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AN AVERAGING INTEGRAL TRANSFORM AND ITS PROPERTIES

S. S. DRAGOMIR[1,2]

Abstract. For a Lebesgue integrable function $f:[a,b]\to\mathbb{C}$ we define the *averaging transform* $S_f:[a,b]\to\mathbb{R}$ by

$$S_f(x):=\begin{cases}\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt \text{ if } x\in\left[a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right],\\ \\ f\left(\frac{a+b}{2}\right) \text{ if } x=\frac{a+b}{2}.\end{cases}$$

In this paper we investigate several properties of the transform S_f concerning differentiability, monotonicity and convexity. Midpoint and trapezoid type inequalities for the transform S_f and its integral mean $\frac{1}{b-a}\int_a^b S_f(x)\,dx$ are obtained. Applications for some integral means of interest are also provided.

1. Introduction

For a *Lebesgue integrable* function $f:[a,b]\to\mathbb{C}$ we define the *averaging transform* $S_f:[a,b]\to\mathbb{R}$ by

$$S_f(x):=\begin{cases}\frac{1}{a+b-2x}\int_x^{a+b-x}f(t)\,dt \text{ if } x\in\left[a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right],\\ \\ f\left(\frac{a+b}{2}\right) \text{ if } x=\frac{a+b}{2}.\end{cases}\tag{1.1}$$

If we denote $F(x):=\int_a^x f(t)\,dt$ then

$$S_f(x)=\frac{F(a+b-x)-F(x)}{a+b-2x}$$

for $x\in\left[a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right]$. Since a point a *continuity* for f in (a,b) is a point a *differentiability* for F, hence if we assume that f is continuous in $\frac{a+b}{2}$, then by *L'Hospital's rule* we have $\lim_{x\to\frac{a+b}{2}}S_f(x)=f\left(\frac{a+b}{2}\right)$, which shows that S_f is also continuous in $\frac{a+b}{2}$.

We observe also that $S_f(a+b-x)=S_f(x)$ for any $x\in[a,b]$, namely S_f is *symmetrical* in the interval $[a,b]$.

The following inequality is well known in literature as the *Hermite-Hadamard inequality* for the convex function $h:[a,b]\to\mathbb{R}$

$$h\left(\frac{\alpha+\beta}{2}\right)\le\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds\le\frac{h(\alpha)+h(\beta)}{2}\tag{HH}$$

for any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$. For a large collection of inequalities related to this result see [11], [9] and the references therein.

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So, if $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then by the (HH) inequality we have the following inequality for the transform S_f

$$f\left(\frac{a+b}{2}\right)\leq S_f(x)\leq\frac{f(a+b-x)+f(x)}{2} \tag{1.2}$$

for any $x\in[a,b]$.

If we take the integral mean $\frac{1}{b-a}\int_a^b$ in (1.2) we also have

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b S_f(x)\,dx\leq\frac{1}{b-a}\int_a^b f(x)\,dx \tag{1.3}$$

that provides a refinement of the second inequality in the (HH) inequality.

If we consider *Bullen's inequality* [11, p. 2] for the convex function $h:[a,b]\to\mathbb{R}$ that shows that the middle term in (HH) is closer to the left term than to the right term, namely

$$0\leq\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds-h\left(\frac{\alpha+\beta}{2}\right)\leq\frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds \tag{B}$$

for any $\alpha,\ \beta\in[a,b]$ with $\alpha\neq\beta$, then we obtain that

$$0\leq S_f(x)-f\left(\frac{a+b}{2}\right)\leq\frac{f(a+b-x)+f(x)}{2}-S_f(x) \tag{1.4}$$

for any $x\in[a,b]$. Moreover, if we take the integral mean in (1.4), then we get

$$\begin{aligned}0&\leq\frac{1}{b-a}\int_a^b S_f(x)\,dx-f\left(\frac{a+b}{2}\right)\\&\leq\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{b-a}\int_a^b S_f(x)\,dx.\end{aligned} \tag{1.5}$$

Some natural examples for S_f are provided by integral means. Let us recall the following means :

(1) The *logarithmic mean*:

$$L=L(a,b):=\begin{cases}a & \text{if } a=b\\ \dfrac{b-a}{\ln b-\ln a} & \text{if } a\neq b\end{cases}\quad a,\ b>0;$$

(2) The *identric mean*:

$$I:=I(a,b)=\begin{cases}a & \text{if } a=b\\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a\neq b\end{cases}\quad a,\ b>0;$$

(3) The *p-logarithmic mean*:

$$L_p=L_p(a,b):=\begin{cases}\left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a\neq b;\\ a & \text{if } a=b\end{cases}$$

where $p\in\mathbf{R}\setminus\{-1,0\}$ and $a,\ b>0$.

It is well known that L_p is *monotonic nondecreasing* over $p \in \mathbf{R}$ with $L_{-1} := L$ and $L_0 := I$.

In particular, we have the inequalities

$$H \leq G \leq L \leq I \leq A, \tag{1.6}$$

where the *arithmetic mean* is

$$A = A(a,b) := \frac{a+b}{2}, \qquad a,\ b \geq 0;$$

the *geometric mean* is

$$G = G(a,b) := \sqrt{ab}, \qquad a,\ b \geq 0;$$

and the *harmonic mean* is

$$H = H(a,b) := \frac{2}{\dfrac{1}{a} + \dfrac{1}{b}}, \qquad a,\ b > 0.$$

If we consider the function $f_{-1} : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f_{-1}(t) = \frac{1}{t}$, then

$$S_{f_{-1}}(x) := \begin{cases} \frac{1}{a+b-2x} \int_x^{a+b-x} \frac{1}{t} dt \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ \left(\frac{a+b}{2}\right)^{-1} \text{ if } x = \frac{a+b}{2}, \end{cases}$$
$$= \begin{cases} L^{-1}(a+b-x, x) \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ A^{-1}(a,b) \text{ if } x = \frac{a+b}{2}. \end{cases}$$

For the function $f_0 : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f_{-0}(t) = \ln t$, we have

$$S_{f_0}(x) := \begin{cases} \frac{1}{a+b-2x} \int_x^{a+b-x} \ln t dt \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ \ln\left(\frac{a+b}{2}\right) \text{ if } x = \frac{a+b}{2}, \end{cases}$$
$$= \begin{cases} \ln I(a+b-x, x) \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ \ln(A(a,b)) \text{ if } x = \frac{a+b}{2}. \end{cases}$$

Also, for the function $f_p : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f_{-0}(t) = t^p$, where $p \neq 0, -1$ we have

$$S_{f_p}(x) := \begin{cases} \frac{1}{a+b-2x} \int_x^{a+b-x} t^p dt \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ \left(\frac{a+b}{2}\right)^p \text{ if } x = \frac{a+b}{2} \end{cases}$$
$$= \begin{cases} L_p^p(a+b-x, x) \text{ if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \\ A^p(a,b) \text{ if } x = \frac{a+b}{2}. \end{cases}$$

Motivated by the above facts, we investigate in the following several properties of the transform S_f concerning differentiability, monotonicity and convexity. Midpoint and trapezoid type inequalities for the transform S_f and its integral mean $\frac{1}{b-a} \int_a^b S_f(x)\, dx$ are obtained. Applications for some integral means of interest are also provided.

2. General Results

The following result concerning the differentiability of the transform S_f holds.

Theorem 1. *Assume that $f:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$, then S_f is differentiable on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ and*

$$S_f'(x)=\frac{2}{a+b-2x}\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right] \tag{2.1}$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$.

If f has finite lateral derivatives $f_+'\left(\frac{a+b}{2}\right)$ and $f_-'\left(\frac{a+b}{2}\right)$, then S_f has lateral derivatives in $\frac{a+b}{2}$ and

$$S_{f+}'\left(\frac{a+b}{2}\right)=\frac{1}{2}\left[f_+'\left(\frac{a+b}{2}\right)-f_-'\left(\frac{a+b}{2}\right)\right]=-S_{f-}'\left(\frac{a+b}{2}\right). \tag{2.2}$$

Moreover, if f is differentiable in $\frac{a+b}{2}$ then S_f is differentiable in $\frac{a+b}{2}$ and $S_f'\left(\frac{a+b}{2}\right)=0$.

Proof. By the continuity of f on $[a,b]$ we have that S_f is differentiable on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ and

$$\begin{aligned}
S_f'(x)&=\frac{F(a+b-x)-F(x)}{a+b-2x}\\
&=\frac{(F'(a+b-x)-F'(x))(a+b-2x)+2(F(a+b-x)-F(x))}{(a+b-2x)^2}\\
&=\frac{(-f(a+b-x)-f(x))(a+b-2x)+2(F(a+b-x)-F(x))}{(a+b-2x)^2}\\
&=\frac{2(F(a+b-x)-F(x))}{(a+b-2x)^2}-\frac{f(a+b-x)+f(x)}{a+b-2x}\\
&=\frac{2}{a+b-2x}\left[\frac{F(a+b-x)-F(x)}{a+b-2x}-\frac{f(a+b-x)+f(x)}{2}\right]
\end{aligned}$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$, which proves the equality (2.1).

From (2.1) for $x>\frac{a+b}{2}$ we have

$$\begin{aligned}
&S_f'(x)\\
&=2\left[\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}-\frac{f(a+b-x)-f\left(\frac{a+b}{2}\right)+f(x)-f\left(\frac{a+b}{2}\right)}{2(a+b-2x)}\right].
\end{aligned} \tag{2.3}$$

By the continuity of S_f' on $\left(\frac{a+b}{2},b\right)$ and Lagrange's mean value theorem we have

$$\lim_{x\to\frac{a+b}{2}+}\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}+}\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}+}S_f'(x).$$

Also

$$\lim_{x\to\frac{a+b}{2}+}\frac{f(a+b-x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=\frac{1}{2}\lim_{x\to\frac{a+b}{2}+}\frac{f\left(\frac{a+b}{2}+\frac{a+b}{2}-x\right)-f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}-x}$$
$$=\frac{1}{2}\lim_{h\to 0+}\frac{f\left(\frac{a+b}{2}-h\right)-f\left(\frac{a+b}{2}\right)}{-h}$$
$$=\frac{1}{2}\lim_{s\to 0-}\frac{f\left(\frac{a+b}{2}+s\right)-f\left(\frac{a+b}{2}\right)}{s}=\frac{1}{2}f'_-\left(\frac{a+b}{2}\right)$$

and

$$\lim_{x\to\frac{a+b}{2}+}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=\frac{1}{2}\lim_{x\to\frac{a+b}{2}+}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}-x}$$
$$=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}+}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}=-\frac{1}{2}f'_+\left(\frac{a+b}{2}\right).$$

Denote $\ell_r:=\lim_{x\to\frac{a+b}{2}+}S'_f(x)$. Then by taking the limit over $x\to\frac{a+b}{2}+$ in the equality (2.3) we get

$$\ell_r=2\left[-\frac{1}{2}\ell_r-\frac{1}{2}f'_-\left(\frac{a+b}{2}\right)+\frac{1}{2}f'_+\left(\frac{a+b}{2}\right)\right]$$

namely

$$2\ell_r=f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right),$$

which implies that

$$\ell_r=\frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right].$$

From (2.1) for $x<\frac{a+b}{2}$ we have

$$S'_f(x)=2\left[\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}-\frac{f(a+b-x)-f\left(\frac{a+b}{2}\right)+f(x)-f\left(\frac{a+b}{2}\right)}{2(a+b-2x)}\right]. \tag{2.4}$$

By the continuity of S'_f on $\left(a,\frac{a+b}{2}\right)$ and Lagrange's mean value theorem we have

$$\lim_{x\to\frac{a+b}{2}-}\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}-}\frac{S_f(x)-f\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}-}S'_f(x).$$

Also

$$\lim_{x\to\frac{a+b}{2}-}\frac{f(a+b-x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=\frac{1}{2}\lim_{x\to\frac{a+b}{2}-}\frac{f\left(\frac{a+b}{2}+\frac{a+b}{2}-x\right)-f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}-x}$$
$$=\frac{1}{2}\lim_{h\to 0+}\frac{f\left(\frac{a+b}{2}+h\right)-f\left(\frac{a+b}{2}\right)}{h}=\frac{1}{2}f'_+\left(\frac{a+b}{2}\right)$$

and

$$\lim_{x\to\frac{a+b}{2}-}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{a+b-2x}=\frac{1}{2}\lim_{x\to\frac{a+b}{2}-}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{\frac{a+b}{2}-x}$$
$$=-\frac{1}{2}\lim_{x\to\frac{a+b}{2}-}\frac{f(x)-f\left(\frac{a+b}{2}\right)}{x-\frac{a+b}{2}}=-\frac{1}{2}f'_-\left(\frac{a+b}{2}\right).$$

Denote $\ell_s:=\lim_{x\to\frac{a+b}{2}-}S'_f(x)$. Then by taking the limit over $x\to\frac{a+b}{2}-$ in the equality (2.4) we get

$$\ell_s=2\left[-\frac{1}{2}\ell_s-\frac{1}{2}f'_+\left(\frac{a+b}{2}\right)+\frac{1}{2}f'_-\left(\frac{a+b}{2}\right)\right]$$

namely

$$2\ell_r=f'_-\left(\frac{a+b}{2}\right)-f'_+\left(\frac{a+b}{2}\right),$$

which implies

$$\ell_r=\frac{1}{2}\left[f'_-\left(\frac{a+b}{2}\right)-f'_+\left(\frac{a+b}{2}\right)\right].$$

□

The following corollary is of interest:

Corollary 1. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous and convex on $[a,b]$, then S_f is non-increasing on $\left[a,\frac{a+b}{2}\right)$ and non-decreasing on $\left(\frac{a+b}{2},b\right]$. S_f is differentiable on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ and has the lateral derivatives*

$$S'_{f+}\left(\frac{a+b}{2}\right)=\frac{1}{2}\left[f'_+\left(\frac{a+b}{2}\right)-f'_-\left(\frac{a+b}{2}\right)\right]=-S'_{f-}\left(\frac{a+b}{2}\right). \tag{2.5}$$

Moreover, if f is differentiable in $\frac{a+b}{2}$ then S_f is differentiable in $\frac{a+b}{2}$ and $S'_f\left(\frac{a+b}{2}\right)=0$.

Proof. We know from Hermite-Hadamard inequality that, if $f:[a,b]\to\mathbb{R}$ is convex on $[a,b]$, then

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f(s)\,ds\le\frac{f(\alpha)+f(\beta)}{2}$$

for any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$.

Therefore

$$S_f(x)-\frac{f(a+b-x)+f(x)}{2}\le 0$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$.

Using the equality (2.1) we then get $S'_f(x)\le 0$ for any $x\in\left(a,\frac{a+b}{2}\right)$ and $S'_f(x)\ge 0$ for any $x\in\left(\frac{a+b}{2},b\right)$. □

We have:

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{C}$ *is continuous on* $[a,b]$ *and differentiable on* (a,b), *then* S_f *is twice differentiable on* $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ *and*

$$S_f''(x)=\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{8}{(a+b-2x)^2}+\frac{f'(a+b-x)-f'(x)}{a+b-2x} \tag{2.6}$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$.

Proof. Since f is differentiable on (a,b), then by (2.1) S_f is twice differentiable on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ and

$$\begin{aligned}
S_f''(x)&=\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]'\frac{2}{a+b-2x}\\
&\quad+\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\left(\frac{2}{a+b-2x}\right)'\\
&=\left[S_f'(x)-\frac{f'(x)-f'(a+b-x)}{2}\right]\frac{2}{a+b-2x}\\
&\quad+\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{4}{(a+b-2x)^2}\\
&=\frac{2}{a+b-2x}S_f'(x)-\frac{f'(x)-f'(a+b-x)}{a+b-2x}\\
&\quad+\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{4}{(a+b-2x)^2}\\
&=\frac{2}{a+b-2x}\frac{2}{a+b-2x}\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\\
&\quad-\frac{f'(x)-f'(a+b-x)}{a+b-2x}\\
&\quad+\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{4}{(a+b-2x)^2}\\
&=\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{8}{(a+b-2x)^2}\\
&\quad+\frac{f'(a+b-x)-f'(x)}{a+b-2x}
\end{aligned}$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$. □

Corollary 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is continuous convex on* $[a,b]$ *and differentiable on* (a,b), *then* S_f *is convex on* $[a,b]$.

Proof. We know that, see [8], if $f:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$ and differentiable on (a,b) then for any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$ we have

$$0\le\frac{f(\alpha)+f(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(s)\,ds\le\frac{1}{8}\left[f'(\beta)-f'(\alpha)\right](\beta-\alpha). \tag{2.7}$$

Therefore, for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ we have

$$\begin{aligned}0&\leq\frac{f(x)+f(a+b-x)}{2}-\frac{1}{a+b-2x}\int_{x}^{a+b-x}f(s)\,ds\\&\leq\frac{1}{8}\left[f'(a+b-x)-f'(x)\right](a+b-2x).\end{aligned}$$

Since

$$\begin{aligned}&S_f''(x)\\&=\left[S_f(x)-\frac{f(a+b-x)+f(x)}{2}\right]\frac{8}{(a+b-2x)^2}+\frac{f'(a+b-x)-f'(x)}{a+b-2x}\\&=\frac{8}{(a+b-2x)^2}\\&\times\left[\frac{1}{8}\left[f'(a+b-x)-f'(x)\right](a+b-2x)-\left[\frac{f(a+b-x)+f(x)}{2}-S_f(x)\right]\right]\\&\geq 0\end{aligned}$$

for any $x\in\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$, we conclude that S_f is convex on both intervals $\left[a,\frac{a+b}{2}\right)$ and $\left(\frac{a+b}{2},b\right]$. Since S_f is continuous on $[a,b]$ and $S_f'\left(\frac{a+b}{2}\right)=0$, it follows that S_f is convex on the whole interval $[a,b]$. □

3. Midpoint and Trapezoid Type Inequalities

We use the following convention for the total variation of a function in the case when $b<a$,

$$\bigvee_a^b(f):=-\bigvee_b^a(f) \tag{3.1}$$

provided the function f is of bounded variation in the classical sense.

We also consider the *Cumulative Variation Function* (CVF) $V_f:[a,b]\to[0,\infty)$ defined by

$$V_f(x)=\bigvee_a^x(f),\ x\in[a,b].$$

It is know that the CVF is monotonic nondecreasing on $[a,b]$ and is continuous in a point $c\in[a,b]$ if and only if the generating function f is continuing in that point. If f is Lipschitzian with the constant $L>0$, i.e.

$$|f(t)-f(s)|\leq L|t-s| \text{ for any } t,\ s\in[a,b],$$

then V_f is also Lipschitzian with the same constant.

We have the following result for functions of bounded variation.

Theorem 3. *Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$. Then*

$$\left|S_f(x)-f\left(\frac{a+b}{2}\right)\right|\leq\frac{1}{2}\left|\bigvee_x^{a+b-x}(f)\right|\leq\frac{1}{2}\bigvee_a^b(f) \tag{3.2}$$

and

$$\left|\frac{f(a+b-x)+f(x)}{2}-S_f(x)\right|\leq\frac{1}{2}\left|\bigvee_x^{a+b-x}(f)\right|\leq\frac{1}{2}\bigvee_a^b(f) \tag{3.3}$$

for any $x \in [a,b]$.

Proof. We use the following midpoint and trapezoid inequalities for functions of bounded variation $h : [\alpha, \beta] \to \mathbb{C}$ [2] (see also [6])

$$\left| \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(s)\,ds - h\left(\frac{\alpha+\beta}{2}\right) \right| \le \frac{1}{2} \bigvee_{\alpha}^{\beta} (h) \tag{3.4}$$

and

$$\left| \frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta - \alpha} \int_{\alpha}^{\beta} h(s)\,ds \right| \le \frac{1}{2} \bigvee_{\alpha}^{\beta} (h), \tag{3.5}$$

where $\bigvee_{\alpha}^{\beta}(h)$ is the total variation of h on the interval $[\alpha, \beta]$. The constant $\frac{1}{2}$ is best possible in (3.4) and (3.5).

Let $x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right]$, then by (3.4), (3.5) and using the convention (3.1) we have

$$\left| S_f(x) - f\left(\frac{a+b}{2}\right) \right| \le \frac{1}{2} \left| \bigvee_{x}^{a+b-x} (f) \right| \tag{3.6}$$

and

$$\left| \frac{f(a+b-x)+f(x)}{2} - S_f(x) \right| \le \frac{1}{2} \left| \bigvee_{x}^{a+b-x} (f) \right|. \tag{3.7}$$

These inequalities become identities if we take $x = \frac{a+b}{2}$.

We observe that

$$\bigvee_{x}^{a+b-x} (f) = V_f(a+b-x) - V_f(x)$$

for $x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right]$ and since V_f is nondecreasing on $[a,b]$ we have

$$V_f(a) - V_f(b) \le V_f(a+b-x) - V_f(x) \le V_f(b) - V_f(a),$$

namely $\left|\bigvee_{x}^{a+b-x}(f)\right| \le \bigvee_{a}^{b}(f)$ and the theorem is proved. □

Remark 1. *Let* $f : [a,b] \to \mathbb{C}$ *be a function of bounded variation on* $[a,b]$. *If we take the integral mean in the inequalities (3.2) and (3.3) and use the property of modulus, we get*

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b S_f(x)\,dx - f\left(\frac{a+b}{2}\right) \right| &\le \frac{1}{2(b-a)} \int_a^b \left| \bigvee_{x}^{a+b-x} (f) \right| dx \\ &\le \frac{1}{2} \bigvee_{a}^{b} (f) \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} \left| \frac{1}{b-a} \int_a^b f(x)\,dx - \frac{1}{b-a} \int_a^b S_f(x)\,dx \right| &\le \frac{1}{2(b-a)} \int_a^b \left| \bigvee_{x}^{a+b-x} (f) \right| dx \\ &\le \frac{1}{2} \bigvee_{a}^{b} (f). \end{aligned} \tag{3.9}$$

Corollary 3. *Let $f : [a, b] \to \mathbb{C}$ be a function of bounded variation on $[a, b]$ and such that the CVF is Lipschitzian with the constant $K > 0$, then*

$$\left|S_f(x) - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{2}\left|\bigvee_x^{a+b-x}(f)\right| \le K\left|x - \frac{a+b}{2}\right| \tag{3.10}$$

and

$$\left|\frac{f(a+b-x)+f(x)}{2} - S_f(x)\right| \le \frac{1}{2}\left|\bigvee_x^{a+b-x}(f)\right| \le K\left|x - \frac{a+b}{2}\right| \tag{3.11}$$

for any $x \in [a, b]$.

Remark 2. *Let $f : [a, b] \to \mathbb{C}$ be as in Corollary 3. If we take the integral mean in the inequalities (3.10) and (3.11) and use the property of modulus, then we get*

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b S_f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| &\le \frac{1}{2(b-a)}\int_a^b\left|\bigvee_x^{a+b-x}(f)\right|dx \\ &\le \frac{1}{4}K(b-a)\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{b-a}\int_a^b S_f(x)\,dx\right| &\le \frac{1}{2(b-a)}\int_a^b\left|\bigvee_x^{a+b-x}(f)\right|dx \\ &\le \frac{1}{4}K(b-a).\end{aligned} \tag{3.13}$$

For *absolutely continuous* functions $h : [\alpha, \beta] \to \mathbb{C}$ we have the following midpoint inequality [12]

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds - h\left(\frac{\alpha+\beta}{2}\right)\right| \le \frac{1}{2}\int_\alpha^\beta |h'(s)|\,ds \tag{3.14}$$

and the trapezoid inequality [4]

$$\left|\frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds\right| \le \frac{1}{2}\int_\alpha^\beta |h'(s)|\,ds. \tag{3.15}$$

The constant $\frac{1}{2}$ is best possible in (3.14) and (3.15).

Then, by a similar argument to the one in Theorem 3 we have from (3.14) and (3.15) the inequalities

$$\left|S_f(x) - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{2}\left|\int_x^{a+b-x}|f'(s)|\,ds\right| \le \frac{1}{2}\int_a^b |f'(s)|\,ds \tag{3.16}$$

and

$$\left|\frac{f(a+b-x)+f(x)}{2} - S_f(x)\right| \le \frac{1}{2}\left|\int_x^{a+b-x}|f'(s)|\,ds\right| \le \frac{1}{2}\int_a^b |f'(s)|\,ds \tag{3.17}$$

for any $x \in [a, b]$.

These imply by integration

$$\left|\frac{1}{b-a}\int_a^b S_f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2(b-a)}\int_a^b \left|\int_x^{a+b-x} |f'(s)|\,ds\right| dx \leq \frac{1}{2}\int_a^b |f'(s)|\,ds \tag{3.18}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{b-a}\int_a^b S_f(x)\,dx\right| \leq \frac{1}{2(b-a)}\int_a^b \left|\int_x^{a+b-x} |f'(s)|\,ds\right| dx \leq \frac{1}{2}\int_a^b |f'(s)|\,ds. \tag{3.19}$$

If the function $h:[\alpha,\beta]\to\mathbb{C}$ is *Lipschitzian* with the constant $L>0$, then we have the midpoint inequality [10] and [3]

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds - h\left(\frac{\alpha+\beta}{2}\right)\right| \leq \frac{1}{4}L(\beta-\alpha) \tag{3.20}$$

and the trapezoid inequality [1] and [13]

$$\left|\frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds\right| \leq \frac{1}{4}L(\beta-\alpha). \tag{3.21}$$

The constant $\frac{1}{4}$ is best possible in (3.20) and (3.21).

Theorem 4. *Let $f:[a,b]\to\mathbb{C}$ be a Lipschitzian function with the constant $L>0$ on $[a,b]$. Then*

$$\left|S_f(x) - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{2}L\left|x-\frac{a+b}{2}\right| \tag{3.22}$$

and

$$\left|\frac{f(a+b-x)+f(x)}{2} - S_f(x)\right| \leq \frac{1}{2}L\left|x-\frac{a+b}{2}\right| \tag{3.23}$$

for any $x\in[a,b]$.

Moreover, we have the integral inequalities

$$\left|\frac{1}{b-a}\int_a^b S_f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{1}{8}L(b-a) \tag{3.24}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{b-a}\int_a^b S_f(x)\,dx\right| \leq \frac{1}{8}L(b-a). \tag{3.25}$$

Proof. If $x \in \left[a, \frac{a+b}{2}\right)$, then by (3.20) for $\alpha = x$ and $\beta = a+b-x$ we have

$$\left|\frac{1}{a+b-2x}\int_x^{a+b-x} f(t)\,dt - f\left(\frac{a+b}{2}\right)\right| \le \frac{1}{4}L(a+b-x-x) = \frac{1}{2}L\left(\frac{a+b}{2}-x\right) = \frac{1}{2}L\left|x-\frac{a+b}{2}\right| \tag{3.26}$$

and by (3.21) for $\alpha = x$ and $\beta = a+b-x$ we have

$$\left|\frac{h(x)+h(a+b-x)}{2} - \frac{1}{a+b-2x}\int_x^{a+b-x} f(t)\,dt\right| \le \frac{1}{2}L\left|x-\frac{a+b}{2}\right|. \tag{3.27}$$

We observe that, if $x \in \left(\frac{a+b}{2}, b\right]$ then by taking $\alpha = a+b-x$ and $\beta = x$ in (3.20) and (3.21) we obtain the same inequalities (3.26) and (3.27).

Finally, $x = \frac{a+b}{2}$ produces equality in (3.22) and (3.23).

The inequalities (3.24) and (3.25) follow by integrating (3.22) and (3.23) and taking into account that

$$\frac{1}{b-a}\int_a^b \left|x-\frac{a+b}{2}\right| dx = \frac{1}{4}(b-a).$$

□

We observe that if the function $f : [a,b] \to \mathbb{C}$ is absolutely continuous and $f' \in L_\infty[a,b]$, namely f' is essentially bounded on $[a,b]$ then f is Lipschitzian with the constant

$$L = \|f'\|_{\infty,[a,b]} := \operatorname*{essup}_{t\in[a,b]} |f'(t)| < \infty$$

and the inequalities in (3.22)-(3.25) may be stated with $\|f'\|_{\infty,[a,b]}$ instead of L.

If the function $h : [\alpha,\beta] \to \mathbb{C}$ is absolutely continuous and $h' \in L_p[\alpha,\beta]$, $p > 1$, namely

$$\|h'\|_{p,[\alpha,\beta]} := \left(\int_\alpha^\beta |h'(s)|^p\,ds\right)^{1/p} < \infty,$$

then we have the following midpoint and trapezoid inequalities [5]

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds - h\left(\frac{\alpha+\beta}{2}\right)\right| \le \frac{1}{2(q+1)^{1/q}}(\beta-\alpha)^{1/q}\|h'\|_{p,[\alpha,\beta]} \tag{3.28}$$

and

$$\left|\frac{h(\alpha)+h(\beta)}{2} - \frac{1}{\beta-\alpha}\int_\alpha^\beta h(s)\,ds\right| \le \frac{1}{2(q+1)^{1/q}}(\beta-\alpha)^{1/q}\|h'\|_{p,[\alpha,\beta]}, \tag{3.29}$$

where $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$.

Using these inequalities we have:

Theorem 5. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Assume that $f'\in L_p[\alpha,\beta]$, $p>1$ and let $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|S_f(x)-f\left(\frac{a+b}{2}\right)\right|\le\frac{1}{2^{1/p}(q+1)^{1/q}}\left|x-\frac{a+b}{2}\right|^{1/q}\|f'\|_{p,[a,b]} \tag{3.30}$$

and

$$\left|\frac{f(a+b-x)+f(x)}{2}-S_f(x)\right|\le\frac{1}{2^{1/p}(q+1)^{1/q}}\left|x-\frac{a+b}{2}\right|^{1/q}\|f'\|_{p,[a,b]} \tag{3.31}$$

for any $x\in[a,b]$.

Moreover, we have the integral inequalities

$$\left|\frac{1}{b-a}\int_a^b S_f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\le\frac{q}{2(q+1)^{1/q+1}}(b-a)^{1/q}\|f'\|_{p,[a,b]} \tag{3.32}$$

and

$$\begin{aligned}\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{b-a}\int_a^b S_f(x)\,dx\right|&\\ \le\frac{q}{2(q+1)^{1/q+1}}(b-a)^{1/q}\|f'\|_{p,[a,b]}&.\end{aligned} \tag{3.33}$$

Proof. The proof of (3.30) and (3.31) follow by (3.28) and (3.29) by employing a similar argument to the one from the proof of Theorem 4.

Taking the integral in (3.30) and (3.31) and observing that

$$\begin{aligned}\frac{1}{b-a}\int_a^b\left|x-\frac{a+b}{2}\right|^{1/q}dx&=\frac{2}{b-a}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^{1/q}dx\\&=\frac{2}{b-a}\frac{\left(\frac{b-a}{2}\right)^{1/q+1}}{1/q+1}=\frac{q(b-a)^{1/q}}{2^{1/q}(q+1)},\end{aligned}$$

we deduce the desired results (3.32) and (3.33). □

4. Further Results for Convex Functions

In [7] we established the following reverse of the first inequality in the Hermite-Hadamard result. This can be stated as

$$0\le\frac{1}{\beta-\alpha}\int_\alpha^\beta f(s)\,ds-f\left(\frac{\alpha+\beta}{2}\right)\le\frac{1}{8}\left[f'(\beta)-f'(\alpha)\right](\beta-\alpha), \tag{4.1}$$

for any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$, provided that $f:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$ and differentiable on (a,b).

With the same assumptions, we have the following reverse of the second inequality in the Hermite-Hadamard result as well [8]

$$0\le\frac{f(\alpha)+f(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta f(s)\,ds\le\frac{1}{8}\left[f'(\beta)-f'(\alpha)\right](\beta-\alpha), \tag{4.2}$$

for any $\alpha,\beta\in[a,b]$ with $\alpha\neq\beta$

Theorem 6. *Assume that $f:[a,b]\to\mathbb{R}$ is continuous convex on $[a,b]$ and differentiable on (a,b). Then*

$$0\le S_f(x)-f\left(\frac{a+b}{2}\right)\le\frac{1}{4}\left[f'(x)-f'(a+b-x)\right]\left(x-\frac{a+b}{2}\right) \tag{4.3}$$

and

$$0\le\frac{f(a+b-x)+f(x)}{2}-S_f(x)\le\frac{1}{4}\left[f'(x)-f'(a+b-x)\right]\left(x-\frac{a+b}{2}\right) \tag{4.4}$$

for any $x\in[a,b]$.

Moreover, we have the integral inequalities

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b S_f(x)\,dx-f\left(\frac{a+b}{2}\right)\\&\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\le\frac{1}{16}\left[f'_-(b)-f'_+(a)\right](b-a)\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}0&\le\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{b-a}\int_a^b S_f(x)\,dx\\&\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right]\le\frac{1}{16}\left[f'_-(b)-f'_+(a)\right](b-a).\end{aligned} \tag{4.6}$$

Proof. The inequalities (4.3) and (4.4) follow by (4.1) and (4.2) for $\alpha=x$ and $\beta=a+b-x$.

Now, observe that

$$\begin{aligned}&\int_a^b\left(x-\frac{a+b}{2}\right)d\left[f(x)+f(a+b-x)\right]\\&=\left[f(x)+f(a+b-x)\right]\left(x-\frac{a+b}{2}\right)\Big|_a^b-\int_a^b\left[f(x)+f(a+b-x)\right]dx\\&=\left[f(b)+f(a)\right]\left(b-\frac{a+b}{2}\right)-\left[f(a)+f(b)\right]\left(a-\frac{a+b}{2}\right)\\&-\int_a^b\left[f(x)+f(a+b-x)\right]dx\\&=\left[f(b)+f(a)\right](b-a)-2\int_a^b f(x)\,dx,\end{aligned}$$

then by integrating (4.3) and (4.4) we deduce the second inequalities in (4.5) and (4.6). The last part is obvious by (4.1) and (4.2). □

5. Some Examples

If we consider the function $f_{-1}:[a,b]\subset(0,\infty)\to\mathbb{R}$, $f_{-1}(t)=\frac{1}{t}$, then

$$S_{f_{-1}}(x):=\begin{cases}L^{-1}(a+b-x,x)\ \text{ if } x\in\left[a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right],\\[2ex] A^{-1}(a,b)\ \text{ if } x=\frac{a+b}{2},\end{cases}$$

and by (4.3) and (4.4) we get

$$0 \le L^{-1}(a+b-x,x) - A^{-1}(a,b) \le \frac{1}{(a+b-x)^2 x^2}\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^2 \tag{5.1}$$

and

$$\begin{aligned} 0 &\le \frac{a+b}{2x(a+b-x)} - L^{-1}(a+b-x,x) \\ &\le \frac{1}{(a+b-x)^2 x^2}\left(\frac{a+b}{2}\right)\left(x-\frac{a+b}{2}\right)^2 \end{aligned} \tag{5.2}$$

for any $x \in [a,b]$.

From (4.5) and (4.6) we also have

$$\begin{aligned} 0 &\le \frac{1}{b-a}\int_a^b L^{-1}(a+b-x,x)\,dx - A^{-1}(a,b) \\ &\le \frac{1}{2}\left[H^{-1}(a,b) - L^{-1}(a,b)\right] \le \frac{1}{16}\frac{b+a}{a^2b^2}(b-a)^2 \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} 0 &\le L^{-1}(a,b) - \frac{1}{b-a}\int_a^b L^{-1}(a+b-x,x)\,dx \\ &\le \frac{1}{2}\left[H^{-1}(a,b) - L^{-1}(a,b)\right] \le \frac{1}{16}\frac{b+a}{a^2b^2}(b-a)^2. \end{aligned} \tag{5.4}$$

For the function $f_0 : [a,b] \subset (0,\infty) \to \mathbb{R}$, $f_{-0}(t) = \ln t$, we have

$$S_{f_0}(x) = \begin{cases} \ln I(a+b-x,x) & \text{if } x \in \left[a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right], \\ \ln (A(a,b)) & \text{if } x = \frac{a+b}{2}, \end{cases}$$

and by (4.3) and (4.4) for the concave function $f_{-0}(t) = \ln t$ we get

$$0 \le \ln A(a,b) - \ln I(a+b-x,x) \le \frac{1}{2x(a+b-x)}\left(x-\frac{a+b}{2}\right)^2 \tag{5.5}$$

and

$$\begin{aligned} 0 &\le \ln I(a+b-x,x) - \ln G(a+b-x,x) \\ &\le \frac{1}{2x(a+b-x)}\left(x-\frac{a+b}{2}\right)^2 \end{aligned} \tag{5.6}$$

for any $x \in [a,b]$.

From (4.5) and (4.6) we also have

$$\begin{aligned} 0 &\le \ln A(a,b) - \frac{1}{b-a}\int_a^b \ln I(a+b-x,x)\,dx \\ &\le \frac{1}{2}\left[\ln I(a,b) - \ln G(a,b)\right] \le \frac{1}{16ab}(b-a)^2 \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} 0 &\le \ln I(a,b) - \frac{1}{b-a}\int_a^b \ln I(a+b-x,x)\,dx \\ &\le \frac{1}{2}\left[\ln I(a,b) - \ln G(a,b)\right] \le \frac{1}{16ab}(b-a)^2. \end{aligned} \tag{5.8}$$

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[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa