}, \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma \right\} \\
&\le \frac{1}{2} p \left(\Gamma^{p-1} - \gamma^{p-1} \right) \max \left\{ \Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu}, \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma \right\}.
\end{aligned} \tag{4.30}
$$

Finally, the following additive reverse of the Hölder inequality can be stated as well:

Corollary 11. *With the assumptions of Proposition 8 we have*

$$
\begin{aligned}
0 &\le \left(\int_\Omega |f|^p \, d\mu \right)^{1/p} \left(\int_\Omega |g|^q \, d\mu \right)^{1/q} - \int_\Omega |fg| \, d\mu \\
&\le 2^{1/p} \cdot \left(\frac{\frac{\gamma^p + \Gamma^p}{2} - \left(\frac{\gamma+\Gamma}{2} \right)^p}{\Gamma - \gamma} \right)^{1/p} \\
&\times \max \left\{ \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} \right)^{1/p}, \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma \right)^{1/p} \right\} \int_\Omega |g|^q \, d\mu \\
&\le \frac{1}{2^{1/p}} p^{1/p} \max \left\{ \left(\Gamma - \frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} \right)^{1/p}, \left(\frac{\int_\Omega |fg| \, d\mu}{\int_\Omega |g|^q \, d\mu} - \gamma \right)^{1/p} \right\} \\
&\times \left(\Gamma^{p-1} - \gamma^{p-1} \right)^{1/p} \int_\Omega |g|^q \, d\mu.
\end{aligned} \tag{4.31}
$$

Remark 18. *As a simpler, however coarser inequality we have the following result:*

$$
\begin{aligned}
0 &\le \left(\int_\Omega |f|^p \, d\mu \right)^{1/p} \left(\int_\Omega |g|^q \, d\mu \right)^{1/q} - \int_\Omega |fg| \, d\mu \\
&\le 2^{1/p} \cdot \left[\frac{\gamma^p + \Gamma^p}{2} - \left(\frac{\gamma + \Gamma}{2} \right)^p \right]^{1/p} \int_\Omega |g|^q \, d\mu,
\end{aligned}
$$

where f and g are as above.

4.3. **Applications for f-Divergence.** The following result holds:

Proposition 10 (Dragomir, 2011 [18])**.** *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < r < 1 < R < \infty$ such that*

$$
r \le \frac{q(x)}{p(x)} \le R \text{ for } \mu\text{-a.e. } x \in \Omega. \tag{4.32}
$$

Then we have the inequalities

$$I_f(p,q) \leq \frac{(R-1)(1-r)}{R-r} \sup_{t\in(r,R)} \Psi_f(t;r,R) \tag{4.33}$$
$$\leq (R-1)(1-r)\frac{f'_-(R)-f'_+(r)}{R-r}$$
$$\leq \frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right],$$

and $\Psi_f(\cdot;r,R):(r,R)\to\mathbb{R}$ *is defined by*

$$\Psi_f(t;r,R) = \frac{f(R)-f(t)}{R-t} - \frac{f(t)-f(r)}{t-r}.$$

We also have the inequality

$$I_f(p,q) \leq \frac{1}{4}(R-r)\frac{f(R)(1-r)+f(r)(R-1)}{(R-1)(1-r)} \tag{4.34}$$
$$\leq \frac{1}{4}(R-r)\left[f'_-(R)-f'_+(r)\right].$$

The proof follows by Theorem 6 by choosing $w(x)=p(x)$, $f(x)=\frac{q(x)}{p(x)}$, $m=r$ and $M=R$ and performing the required calculations. The details are omitted.

Utilising the same approach and Theorem 7 we can also state that:

Proposition 11 (Dragomir, 2011 [18])**.** *With the assumptions of Proposition 10 we have*

$$I_f(p,q) \leq 2\max\left\{\frac{R-1}{R-r},\frac{1-r}{R-r}\right\}\left[\frac{f(r)+f(R)}{2}-f\left(\frac{r+R}{2}\right)\right] \tag{4.35}$$
$$\leq \frac{1}{2}\max\{R-1,1-r\}\left[f'_-(R)-f'_+(r)\right].$$

The above results can be utilized to obtain various inequalities for the divergence measures in Information Theory that are particular instances of f-divergence.

Consider the Kullback-Leibler divergence

$$D_{KL}(p,q) := \int_\Omega p(x)\ln\left[\frac{p(x)}{q(x)}\right]d\mu(x),\quad p,\ q\in\mathcal{P},$$

which is an f-divergence for the convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=-\ln t$.

If $p,\ q\in\mathcal{P}$ such that there exists the constants $0<r<1<R<\infty$ with

$$r \leq \frac{q(x)}{p(x)} \leq R \text{ for } \mu\text{-a.e. } x\in\Omega. \tag{4.36}$$

then we get from (4.33) that

$$D_{KL}(p,q) \leq \frac{(R-1)(1-r)}{rR}, \tag{4.37}$$

from (4.34) that

$$D_{KL}(p,q) \leq \frac{1}{4}(R-r)\ln\left[R^{-\frac{1}{R-1}}r^{-\frac{1}{1-r}}\right]$$

and from (4.35) that

$$D_{KL}(p,q) \le 2\max\left\{\frac{R-1}{R-r},\frac{1-r}{R-r}\right\}\ln\left(\frac{A(r,R)}{G(r,R)}\right) \le \frac{1}{2}\max\{R-1,1-r\}\left(\frac{R-r}{rR}\right), \tag{4.38}$$

where $A(r,R)$ is the arithmetic mean and $G(r,R)$ is the geometric mean of the positive numbers r and R.

5. Superadditivity and Monotonicity Properties

5.1. **General Results.** For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge 0$ for μ -a.e. $x\in\Omega$ and $\int_\Omega w d\mu>0$ we consider the functional

$$J(w;\Phi,f) := \int_\Omega w(\Phi\circ f)\,d\mu - \Phi\left(\frac{\int_\Omega wf d\mu}{\int_\Omega w d\mu}\right)\int_\Omega w d\mu \ge 0, \tag{5.1}$$

where $\Phi: I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in L_w(\Omega,\mu)$.

Theorem 8 (Dragomir, 2011 [21])**.** *Let $w_i:\Omega\to\mathbb{R}$, with $w_i(x)\ge 0$ for μ-a.e. $x\in\Omega$ and $\int_\Omega w_i d\mu>0$, $i\in\{1,2\}$. If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that f, $\Phi\circ f\in L_{w_1}(\Omega,\mu)\cap L_{w_2}(\Omega,\mu)$, then*

$$J(w_1+w_2;\Phi,f)\ge J(w_1;\Phi,f)+J(w_2;\Phi,f)\ge 0 \tag{5.2}$$

i.e., J is a superadditive functional of weights.

Moreover, if $w_2\ge w_1\ge 0$ μ-a.e. on Ω, then

$$J(w_2;\Phi,f)\ge J(w_1;\Phi,f)\ge 0, \tag{5.3}$$

i.e., J is a monotonic nondecreasing functional of weights.

Proof. Utilising the convexity property of Φ we have successively

$$
\begin{aligned}
&J(w_1+w_2;\Phi,f) \qquad (5.4)\\
&=\int_\Omega (w_1+w_2)(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega (w_1+w_2)\,f d\mu}{\int_\Omega (w_1+w_2)\,d\mu}\right)\int_\Omega (w_1+w_2)\,d\mu\\
&=\int_\Omega w_1(\Phi\circ f)\,d\mu+\int_\Omega w_2(\Phi\circ f)\,d\mu\\
&-\Phi\left(\frac{\int_\Omega w_1 d\mu\cdot\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}+\int_\Omega w_2 d\mu\cdot\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}}{\int_\Omega (w_1+w_2)\,d\mu}\right)\int_\Omega (w_1+w_2)\,d\mu\\
&\ge\int_\Omega w_1(\Phi\circ f)\,d\mu+\int_\Omega w_2(\Phi\circ f)\,d\mu\\
&-\left[\frac{\int_\Omega w_1 d\mu}{\int_\Omega (w_1+w_2)\,d\mu}\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+\frac{\int_\Omega w_2 d\mu}{\int_\Omega (w_1+w_2)\,d\mu}\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\right]\\
&\times\int_\Omega (w_1+w_2)\,d\mu\\
&=\int_\Omega w_1(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu\\
&+\int_\Omega w_2(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\int_\Omega w_2 d\mu\\
&=J(w_1;\Phi,f)+J(w_2;\Phi,f)
\end{aligned}
$$

which proves the superadditivity property.

Now, if $w_2\ge w_1\ge 0$, then on applying the superadditivity property we have

$$
\begin{aligned}
J(w_2;\Phi,f)&=J(w_1+(w_2-w_1);\Phi,f)\ge J(w_1;\Phi,f)+J(w_2-w_1;\Phi,f)\\
&\ge J(w_1;\Phi,f)
\end{aligned}
$$

since by the Jensen's inequality for the positive weights we have $J(w_2-w_1;\Phi,f)\ge 0$. $\square$

The above theorem has a simple however interesting consequence that provides both a refinement and a reverse for the Jensen's integral inequality:

Corollary 12. *Let $w_i:\Omega\to\mathbb{R}$, with $w_i(x)\ge 0$ for μ-a.e. $x\in\Omega$, $\int_\Omega w_i d\mu>0$, $i\in\{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$0\le\gamma\le\frac{w_2}{w_1}\le\Gamma<\infty\ \ \mu\text{-a.e. on }\Omega. \tag{5.5}$$

If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in L_{w_1}(\Omega,\mu)\cap L_{w_2}(\Omega,\mu)$, then

$$
\begin{aligned}
0&\le\gamma\left[\int_\Omega w_1(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu\right] \qquad (5.6)\\
&\le\int_\Omega w_2(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\int_\Omega w_2 d\mu\\
&\le\Gamma\left[\int_\Omega w_1(\Phi\circ f)\,d\mu-\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\int_\Omega w_1 d\mu\right]
\end{aligned}
$$

or, equivalently,

$$0 \leq \gamma \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1 (\Phi\circ f)\, d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right] \tag{5.7}$$
$$\leq \frac{\int_\Omega w_2 (\Phi\circ f)\, d\mu}{\int_\Omega w_2 d\mu} - \Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)$$
$$\leq \Gamma \frac{\int_\Omega w_1 d\mu}{\int_\Omega w_2 d\mu}\left[\frac{\int_\Omega w_1 (\Phi\circ f)\, d\mu}{\int_\Omega w_1 d\mu} - \Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)\right].$$

Proof. From (5.5) we have $\gamma w_1 \leq w_2 \leq \Gamma w_1 < \infty$ μ-a.e. on Ω and by the monotonicity property (5.3) we get

$$J(\Gamma w_1; \Phi, f) \geq J(w_2; \Phi, f) \geq J(\gamma w_1; \Phi, f). \tag{5.8}$$

Since the the functional is positive homogeneous, namely $J(\alpha w; \Phi, f) = \alpha J(w; \Phi, f)$, then we get from (5.8) the desired result (5.6). □

Remark 19. *Assume that* $\mu(\Omega) < \infty$ *and let* $w : \Omega \to \mathbb{R}$*, with* $w(x) \geq 0$ *for* μ*-a.e.* $x \in \Omega$*,* $\int_\Omega w d\mu > 0$ *and* w *is essentially bounded, i.e.* $\operatorname{essinf}_{x\in\Omega} w(x)$ *and* $\operatorname{essup}_{x\in\Omega} w(x)$ *are finite. If* $\Phi : I \to \mathbb{R}$ *is a continuous convex function on the interval of real numbers* I*,* $f : \Omega \to \mathbb{R}$ *is* μ*-measurable and such that* $f,\ \Phi\circ f \in L_w(\Omega,\mu) \cap L(\Omega,\mu)$*, then*

$$0 \leq \frac{\operatorname{essinf}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi\circ f)\, d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right] \tag{5.9}$$
$$\leq \frac{\int_\Omega w (\Phi\circ f)\, d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right)$$
$$\leq \frac{\operatorname{essup}_{x\in\Omega} w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega w d\mu}\left[\frac{\int_\Omega (\Phi\circ f)\, d\mu}{\mu(\Omega)} - \Phi\left(\frac{\int_\Omega f d\mu}{\mu(\Omega)}\right)\right].$$

This result can be used to provide the following result related to the Hermite-Hadamard inequality for convex functions that states that

$$\frac{1}{b-a}\int_a^b \Phi(t)\, dt \geq \Phi\left(\frac{a+b}{2}\right)$$

for any convex function $\Phi : [a,b] \to \mathbb{R}$.

Indeed , if $w : [a,b] \to [0,\infty)$ is Lebesgue integrable, then we have

$$0 \leq \frac{\operatorname{essinf}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\, dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right)\right] \tag{5.10}$$
$$\leq \frac{\int_a^b w(t)\,\Phi(t)\, dt}{\int_a^b w(t)\, dt} - \Phi\left(\frac{\int_\Omega w(t)\, t dt}{\int_a^b w(t)\, dt}\right)$$
$$\leq \frac{\operatorname{essup}_{x\in[a,b]} w(x)}{\frac{1}{b-a}\int_a^b w(t)\, dt}\left[\frac{1}{b-a}\int_a^b \Phi(t)\, dt - \Phi\left(\frac{a+b}{2}\right)\right].$$

Now we consider another functional depending on the weights

$$K(w; \Phi, f) := \frac{J(w; \Phi, f)}{\int_\Omega w d\mu} = \frac{\int_\Omega w (\Phi\circ f)\, d\mu}{\int_\Omega w d\mu} - \Phi\left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu}\right) \geq 0$$

and the composite functional

$$L(w;\Phi,f) := \left(\int_\Omega w\,d\mu\right)\ln\left[K(w;\Phi,f)+1\right] \geq 0,$$

where $\Phi : I \to \mathbb{R}$ is a continuous convex function on the interval of real numbers I and $f : \Omega \to \mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f \in L_w(\Omega,\mu)$.

Theorem 9 (Dragomir, 2011 [21])**.** *With the assumptions of Theorem 8, L is a superadditive and monotonic nondecreasing functional of weights.*

Proof. Let $w_i : \Omega \to \mathbb{R}$, with $w_i(x) \geq 0$ for μ-a.e. $x \in \Omega$ and $\int_\Omega w_i d\mu > 0$, $i \in \{1,2\}$ such that $f,\ \Phi\circ f \in L_{w_1}(\Omega,\mu) \cap L_{w_2}(\Omega,\mu)$.

Utilising the superadditivity property of J we have

$$\begin{aligned}
&L(w_1+w_2;\Phi,f) \\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[K(w_1+w_2;\Phi,f)+1\right] \\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[\frac{J(w_1+w_2;\Phi,f)}{\int_\Omega (w_1+w_2)\,d\mu}+1\right] \\
&\geq \left(\int_\Omega (w_1+w_2)\,d\mu\right)\ln\left[\frac{J(w_1;\Phi,f)+J(w_2;\Phi,f)}{\int_\Omega (w_1+w_2)\,d\mu}+1\right] \\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right) \\
&\quad\times\ln\left[\frac{\int_\Omega w_1 d\mu\cdot\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+\int_\Omega w_2 d\mu\cdot\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}}{\int_\Omega (w_1+w_2)\,d\mu}+1\right] \\
&= \left(\int_\Omega (w_1+w_2)\,d\mu\right) \\
&\quad\times\ln\left[\frac{\int_\Omega w_1 d\mu\cdot\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+\int_\Omega w_2 d\mu\cdot\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\int_\Omega (w_1+w_2)\,d\mu}\right] \\
&:= A.
\end{aligned}\tag{5.11}$$

By the weighted arithmetic mean - geometric mean inequality we have

$$\begin{aligned}
&\frac{\int_\Omega w_1 d\mu\cdot\left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)+\int_\Omega w_2 d\mu\cdot\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)}{\int_\Omega (w_1+w_2)\,d\mu} \\
&\geq \left(\frac{J(w_1;\Phi,f)}{\int_\Omega w_1 d\mu}+1\right)^{\frac{\int_\Omega w_1 d\mu}{\int_\Omega (w_1+w_2)d\mu}}\left(\frac{J(w_2;\Phi,f)}{\int_\Omega w_2 d\mu}+1\right)^{\frac{\int_\Omega w_2 d\mu}{\int_\Omega (w_1+w_2)d\mu}},
\end{aligned}$$

therefore, by taking the logarithm and utilizing the definition of the functional K, we get the inequality

$$\begin{aligned}
A &\geq \left(\int_\Omega w_1 d\mu\right)\ln\left(K(w_1;\Phi,f)+1\right)+\left(\int_\Omega w_2 d\mu\right)\ln\left(K(w_2;\Phi,f)+1\right) \\
&= L(w_1;\Phi,f)+L(w_2;\Phi,f).
\end{aligned}\tag{5.12}$$

Utilising (5.11) and (5.12) we deduce the superadditivity of the functional L as a function of weights.

Since $L(w;\Phi,f)\geq 0$ for any weight w and it is superadditive, by employing a similar argument to the one in the proof of Theorem 8 we conclude that it is also monotonic nondecreasing as a function of weights. $\square$

The following result provides another refinement and reverse of the Jensen inequality:

Corollary 13. *Let $w_i:\Omega\to\mathbb{R}$ with $w_i(x)\geq 0$ for μ-a.e. $x\in\Omega$, $\int_\Omega w_i d\mu>0$, $i\in\{1,2\}$ and there exists the nonnegative constants γ, Γ such that*

$$0\leq\gamma\leq\frac{w_2}{w_1}\leq\Gamma<\infty\ \mu\text{-a.e. on }\Omega.$$

If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in L_{w_1}(\Omega,\mu)\cap L_{w_2}(\Omega,\mu)$, then

$$\begin{aligned}(5.13)\qquad 0&\leq\left[\frac{\int_\Omega w_1(\Phi\circ f)\,d\mu}{\int_\Omega w_1 d\mu}-\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1\\ &\leq\frac{\int_\Omega w_2(\Phi\circ f)\,d\mu}{\int_\Omega w_2 d\mu}-\Phi\left(\frac{\int_\Omega w_2 f d\mu}{\int_\Omega w_2 d\mu}\right)\\ &\leq\left[\frac{\int_\Omega w_1(\Phi\circ f)\,d\mu}{\int_\Omega w_1 d\mu}-\Phi\left(\frac{\int_\Omega w_1 f d\mu}{\int_\Omega w_1 d\mu}\right)+1\right]^{\Gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1.\end{aligned}$$

Proof. Since L is monotonic nondecreasing and positive homogeneous as a function of weights, we have

$$\gamma L(w_1;\Phi,f)\leq L(w_2;\Phi,f)\leq\Gamma L(w_1;\Phi,f),$$

namely

$$\begin{aligned}[K(w_1;\Phi,f)+1]^{\gamma\left(\int_\Omega w_1 d\mu\right)}&\leq[K(w_2;\Phi,f)+1]^{\left(\int_\Omega w_2 d\mu\right)}\\ &\leq[K(w_1;\Phi,f)+1]^{\Gamma\left(\int_\Omega w_1 d\mu\right)},\end{aligned}$$

which provides that

$$\begin{aligned}[K(w_1;\Phi,f)+1]^{\gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1&\leq K(w_2;\Phi,f)\\ &\leq[K(w_1;\Phi,f)+1]^{\Gamma\frac{\left(\int_\Omega w_1 d\mu\right)}{\left(\int_\Omega w_2 d\mu\right)}}-1.\end{aligned}$$

$\square$

Remark 20. *Assume that $\mu(\Omega)<\infty$ and let $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. $x\in\Omega$, $\int_\Omega w d\mu>0$ and w is essentially bounded, i.e.* $\operatorname{essinf}_{x\in\Omega}w(x)$ *and* $\operatorname{essup}_{x\in\Omega}w(x)$ *are finite. If $\Phi:I\to\mathbb{R}$ is a continuous convex function on the interval of real numbers I, $f:\Omega\to\mathbb{R}$ is μ-measurable and such that $f,\ \Phi\circ f\in$*

$L_w(\Omega,\mu)\cap L(\Omega,\mu)$, *then*

$$0\le\left[\frac{\int_\Omega(\Phi\circ f)\,d\mu}{\mu(\Omega)}-\Phi\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\inf_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega wd\mu\right)}}-1 \tag{5.14}$$
$$\le\frac{\int_\Omega w(\Phi\circ f)\,d\mu}{\int_\Omega wd\mu}-\Phi\left(\frac{\int_\Omega wfd\mu}{\int_\Omega wd\mu}\right)$$
$$\le\left[\frac{\int_\Omega(\Phi\circ f)\,d\mu}{\mu(\Omega)}-\Phi\left(\frac{\int_\Omega fd\mu}{\mu(\Omega)}\right)+1\right]^{\frac{ess\sup_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\left(\int_\Omega wd\mu\right)}}-1.$$

In particular, if $w:[a,b]\to[0,\infty)$ *is Lebesgue integrable, then we have the following result related to the Hermite-Hadamard inequality for the convex function* $\Phi:[a,b]\to\mathbb{R}$

$$0\le\left[\frac{1}{b-a}\int_a^b\Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\operatorname{essinf}_{x\in[a,b]}w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1 \tag{5.15}$$
$$\le\frac{\int_a^b w(t)\,\Phi(t)\,dt}{\int_a^b w(t)\,dt}-\Phi\left(\frac{\int_\Omega w(t)\,tdt}{\int_a^b w(t)\,dt}\right)$$
$$\le\left[\frac{1}{b-a}\int_a^b\Phi(t)\,dt-\Phi\left(\frac{a+b}{2}\right)+1\right]^{\frac{\operatorname{essup}_{x\in[a,b]}w(x)}{\frac{1}{b-a}\int_a^b w(t)dt}}-1.$$

5.2. **Applications for the Hölder Inequality.** Assume that $p>1$. If $h:\Omega\to\mathbb{R}$ is μ-measurable, $\mu(\Omega)<\infty$, $|h|,|h|^p\in L_w(\Omega,\mu)\cap L(\Omega,\mu)$, then by (5.9) we have the bounds

$$0\le\frac{\operatorname{essinf}_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega wd\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega|h|^p\,d\mu-\left(\frac{1}{\mu(\Omega)}\int_\Omega|h|\,d\mu\right)^p\right] \tag{5.16}$$
$$\le\frac{1}{\int_\Omega wd\mu}\int_\Omega w|h|^p\,d\mu-\left(\frac{1}{\int_\Omega wd\mu}\int_\Omega w|h|\,d\mu\right)^p$$
$$\le\frac{\operatorname{essup}_{x\in\Omega}w(x)}{\frac{1}{\mu(\Omega)}\int_\Omega wd\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega|h|^p\,d\mu-\left(\frac{1}{\mu(\Omega)}\int_\Omega|h|\,d\mu\right)^p\right].$$

Proposition 12 (Dragomir, 2011 [21])**.** *If* $f\in L_p(\Omega,\mu)$, $g\in L_q(\Omega,\mu)$ *with* $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\mu(\Omega)<\infty$ *and there exists the constants* $\delta,\ \Delta>0$ *and such that*

$$\delta\le|g|\le\Delta\ \mu\text{-}a.e\text{ on }\Omega, \tag{5.17}$$

then we have

$$0\le\frac{\delta^q}{\frac{1}{\mu(\Omega)}\int_\Omega|g|^q\,d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega\frac{|f|^p}{|g|^q}d\mu-\left(\frac{1}{\mu(\Omega)}\int_\Omega\frac{|f|}{|g|^{q-1}}d\mu\right)^p\right] \tag{5.18}$$
$$\le\frac{\int_\Omega|f|^p\,d\mu}{\int_\Omega|g|^q\,d\mu}-\left(\frac{\int_\Omega|fg|\,d\mu}{\int_\Omega|g|^q\,d\mu}\right)^p$$
$$\le\frac{\Delta^q}{\frac{1}{\mu(\Omega)}\int_\Omega|g|^q\,d\mu}\left[\frac{1}{\mu(\Omega)}\int_\Omega\frac{|f|^p}{|g|^q}d\mu-\left(\frac{1}{\mu(\Omega)}\int_\Omega\frac{|f|}{|g|^{q-1}}d\mu\right)^p\right].$$

Proof. The inequalities (5.18) follows from (5.16) by choosing

$$h = \frac{|f|}{|g|^{q-1}} \text{ and } w = |g|^q .$$

The details are omitted. □

Remark 21. *We observe that for $p = q = 2$ we have from (5.18) the following reverse of the Cauchy-Bunyakovsky-Schwarz inequality*

$$\begin{aligned} (5.19) \qquad 0 &\leq \delta^2 \mu(\Omega) \left[\frac{1}{\mu(\Omega)} \int_\Omega \left| \frac{f}{g} \right|^2 d\mu - \left(\frac{1}{\mu(\Omega)} \int_\Omega \left| \frac{f}{g} \right| d\mu \right)^2 \right] \int_\Omega |g|^2 \, d\mu \\ &\leq \int_\Omega |g|^2 \, d\mu \int_\Omega |f|^2 \, d\mu - \left(\int_\Omega |fg| \, d\mu \right)^2 \\ &\leq \Delta^2 \mu(\Omega) \left[\frac{1}{\mu(\Omega)} \int_\Omega \left| \frac{f}{g} \right|^2 d\mu - \left(\frac{1}{\mu(\Omega)} \int_\Omega \left| \frac{f}{g} \right| d\mu \right)^2 \right] \int_\Omega |g|^2 \, d\mu, \end{aligned}$$

provided that f, $g \in L_2(\Omega, \mu)$ and g satisfies the bounds (5.17).

Similar results can be stated by utilizing the inequality (5.13), however the details are not presented here.

5.3. **Applications for f-Divergence Measures.** The following result holds:

Proposition 13 (Dragomir, 2011 [21])**.** *Let $f : (0, \infty) \to \mathbb{R}$ be a convex function with the property that $f(1) = 0$. Assume that $p, q \in \mathcal{P}$ and there exists the constants $0 < s < 1 < S < \infty$ such that*

$$(5.20) \qquad s \leq \frac{p(x)}{q(x)} \leq S \text{ for } \mu\text{-a.e. } x \in \Omega.$$

Then we have the inequalities

$$\begin{aligned} (5.21) \qquad & s \left[I_{f\left(\frac{1}{\cdot}\right)}(q,p) - f\left(D_{\chi^2}(p,q) + 1\right) \right] \\ &\leq I_f(p,q) \\ &\leq S \left[I_{f\left(\frac{1}{\cdot}\right)}(q,p) - f\left(D_{\chi^2}(p,q) + 1\right) \right]. \end{aligned}$$

Proof. If we use the inequality (5.6) we get

$$\begin{aligned} (5.22) \qquad & s \left[\int_\Omega q f\left(\frac{q}{p}\right) d\mu - f\left(\int_\Omega \frac{q^2}{p} d\mu \right) \right] \\ &\leq \int_\Omega p f\left(\frac{q}{p}\right) d\mu \\ &\leq S \left[\int_\Omega q f\left(\frac{q}{p}\right) d\mu - f\left(\int_\Omega \frac{q^2}{p} d\mu \right) \right]. \end{aligned}$$

Since

$$\int_\Omega \frac{q^2}{p} d\mu = D_{\chi^2}(p,q) + 1$$

and

$$\int_\Omega q f\left(\frac{q}{p}\right) d\mu = I_{f\left(\frac{1}{\cdot}\right)}(q,p),$$

then from (5.22) we deduce the desired result (5.21). □

Consider the Kullback-Leibler divergence

$$D_{KL}(p,q) := \int_{\Omega} p(x) \ln \left[\frac{p(x)}{q(x)} \right] d\mu(x), \quad p,\ q \in \mathcal{P},$$

which is an f-divergence for the convex function $f : (0,\infty) \to \mathbb{R}$, $f(t) = -\ln t$.

If $p,\ q \in \mathcal{P}$ such that there exists the constants $0 < s < 1 < S < \infty$ with

$$s \leq \frac{p(x)}{q(x)} \leq S \text{ for } \mu\text{-a.e. } x \in \Omega.. \tag{5.23}$$

then we get from (5.21) that

$$\begin{aligned} & s \left[\ln \left(D_{\chi^2}(p,q) + 1 \right) - D_{KL}(q,p) \right] \\ & \leq D_{KL}(p,q) \\ & \leq S \left[\ln \left(D_{\chi^2}(p,q) + 1 \right) - D_{KL}(q,p) \right]. \end{aligned} \tag{5.24}$$

Similar results for f-divergence measures can be stated by utilizing the inequality (5.13), however the details are not presented here.

6. Inequalities for Selfadjoint Operators

6.1. **Preliminary Facts.** The above integral inequalities can be used to obtain various reverses of Jensen's inequality for convex functions of selfadjoint operators on complex Hilbert spaces. In order to state these results, we need the following preparations.

Let A be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $\mathrm{Sp}(A)$ included in the interval $[m,M]$ for some real numbers $m < M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f : [m,M] \to \mathbb{R}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral* (see for instance [31, p. 257]):

$$\langle f(A)x, y \rangle = \int_{m-0}^{M} f(\lambda)\, d\langle E_\lambda x, y \rangle, \tag{6.1}$$

and

$$\|f(A)x\|^2 = \int_{m-0}^{M} |f(\lambda)|^2\, d\|E_\lambda x\|^2, \tag{6.2}$$

for any $x, y \in H$.

The function $g_{x,y}(\lambda) := \langle E_\lambda x, y \rangle$ is of *bounded variation* on the interval $[m,M]$ and $g_{x,y}(m-0) = 0$ while $g_{x,y}(M) = \langle x, y \rangle$ for any $x, y \in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x, x \rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$ for any $x \in H$.

The following result that provides an operator version for the Jensen inequality:

Theorem 10 (Mond-Pečarić, 1993, [41])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that $\mathrm{Sp}(A) \subseteq [m,M]$ for some scalars m, M with $m < M$. If Φ is a convex function on $[m,M]$, then*

$$\Phi(\langle Ax, x \rangle) \leq \langle \Phi(A)x, x \rangle \tag{MP}$$

for each $x \in H$ with $\|x\| = 1$.

As a special case of Theorem 10 we have the following Hölder-McCarthy inequality:

Theorem 11 (Hölder-McCarthy, 1967, [39])**.** *Let A be a selfadjoint positive operator on a Hilbert space H. Then for all $x \in H$ with $\|x\| = 1$,*
(i) $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r > 1$;
(ii) $\langle A^r x, x\rangle \leq \langle Ax, x\rangle^r$ *for all* $0 < r < 1$;
(iii) If A is invertible, then $\langle A^r x, x\rangle \geq \langle Ax, x\rangle^r$ *for all* $r < 0$.

The following reverse for the (MP) inequality that generalizes the scalar Lah-Ribarić inequality for convex functions is well known, see for instance [27, p. 57]:

Theorem 12. *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If Φ is a convex function on $[m, M]$, then*

$$\langle \Phi(A) x, x\rangle \leq \frac{M - \langle Ax, x\rangle}{M - m}\Phi(m) + \frac{\langle Ax, x\rangle - m}{M - m}\Phi(M) \tag{LR}$$

for each $x \in H$ with $\|x\| = 1$.

In [24] we obtained the following weighted version of (MP) and (LR).

Theorem 13 (Dragomir, 2014 [24])**.** *Let A be a selfadjoint operator on the Hilbert space H and assume that* $\operatorname{Sp}(A) \subseteq [m, M]$ *for some scalars m, M with $m < M$. If $\Phi : [k, K] \subset \mathbb{R} \to \mathbb{R}$ is a continuous convex function on the interval $[k, K]$, $w : [m, M] \to [0, \infty)$ is continuous on $[m, M]$, $f : [m, M] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function on the interval $[m, M]$ and with the property that*

$$k \leq f(t) \leq K \text{ for any } t \in [m, M], \tag{6.3}$$

then

$$\begin{aligned} &\Phi\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) \\ &\leq \frac{\langle w(A)(\Phi \circ f)(A) x, x\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\Phi(k) + \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right)\Phi(K)}{K - k}, \end{aligned} \tag{6.4}$$

for any $x \in H$ with $\langle w(A) x, x\rangle \neq 0$.

For various particular instances of (6.4) that are of interest being related to Hölder-McCarthy's inequalities mentioned above, see [24].

For classical and recent result concerning inequalities for continuos functions of selfadjoint operators, see the recent monographs, [27], [22] and [23].

6.2. **Reverses for Functions of Operators.** We have the following results:

Theorem 14 (Dragomir, 2015 [25])**.** *Let A be a selfadjoint operator on the Hilbert space H such that* $\operatorname{Sp}(A) \subseteq [k, K]$ *for some scalars k, K with $k < K$. Assume that $\Phi : [k, K] \subset \mathbb{R} \to \mathbb{R}$ is a continuous convex function on the interval $[k, K]$, $w : [k, K] \to [0, \infty)$ is continuous on $[k, K]$, $f : [k, K] \subset \mathbb{R} \to \mathbb{R}$ is a continuous function on the interval $[k, K]$ and satisfies the property (6.3)* *(i) If Φ is continuously differentiable on (k, K), then we have*

$$
\begin{aligned}
(6.5)\quad 0 &\leq \frac{\langle w(A)(\Phi \circ f)(A)x, x\rangle}{\langle w(A)x, x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right) \\
&\leq \frac{\langle (\Phi' \circ f)(A)f(A)w(A)x, x\rangle}{\langle w(A)x, x\rangle} \\
&\quad - \frac{\langle (\Phi' \circ f)(A)w(A)x, x\rangle}{\langle w(A)x, x\rangle}\frac{\langle f(A)w(A)x, x\rangle}{\langle w(A)x, x\rangle} \\
&\leq \frac{1}{2}\left[\Phi'_-(K) - \Phi'_+(k)\right]\frac{\left\langle \left| f(A) - \frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H \right| x, x\right\rangle}{\langle w(A)x, x\rangle} \\
&\leq \frac{1}{2}\left[\Phi'_-(K) - \Phi'_+(k)\right]\left[\frac{\langle f^2(A)w(A)x, x\rangle}{\langle w(A)x, x\rangle} - \left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\Phi'_-(K) - \Phi'_+(k)\right](K - k)
\end{aligned}
$$

for any $x \in H$ with $\langle w(A)x, x\rangle \neq 0$.

(ii) If we consider the function $\Psi_\Phi(\cdot; k, K) : (k, K) \to \mathbb{R}$ defined by

$$
\Psi_\Phi(t; k, K) = \frac{\Phi(K) - \Phi(t)}{K - t} - \frac{\Phi(t) - \Phi(k)}{t - k},
$$

then

$$
\begin{aligned}
(6.6)\quad 0 &\leq \frac{\langle w(A)(\Phi \circ f)(A)x, x\rangle}{\langle w(A)x, x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right) \\
&\leq \frac{\left(K - \frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle} - k\right)}{K - k}\sup_{t \in (k,K)}\Psi_\Phi(t; k, K) \\
&\leq \left(K - \frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right)\left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle} - k\right)\frac{\Phi'_-(K) - \Phi'_+(k)}{K - k} \\
&\leq \frac{1}{4}\left[\Phi'_-(K) - \Phi'_+(k)\right](K - k)
\end{aligned}
$$

and

$$
\begin{aligned}
(6.7)\quad 0 &\leq \frac{\langle w(A)(\Phi \circ f)(A)x, x\rangle}{\langle w(A)x, x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}\right) \\
&\leq \frac{1}{4}(K - k)\Psi_\Phi\left(\frac{\langle w(A)f(A)x, x\rangle}{\langle w(A)x, x\rangle}; k, K\right) \\
&\leq \frac{1}{4}\left[\Phi'_-(K) - \Phi'_+(k)\right](K - k)
\end{aligned}
$$

for any $x \in H$ with $\langle w(A)x, x\rangle \neq 0$.

(iii) We have the inequalities

$$
\begin{aligned}
(6.8) \qquad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right) \\
&\le 2\max\left\{\frac{K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}}{K-k}, \frac{\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k}{K-k}\right\} \\
&\times\left[\frac{\Phi(k)+\Phi(K)}{2} - \Phi\left(\frac{k+K}{2}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
(6.9) \qquad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right) \\
&\le 2\left[\frac{\Phi(k)+\Phi(K)}{2} - \Phi\left(\frac{k+K}{2}\right)\right]
\end{aligned}
$$

for any $x\in H$ with $\langle w(A)x,x\rangle \neq 0$.

(iv) We also have the inequalities

$$
\begin{aligned}
(6.10) \qquad 0 &\le \frac{\langle w(A)(\Phi\circ f)(A)x,x\rangle}{\langle w(A)x,x\rangle} - \Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right) \\
&\le \frac{1}{2}\Psi_\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)\frac{\left\langle \left| f(A) - \frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H\right| x,x\right\rangle}{\langle w(A)x,x\rangle} \\
&\le \frac{1}{2}\Psi_\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right) \\
&\times\left[\frac{\langle f^2(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle} - \left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\Psi_\Phi\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)(K-k)
\end{aligned}
$$

for any $x\in H$ with $\langle w(A)x,x\rangle \neq 0$.

Proof. (i) Let $\{E_\lambda\}_\lambda$ be the spectral family of the operator A. Let $\varepsilon>0$ and write the inequality (1.17) on the interval $[k-\varepsilon,K]$ and for the monotonic nondecreasing function $g(t)=\langle E_t x,x\rangle$, $x\in H$ with $\langle w(A)x,x\rangle \neq 0$, to get

$$
\begin{aligned}
0 &\le \frac{\int_{k-\varepsilon}^{K}(\Phi\circ f)(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle}-\Phi\left(\frac{\int_{k-\varepsilon}^{K}f(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle}\right) \\
&\le \frac{\int_{k-\varepsilon}^{K}(\Phi'\circ f)(t)\,f(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle} \\
&\quad -\frac{\int_{k-\varepsilon}^{K}(\Phi'\circ f)(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle}\frac{\int_{k-\varepsilon}^{K}f(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle} \\
&\le \frac{1}{2}\frac{\left[\Phi'_{-}(K)-\Phi'_{+}(k)\right]}{\int_{k-\varepsilon}^{K}w(t)\,d\langle E_t x,x\rangle} \\
&\quad \times \int_{k-\varepsilon}^{K}\left|f(t)-\frac{\int_{k-\varepsilon}^{K}f(s)\,w(s)\,d\langle E_s x,x\rangle}{\int_{k-\varepsilon}^{K}w(s)\,d\langle E_s x,x\rangle}\right| w(t)\,d\langle E_t x,x\rangle \\
&\le \frac{1}{2}\left[\Phi'_{-}(K)-\Phi'_{+}(k)\right] \\
&\quad \times \left[\frac{\int_{k-\varepsilon}^{K}f^{2}(t)\,w(t)\,d\langle E_t x,x\rangle}{\int_{k-\varepsilon}^{K}w(s)\,d\langle E_s x,x\rangle}-\left(\frac{\int_{k-\varepsilon}^{K}f(s)\,w(s)\,d\langle E_s x,x\rangle}{\int_{k-\varepsilon}^{K}w(s)\,d\langle E_s x,x\rangle}\right)^{2}\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\left[\Phi'_{-}(K)-\Phi'_{+}(k)\right](K-k).
\end{aligned} \tag{6.11}
$$

Letting $\varepsilon\to 0+$ and using the spectral representation theorem summarized in (6.1) we get the required inequality (6.5).

(ii) Follows by the first part of Theorem 6 , (iii) follows by Theorem 7 while (iv) follows by the second part of Theorem 6. The details are omitted. □

We have the following generalization and reverse for the Hölder-McCarthy inequality:

Corollary 14 (Dragomir, 2015 [25]). *Let A be a selfadjoint operator on the Hilbert space H such that $\operatorname{Sp}(A)\subseteq[k,K]$ for some scalars k, K with $k<K$. Assume that $w:[k,K]\to[0,\infty)$ is continuous on $[k,K]$, $f:[k,K]\subset\mathbb{R}\to\mathbb{R}$ is a continuous function on the interval $[k,K]$ and satisfies the property (6.3) with $k>0$. Assume also that $p\in(-\infty,0)\cup(1,\infty)$.*

(i) We have

$$0 \leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \tag{6.12}$$
$$\leq p\left[\frac{\langle f^p(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \frac{\left\langle f^{p-1}(A) w(A) x, x\right\rangle}{\langle w(A) x, x\rangle}\frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle}\right]$$
$$\leq \frac{1}{2}p\left(K^{p-1} - k^{p-1}\right)\frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle}$$
$$\leq \frac{1}{2}p\left(K^{p-1} - k^{p-1}\right)\left[\frac{\left\langle f^2(A) w(A) x, x\right\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K-k)$$

for any $x \in H$ with $\langle w(A) x, x\rangle \neq 0$.

(ii) If we consider the function $\Psi_p(\cdot; k, K) : (k, K) \to \mathbb{R}$ defined by

$$\Psi_p(t; k, K) = \frac{K^p - t^p}{K - t} - \frac{t^p - k^p}{t - k},$$

then

$$0 \leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \tag{6.13}$$
$$\leq \frac{\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right)}{K - k}\sup_{t\in(k,K)}\Psi_p(t; k, K)$$
$$\leq p\frac{K^{p-1} - k^{p-1}}{K - k}\left(K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k\right)$$
$$\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K-k)$$

and

$$0 \leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \tag{6.14}$$
$$\leq \frac{1}{4}(K-k)\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)$$
$$\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K-k)$$

for any $x \in H$ with $\langle w(A) x, x\rangle \neq 0$.

(iii) We have the inequalities

$$0 \leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \tag{6.15}$$
$$\leq 2\max\left\{\frac{K - \frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}}{K - k}, \frac{\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle} - k}{K - k}\right\}$$
$$\times\left[\frac{k^p + K^p}{2} - \left(\frac{k + K}{2}\right)^p\right]$$

and

$$0 \leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \leq 2\left[\frac{k^p + K^p}{2} - \left(\frac{k+K}{2}\right)^p\right] \tag{6.16}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

(iv) *We also have the inequalities*

$$\begin{aligned} 0 &\leq \frac{\langle w(A) f^p(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^p \\ &\leq \frac{1}{2}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)\frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{1}{2}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right) \\ &\quad \times \left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\Psi_p\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}; k, K\right)(K-k) \end{aligned} \tag{6.17}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$.

If $p \in (0,1)$, then by taking $\Phi(t) = -t^p$ we can get similar inequalities. However the details are omitted.

If we take $\Phi(t) = -\ln t$, $t > 0$ in Theorem 14 then we get the following logarithmic inequalities:

Corollary 15 (Dragomir, 2015 [25])**.** *Let A be a selfadjoint operator on the Hilbert space H such that* $\mathrm{Sp}(A) \subseteq [k, K]$ *for some scalars k, K with $k < K$. Assume that* $w : [k, K] \to [0, \infty)$ *is continuous on* $[k, K]$, $f : [k, K] \subset \mathbb{R} \to \mathbb{R}$ *is a continuous function on the interval* $[k, K]$ *and satisfies the property (6.3) with* $k > 0$.

(i) *We have*

$$\begin{aligned} 0 &\leq \ln\left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right) - \frac{\langle w(A) \ln f(A) x, x\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{\langle f^{-1}(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle}\frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - 1 \\ &\leq \frac{1}{2}\frac{K-k}{kK}\frac{\left\langle \left| f(A) - \frac{\langle f(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} 1_H \right| x, x\right\rangle}{\langle w(A) x, x\rangle} \\ &\leq \frac{1}{2}\frac{K-k}{kK}\left[\frac{\langle f^2(A) w(A) x, x\rangle}{\langle w(A) x, x\rangle} - \left(\frac{\langle w(A) f(A) x, x\rangle}{\langle w(A) x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\frac{(K-k)^2}{kK} \end{aligned} \tag{6.18}$$

for any $x \in H$ *with* $\langle w(A) x, x\rangle \neq 0$, *(ii) If we consider the function* $\Psi_{-\ln}(\cdot;k,K):(k,K)\to\mathbb{R}$ *defined by*

$$\Psi_{-\ln}(t;k,K)=\frac{\ln t-\ln k}{t-k}-\frac{\ln K-\ln t}{K-t},$$

then

$$0\le\ln\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)-\frac{\langle w(A)\ln f(A)x,x\rangle}{\langle w(A)x,x\rangle}\tag{6.19}$$
$$\le\frac{\left(K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k\right)}{K-k}\sup_{t\in(k,K)}\Psi_{-\ln}(t;k,K)$$
$$\le\frac{1}{Kk}\left(K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k\right)\le\frac{1}{4}\frac{(K-k)^2}{kK}$$

and

$$0\le\ln\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)-\frac{\langle w(A)\ln f(A)x,x\rangle}{\langle w(A)x,x\rangle}\tag{6.20}$$
$$\le\frac{1}{4}(K-k)\Psi_{-\ln}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)\le\frac{1}{4}\frac{(K-k)^2}{kK}$$

for any $x\in H$ *with* $\langle w(A)x,x\rangle\neq0$.

(iii) We have the inequalities

$$0\le\ln\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)-\frac{\langle w(A)\ln f(A)x,x\rangle}{\langle w(A)x,x\rangle}\tag{6.21}$$
$$\le2\max\left\{\frac{K-\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}}{K-k},\frac{\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}-k}{K-k}\right\}\ln\left(\frac{k+K}{2\sqrt{kK}}\right)$$

and

$$0\le\ln\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)-\frac{\langle w(A)\ln f(A)x,x\rangle}{\langle w(A)x,x\rangle}\le\ln\left(\frac{k+K}{2\sqrt{kK}}\right)^2\tag{6.22}$$

for any $x\in H$ *with* $\langle w(A)x,x\rangle\neq0$.

(iv) We also have the inequalities

$$0\le\ln\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)-\frac{\langle w(A)\ln f(A)x,x\rangle}{\langle w(A)x,x\rangle}\tag{6.23}$$
$$\le\frac{1}{2}\Psi_{-\ln}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)\frac{\left\langle\left|f(A)-\frac{\langle f(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}1_H\right|x,x\right\rangle}{\langle w(A)x,x\rangle}$$
$$\le\frac{1}{2}\Psi_{-\ln}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)$$
$$\times\left[\frac{\langle f^2(A)w(A)x,x\rangle}{\langle w(A)x,x\rangle}-\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\le\frac{1}{4}\Psi_{-\ln}\left(\frac{\langle w(A)f(A)x,x\rangle}{\langle w(A)x,x\rangle};k,K\right)(K-k)$$

for any $x\in H$ *with* $\langle w(A)x,x\rangle\neq0$.

6.3. **Some Examples.** If we choose $w(t)=1$ and $f(t)=t$ with $t\in[k,K]\subset[0,\infty)$ then we get from Corollary 14 that

$$\begin{aligned}0&\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\leq p\left[\langle A^{p}x,x\rangle-\left\langle A^{p-1}x,x\right\rangle\langle Ax,x\rangle\right]\\&\leq\frac{1}{2}p\left(K^{p-1}-k^{p-1}\right)\langle|A-\langle Ax,x\rangle 1_{H}|\,x,x\rangle\\&\leq\frac{1}{2}p\left(K^{p-1}-k^{p-1}\right)\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]^{\frac{1}{2}}\\&\leq\frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)\,,\end{aligned}\tag{6.24}$$

$$\begin{aligned}0&\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\\&\leq\frac{(K-\langle Ax,x\rangle)(\langle Ax,x\rangle-k)}{K-k}\sup_{t\in(k,K)}\Psi_{p}\left(t;k,K\right)\\&\leq p\frac{K^{p-1}-k^{p-1}}{K-k}\left(K-\langle Ax,x\rangle\right)\left(\langle Ax,x\rangle-k\right)\\&\leq\frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)\,,\end{aligned}\tag{6.25}$$

$$\begin{aligned}0&\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\leq\frac{1}{4}(K-k)\,\Psi_{p}\left(\langle Ax,x\rangle;k,K\right)\\&\leq\frac{1}{4}p\left(K^{p-1}-k^{p-1}\right)(K-k)\,,\end{aligned}\tag{6.26}$$

$$\begin{aligned}0&\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\\&\leq 2\max\left\{\frac{K-\langle Ax,x\rangle}{K-k},\frac{\langle Ax,x\rangle-k}{K-k}\right\}\left[\frac{k^{p}+K^{p}}{2}-\left(\frac{k+K}{2}\right)^{p}\right],\end{aligned}\tag{6.27}$$

$$0\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\leq 2\left[\frac{k^{p}+K^{p}}{2}-\left(\frac{k+K}{2}\right)^{p}\right]\tag{6.28}$$

and

$$\begin{aligned}0&\leq\langle A^{p}x,x\rangle-\langle Ax,x\rangle^{p}\leq\frac{1}{2}\Psi_{p}\left(\langle Ax,x\rangle;k,K\right)\langle|A-\langle Ax,x\rangle 1_{H}|\,x,x\rangle\\&\leq\frac{1}{2}\Psi_{p}\left(\langle Ax,x\rangle;k,K\right)\left[\left\langle A^{2}x,x\right\rangle-\langle Ax,x\rangle^{2}\right]^{\frac{1}{2}}\\&\leq\frac{1}{4}\Psi_{p}\left(\langle Ax,x\rangle;k,K\right)(K-k)\end{aligned}\tag{6.29}$$

for any $x\in H$, $\|x\|=1$.

If we choose $w(t) = t^q$, $q \neq 0$ and $f(t) = t$ with $t \in [k, K] \subset [0, \infty)$ then we get from Corollary 14 that

$$
\begin{aligned}
(6.30)\qquad 0 &\leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \\
&\leq p\left[\frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \frac{\langle A^{p+q-1}x, x\rangle}{\langle A^q x, x\rangle}\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right] \\
&\leq \frac{1}{2}p\left(K^{p-1} - k^{p-1}\right)\frac{\left\langle \left|A - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}1_H\right| x, x\right\rangle}{\langle A^q x, x\rangle} \\
&\leq \frac{1}{2}p\left(K^{p-1} - k^{p-1}\right)\left[\frac{\langle A^{q+2}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K - k),
\end{aligned}
$$

$$
\begin{aligned}
(6.31)\qquad 0 &\leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \\
&\leq \frac{\left(K - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}\right)\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle} - k\right)}{K - k}\sup_{t\in(k,K)}\Psi_p(t; k, K) \\
&\leq p\frac{K^{p-1} - k^{p-1}}{K - k}\left(K - \frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle} - k\right) \\
&\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K - k),
\end{aligned}
$$

$$
\begin{aligned}
(6.32)\qquad 0 &\leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \leq \frac{1}{4}(K - k)\Psi_p\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right) \\
&\leq \frac{1}{4}p\left(K^{p-1} - k^{p-1}\right)(K - k),
\end{aligned}
$$

$$
\begin{aligned}
(6.33)\qquad 0 &\leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \\
&\leq 2\max\left\{\frac{K - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}}{K - k}, \frac{\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle} - k}{K - k}\right\}\left[\frac{k^p + K^p}{2} - \left(\frac{k + K}{2}\right)^p\right],
\end{aligned}
$$

$$
(6.34)\qquad 0 \leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \leq 2\left[\frac{k^p + K^p}{2} - \left(\frac{k + K}{2}\right)^p\right]
$$

and

$$0 \leq \frac{\langle A^{p+q}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^p \tag{6.35}$$
$$\leq \frac{1}{2}\Psi_p\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right)\frac{\left\langle\left|A - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}1_H\right| x, x\right\rangle}{\langle A^q x, x\rangle}$$
$$\leq \frac{1}{2}\Psi_p\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right)\left[\frac{\langle A^{q+2}x, x\rangle}{\langle A^q x, x\rangle} - \left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\Psi_p\left(\frac{\langle A^{q+1}x, x\rangle}{\langle A^q x, x\rangle}; k, K\right)(K - k)$$

for any $x \in H \setminus \{0\}$.

If we choose $w(t) = 1$ and $f(t) = t$ with $t \in [k, K] \subset [0, \infty)$ then we get from Corollary 15 that

$$0 \leq \ln\langle Ax, x\rangle - \langle\ln Ax, x\rangle \leq \langle A^{-1}x, x\rangle\langle Ax, x\rangle - 1 \tag{6.36}$$
$$\leq \frac{1}{2}\frac{K-k}{kK}\langle|A - \langle Ax, x\rangle 1_H| x, x\rangle$$
$$\leq \frac{1}{2}\frac{K-k}{kK}\left[\langle A^2x, x\rangle - \langle Ax, x\rangle^2\right]^{\frac{1}{2}} \leq \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \leq \ln\langle Ax, x\rangle - \langle\ln Ax, x\rangle \tag{6.37}$$
$$\leq \frac{(K - \langle Ax, x\rangle)(\langle Ax, x\rangle - k)}{K-k}\sup_{t\in(k,K)}\Psi_{-\ln}(t; k, K)$$
$$\leq \frac{1}{Kk}(K - \langle Ax, x\rangle)(\langle Ax, x\rangle - k) \leq \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \leq \ln\langle Ax, x\rangle - \langle\ln Ax, x\rangle \leq \frac{1}{4}(K-k)\Psi_{-\ln}(\langle Ax, x\rangle; k, K) \tag{6.38}$$
$$\leq \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \leq \ln\langle Ax, x\rangle - \langle\ln Ax, x\rangle \tag{6.39}$$
$$\leq 2\max\left\{\frac{K - \langle Ax, x\rangle}{K-k}, \frac{\langle Ax, x\rangle - k}{K-k}\right\}\ln\left(\frac{k+K}{2\sqrt{kK}}\right),$$

$$0 \leq \ln\langle Ax, x\rangle - \langle\ln Ax, x\rangle \leq \ln\left(\frac{k+K}{2\sqrt{kK}}\right)^2 \tag{6.40}$$

and

$$0 \le \ln \langle Ax, x\rangle - \langle \ln Ax, x\rangle \tag{6.41}$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\langle Ax, x\rangle; k, K\right)\langle \left| f(A) - \langle Ax, x\rangle 1_H\right| x, x\rangle$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\langle Ax, x\rangle; k, K\right)\left[\left\langle A^2 x, x\right\rangle - \langle Ax, x\rangle^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\Psi_{-\ln}\left(\langle Ax, x\rangle; k, K\right)(K-k)$$

for any $x \in H$ with $\|x\| = 1$.

If we choose $w(t) = t^q$, $q \neq 0$ and $f(t) = t$ with $t \in [k, K] \subset [0, \infty)$ then we get from Corollary 15 that

$$0 \le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \tag{6.42}$$
$$\le \frac{\left\langle A^{q-1}x, x\right\rangle}{\langle A^q x, x\rangle}\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - 1 \le \frac{1}{2}\frac{K-k}{kK}\frac{\left\langle \left| A - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}1_H\right| x, x\right\rangle}{\langle A^q x, x\rangle}$$
$$\le \frac{1}{2}\frac{K-k}{kK}\left[\frac{\left\langle A^{q+2}x, x\right\rangle}{\langle A^q x, x\rangle} - \left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)^2\right]^{\frac{1}{2}} \le \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \tag{6.43}$$
$$\le \frac{\left(K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k\right)}{K-k}\sup_{t\in(k,K)}\Psi_{-\ln}(t; k, K)$$
$$\le \frac{1}{Kk}\left(K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right)\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k\right) \le \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \tag{6.44}$$
$$\le \frac{1}{4}(K-k)\,\Psi_{-\ln}\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}; k, K\right) \le \frac{1}{4}\frac{(K-k)^2}{kK},$$

$$0 \le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \tag{6.45}$$
$$\le 2\max\left\{\frac{K - \frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}}{K-k}, \frac{\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle} - k}{K-k}\right\}\ln\left(\frac{k+K}{2\sqrt{kK}}\right),$$

$$0 \le \ln\left(\frac{\left\langle A^{q+1}x, x\right\rangle}{\langle A^q x, x\rangle}\right) - \frac{\langle A^q \ln Ax, x\rangle}{\langle A^q x, x\rangle} \le \ln\left(\frac{k+K}{2\sqrt{kK}}\right)^2, \tag{6.46}$$

and

$$(6.47)\qquad 0 \le \ln\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}\right) - \frac{\langle A^q \ln Ax,x\rangle}{\langle A^q x,x\rangle}$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle};k,K\right)\frac{\left\langle \left|A - \frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}1_H\right| x,x\right\rangle}{\langle A^q x,x\rangle}$$
$$\le \frac{1}{2}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle};k,K\right)\left[\frac{\langle A^{q+2}x,x\rangle}{\langle A^q x,x\rangle} - \left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle}\right)^2\right]^{\frac{1}{2}}$$
$$\le \frac{1}{4}\Psi_{-\ln}\left(\frac{\langle A^{q+1}x,x\rangle}{\langle A^q x,x\rangle};k,K\right)(K-k)$$

for any $x \in H \setminus \{0\}$.

References

[1] S. M. Ali and S. D. Silvey, A general class of coefficients of divergence of one distribution from another, *J. Roy. Statist. Soc. Sec B,* **28** (1966), 131-142.

[2] A. Bhattacharyya, On a measure of divergence between two statistical populations defined by their probability distributions, *Bull. Calcutta Math. Soc.,* **35** (1943), 99-109.

[3] M. Beth Bassat, f-entropies, probability of error and feature selection, *Inform. Control*, **39** (1978), 227-242.

[4] I. Burbea and C. R. Rao, On the convexity of some divergence measures based on entropy function, *IEEE Trans. Inf. Th.,* **28** (3) (1982), 489-495.

[5] C. H. Chen, *Statistical Pattern Recognition,* Rocelle Park, New York, Hoyderc Book Co., 1973.

[6] X. L. Cheng and J. Sun, A note on the perturbed trapezoid inequality, *J. Ineq. Pure. & Appl. Math.,* **3** (2) (2002), Article 29.

[7] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications. *Tamkang J. Math.* **38** (2007), no. 1, 37–49. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 14. [Online http://rgmia.org/papers/v5n2/RGIApp.pdf].

[8] C. K. Chow and C. N. Lin, Approximating discrete probability distributions with dependence trees, *IEEE Trans. Inf. Th.,* **14** (3) (1968), 462-467.

[9] I. Csiszár, Information-type measures of difference of probability distributions and indirect observations, *Studia Math. Hungarica,* **2** (1967), 299-318.

[10] I. Csiszár, On topological properties of f-divergences, *Studia Math. Hungarica,* **2** (1967), 329-339.

[11] I. Csiszár and J. Körner, *Information Theory: Coding Theorem for Discrete Memoryless Systems,* Academic Press, New York, 1981.

[12] S. S. Dragomir, A converse result for Jensen's discrete inequality via Grüss' inequality and applications in information theory. *An. Univ. Oradea Fasc. Mat.* **7** (1999/2000), 178–189.

[13] S. S. Dragomir, On a reverse of Jessen's inequality for isotonic linear functionals, *J. Ineq. Pure & Appl. Math.*, **2** (2001), No. 3, Article 36.

[14] S. S. Dragomir, A Grüss type inequality for isotonic linear functionals and applications. *Demonstratio Math.* **36** (2003), No. 3, 551–562. Preprint *RGMIA Res. Rep. Coll.* **5** (2002), Suplement, Art. 12. [Online http://rgmia.org/papers/v5e/GTIILFApp.pdf].

[15] S. S. Dragomir, A converse inequality for the Csiszár Φ-divergence. *Tamsui Oxf. J. Math. Sci.* **20** (2004), No. 1, 35–53. Preprint in S. S. Dragomir (Ed.): *Inequalities for Csiszár f-Divergence in Information Theory,* RGMIA Monographs, Victoria University, 2000, [Online http://rgmia.org/papers/Csiszar/Csiszar.pdf].

[16] S. S. Dragomir, Bounds for the normalized Jensen functional, *Bull. Austral. Math. Soc.* **74** (3) (2006), 471-476.

[17] S. S. Dragomir, Bounds for the deviation of a function from the chord generated by its extremities. *Bull. Aust. Math. Soc.* **78** (2008), No. 2, 225–248.

[18] S. S. Dragomir, Some reverses of the Jensen inequality with applications. *Bull. Aust. Math. Soc.* **87** (2013), no. 2, 177–194. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 72. [Online http://rgmia.org/papers/v14/v14a72.pdf].

[19] S. S. Dragomir, Reverses of the Jensen inequality in terms of first derivative and applications. *Acta Math. Vietnam.* **38** (2013), No. 3, 429–446. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 71. [Online http://rgmia.org/papers/v14/v14a71.pdf].

[20] S. S. Dragomir, A refinement and a divided difference reverse of Jensen's inequality with applications. *Rev. Colombiana Mat.* **50** (2016), No. 1, 17–39. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 74. [Online http://rgmia.org/papers/v14/v14a74.pdf].

[21] S. S. Dragomir, Superadditivity of the Jensen integral inequality with applications. *Miskolc Math. Notes* **13** (2012), No. 2, 303–316. Preprint *RGMIA Res. Rep. Coll.* **14** (2011), Art. 75. [Online http://rgmia.org/papers/v14/v14a75.pdf].

[22] S. S. Dragomir, *Operator Inequalities of the Jensen, Čebyšev and Grüss Type.* SpringerBriefs in Mathematics. Springer, New York, 2012. xii+121 pp. ISBN: 978-1-4614-1520-6.

[23] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* SpringerBriefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[24] S. S. Dragomir, Jensen type weighted inequalities for functions of selfadjoint and unitary operators, *Ital. J. Pure Appl. Math.* No. **32** (2014), 247–264.

[25] S. S. Dragomir, Weighted reverse inequalities of Jensen type for functions of selfadjoint operators. *Transylv. J. Math. Mech.* **8** (2016), No. 1, 29–44. Preprint *RGMIA Res. Rep. Coll.* **18** (2015), Art. 110. [Online http://rgmia.org/papers/v18/v18a110.pdf].

[26] S. S. Dragomir and N. M. Ionescu, Some converse of Jensen's inequality and applications. *Rev. Anal. Numér. Théor. Approx.* **23** (1994), no. 1, 71–78.

[27] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.

[28] D. V. Gokhale and S. Kullback, *Information in Contingency Tables*, New York, Marcel Decker, 1978.

[29] J. H. Havrda and F. Charvat, Quantification method classification process: concept of structural α-entropy, *Kybernetika*, **3** (1967), 30-35.

[30] E. Hellinger, Neue Bergrüirdung du Theorie quadratisher Formerus von uneudlichvieleu Veränderlicher, *J. für Reine and Augeur. Math.,* **36** (1909), 210-271.

[31] G. Helmberg, *Introduction to Spectral Theory in Hilbert Space,* John Wiley, New York, 1969.

[32] H. Jeffreys, An invariant form for the prior probability in estimating problems, *Proc. Roy. Soc. London*, **186** A (1946), 453-461.

[33] T. T. Kadota and L. A. Shepp, On the best finite set of linear observables for discriminating two Gaussian signals, *IEEE Trans. Inf. Th.,* **13** (1967), 288-294.

[34] T. Kailath, The divergence and Bhattacharyya distance measures in signal selection, *IEEE Trans. Comm. Technology.,* Vol COM-15 (1967), 52-60.

[35] J. N. Kapur, A comparative assessment of various measures of directed divergence, *Advances in Management Studies*, **3** (1984), 1-16.

[36] D. Kazakos and T. Cotsidas, A decision theory approach to the approximation of discrete probability densities, *IEEE Trans. Perform. Anal. Machine Intell.,* **1** (1980), 61-67.

[37] S. Kullback and R .A. Leibler, On information and sufficiency, *Annals Math. Statist.,* **22** (1951), 79-86.

[38] J. Lin, Divergence measures based on the Shannon entropy, *IEEE Trans. Inf. Th.,* **37** (1) (1991), 145-151.

[39] C. A. McCarthy, c_p, *Israel J. Math.*, **5**(1967), 249-271.

[40] M. Mei, The theory of genetic distance and evaluation of human races, *Japan J. Human Genetics*, **23** (1978), 341-369.

[41] B. Mond and J. Pečarić, Convex inequalities in Hilbert space, *Houston J. Math.*, **19**(1993), 405-420.

[42] C. P. Niculescu, An extension of Chebyshev's inequality and its connection with Jensen's inequality. *J. Inequal. Appl.* **6** (2001), No. 4, 451–462.

[43] E. C. Pielou, *Ecological Diversity,* Wiley, New York, 1975.

[44] C. R. Rao, Diversity and dissimilarity coefficients: a unified approach, *Theoretic Population Biology*, **21** (1982), 24-43.
[45] A. Rényi, On measures of entropy and information, *Proc. Fourth Berkeley Symp. Math. Stat. and Prob., University of California Press*, **1** (1961), 547-561.
[46] A.W. Roberts and D. E. Varberg, *Convex Functions,* Academic Press, 1973.
[47] A. Sen, *On Economic Inequality,* Oxford University Press, London 1973.
[48] B. D. Sharma and D. P. Mittal, New non-additive measures of relative information, *Journ. Comb. Inf. Sys. Sci.,* **2** (4)(1977), 122-132.
[49] H. Shioya and T. Da-Te, A generalisation of Lin divergence and the derivative of a new information divergence, *Elec. and Comm. in Japan*, **78** (7) (1995), 37-40.
[50] S. Simić, On a global upper bound for Jensen's inequality, *J. Math. Anal. Appl.* **343** (2008), 414-419. ć
[51] I. J. Taneja, *Generalised Information Measures and their Applications* (http://www.mtm.ufsc.br/~taneja/bhtml/bhtml.html).
[52] F. Topsoe, Some inequalities for information divergence and related measures of discrimination, Preprint *RGMIA Res. Rep. Coll.,* **2** (1) (1999), 85-98.
[53] H. Theil, *Economics and Information Theory*, North-Holland, Amsterdam, 1967.
[54] H. Theil, *Statistical Decomposition Analysis*, North-Holland, Amsterdam, 1972.
[55] I. Vajda, *Theory of Statistical Inference and Information,* Dordrecht-Boston, Kluwer Academic Publishers, 1989.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa