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Mulitivalued Nonexpansive Mappings Characterized By A Fixed Point Property

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Abstract

We establish a characterization for the class of multivalued nonexpansive mappings with nonempty fixed point-sets as an extension of the well known Nadler's contraction mapping principle to multivalued nonexpansive mappings in Banach spaces. Our results also establish a fixed point property of Banach spaces for a wide class of contractive operators in Banach spaces.

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1 Introduction

The popular Nadler's [6] fixed point result for multivalued contractions states that if a multivalued operator T from a complete metric space (X, d) into the collection $\mathcal{CB}(X)$ of its closed bounded subsets satisfies the condition

$$H(Tx, Ty) \leq Ld(x, y);\ L \in (0, 1), \tag{1}$$

then the fixed point-set $Fix(T)$ of T is not empty. Here, we recall the Hausdorff metric $H(A, B)$, (induced by the metric d), on a the collection $\mathcal{CB}(X) \times \mathcal{CB}(X)$ of nonempty closed

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bounded subsets of a metric space (X,d) defined as follows:

$$H(A,B) \quad = \quad \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\};\ \ A,B \ \in \ \mathcal{CB}(X) \tag{2}$$

where $D(x,B)$ is given by

$$D(x,B) \quad = \quad \inf\{d(x,y) : y \in B\} \tag{3}$$

In normed linear spaces (3) takes the form

$$D(x,B) \quad = \quad \inf\{\|x-y\| : y \in B\}. \tag{4}$$

Nadler's contraction theorem has so many extensions (see [1, 2] and the references therein), based on modifications of the contractive condition (1), among which a result due to M. Berinde and V. Berinde [3] concerning fixed points of multivalued almost contractions, is of interest in this praxis.

Definition 1 *[3]*Let (X,d) be a metric space $T : X \longrightarrow \mathcal{P}(X)$ be multivalued operetor. T is said to be a multivalued $(\delta, L)-$weak contraction if and only if there exist $\delta \in (0,1)$ and $L \geq 0$ such that

$$H(Tx,Ty) \quad \leq \quad \delta d(x,y) + LD(y,Tx). \tag{5}$$

Following [4] $(\delta, L)-$weak contractions are called *almost contractions.* M. Berinde and V. Berinde [3] showed that multivalued almost contractions defined on a complete metric spaces have nonempty fixed point-sets. They also used many important examples to illustrate that the almost contraction condition (5) generalizes the Nadler's condition (1) and many others. We desire to obtain further extension of Nadler's result and several of its extensions via applications of the result in [3] mentioned above. In particular, we use a fixed point property of Banach spaces to characterize a class of nonexpansive mappings for which the following condition is satisfied:

$$D(y,Tx) \leq M\|x-y\| \tag{6}$$

for some constant $M \geq 1$ and all distinct elements x and y in a certain open subset of a Banach space.

The following lemma proves invaluable in the sequel:

Lemma 2 *[10] Let A and B be subsets of a metric space (X,d) and $q > 1$. Then for every $a \in A$ there exists an element $b \in B$ such that*

$$d(a,b) \quad \leq \quad qH(A,B) \tag{7}$$

Other interesting metrical studies of fixed points of multivalued operators can be found in [7, 5, 10, 8, 9].

2 Main Results

We shall prove a characterization of a class of multivalued nonexpansive mappings by a fixed point property of arbitrary Banach spaces (with respect to this class). Specifically we shall prove the following result:

Theorem 3 *Let K be a closed convex subset of a Banach space E and $T : K \longrightarrow K$ a a multivalued nonexpansive operator. Then T has a fixed point in K if and only if T satisfies the mixed orbital condition (6) in some open subset $K_1 \subset K$ and the Krasnoselskii iteration scheme $x_{n+1} \in \lambda x_n + (1-\lambda)T^n x_0, n \geq 0$ converges to a fixed point of T in K; for some $x_0 \in K_1$ and $y \in Tx_0$ such that $x_1 = y$.*

Lemma 4 *Let $T : V \longrightarrow V$ be a multivalued self-map of a normed linear space V. If T is such that $\|x - y\| \leq D(y, Tx)$ for any distinct $x, y \in V$ satisfying $x, y \notin Fix(T)$, then $Tx \neq Ty$.*

PROOF
Given a multivalued nonexpansive mapping $T : V \longrightarrow \mathcal{CB}(V)$ of a normed linear space V with $\|x - y\| \leq D(y, Tx)$. Let x and y be distinct elements of the normed linear space V with $x, y \notin Fix(T)$.

Let Z be defined by $Z = \{\lambda x + (1 - \lambda z) : z \in Ty\}$. Clearly, Z is the collection of all convex combinations of the point x and the elements of the set Ty, and $D(y, Z) \leq D(y, Ty)$. It follows that there exists $z \in Ty$ such that $\|y - [\lambda x + (1-\lambda)z]\| \leq \|y - z\|$. It also follow that when for all $\lambda \in [0, 1]$ the following holds:

$$\|x - y\| + \lambda\|x - z\| \leq \|y - z\|, \tag{8}$$

then we obtain $z = x$ yielding $x \in Ty$. In this case $Ty = Tx$ leads to the contracdiction $x \in Fix(T)$. End of proof. □

Lemma 5 *Let K be a closed convex subset of a real Banach space E and $T : K \longrightarrow K$ a multivalued nonexpansive mapping. If $Fix(T) \neq \emptyset$, then for any initial guess $x_0 \in K$ the multivalued Krasnoselskii iteration scheme $x_{n+1} \in \lambda x_n + (1-\lambda)Tx_n$ converges to a fixed point of T for any $\lambda \in (0, 1)$ and for some $y \in \lambda x_0 + (1-\lambda)Tx_0$ such that $x_1 = y$.*

PROOF
Let $T : K \longrightarrow K$ be a multivalued nonexpansive operator, $\lambda \in (0, 1)$ and $z \in K$ fixed, then the mapping $x \mapsto \lambda z + (1-\lambda)Tx, x \in K$ is a multivalued contraction mapping of the convex

set K into K. In partitcular, given that $Fix(T) \neq \emptyset$, the mapping $S_\lambda^* : K \longrightarrow K$ given by $S_\lambda^* x = \lambda p + (1-\lambda)Tx,\ p \in Fix(T)$ is a multivalued contraction and $p \in Fix(S_\lambda^*)$.

Observe that the operator S_λ^* and the multivalued averaged operator S_λ given by $S_\lambda x = \lambda x + (1-\lambda)Tx, x \in K$ (which yields the multivalued Krasnoselskii scheme with initial guess x) have the following orbital relation:

$$H(S_\lambda x, S_\lambda y) \quad \leq \quad \lambda\|x-y\| + H(S_\lambda^* x, S_\lambda^* y). \tag{9}$$

Let $x_{n+1} = S_\lambda x_n$ for any $x_0 \in K$ and for some $y \in \lambda x_0 + (1-\lambda)Tx_0$ such that $x_1 = y$ and such that $x_k \in \lambda x_{k-1} + (1-\lambda)Tx_{k-1}; k = 1, ..., n$. Applying (9) and in view of Lemma 2 choosing $q > 1$ such that $\lambda q < 1$ we obtain for $n \geq N \in \mathbb{N}$ the following:

$$\begin{aligned} H(S_\lambda x_n, S_\lambda x_{n-1}) &\leq \|x_{n+1} - x_n\| \leq \lambda\|x_n - x_{n-1}\| + H(S_\lambda^* x_n, S_\lambda^* x_{n-1}) \\ &\leq \lambda q H(S_\lambda x_{n-1}, S_\lambda x_{n-2}) + H(S_\lambda^* x_n, S_\lambda^* x_{n-1}) \\ &\leq \lambda^2 q\|x_{n-1} - x_{n-2}\| + \lambda q H(S_\lambda^* x_{n-1}, S_\lambda^* x_{n-2}) + H(S_\lambda^* x_n, S_\lambda^* x_{n-1}) \\ &\leq h^2\|x_{n-1} - x_{n-2}\| + hH(S_\lambda^* x_{n-1}, S_\lambda^* x_{n-2}) + H(S_\lambda^* x_n, S_\lambda^* x_{n-1}) \\ &\leq h^n\|x_1 - x_0\| + \sum_{j=1}^{n} h^{n-j} H(S_\lambda^* x_j, S_\lambda^* x_{j-1});\ \text{where } h = \lambda q. \end{aligned} \tag{10}$$

It is obvious from (10) that $\lim_{N\to\infty} H(S_\lambda x_n, S_\lambda x_{n-1}) = 0$ since S_λ^* is known to be a contraction. Application of (10) yields for $n, m \geq N \in \mathbb{N}$ with $m = n + l;\ l = 0, 1, ..., m - n$:

$$\begin{aligned} H(S_\lambda x_n, S_\lambda x_m) &\leq \sum_{i=0}^{l} h^{n+i}\|x_1 - x_0\| + \sum_{i=0}^{l}\sum_{j=1}^{n} h^{n+i-j} H(S_\lambda^* x_j, S_\lambda^* x_{j-1}) \\ &\leq \frac{h^{n+l}}{1-h}\|x_1 - x_0\| + \sum_{i=0}^{l}\sum_{j=1}^{n} h^{n+i-j} H(S_\lambda^* x_j, S_\lambda^* x_{j-1}). \end{aligned} \tag{11}$$

By (11) the sequence $\{S_\lambda x_n\}_{n=1}^{\infty}$ generated by the process described above is a Cauchy sequence in a complete metric space and it is trivial to prove that a fixed point $p \in Tp$ is the limit of the sequence. This completes the proof. □

Theorem 6 *Let K be a closed convex subset of a real Banach space E and $T : K \longrightarrow K$ a multivalued nonexpansive mapping. If $Fix(T) \neq \emptyset$, then there exists an open subset $K_1 \subset K$ such that T satisfies the condition (6) stated below:*

$$D(y, Tx) \quad \leq \quad M\|x - y\|$$

for some $M \geq 1$ and for all $x, y \in K_1; x \neq y, x, y \notin Fix(T)$.

PROOF

If T is such that $D(y, Tx) \leq \|x - y\|$ then the proof is done. On the other hand, given a closed convex subset K of real Banach space E and $T : K \longrightarrow K$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. Let Y denote the collection of ellements of K satisfying $\|x - y\| \leq D(y, Tx), x \neq y, x, y \notin Fix(T)$. We shall derive an open subset $K_1 \subset K$ in which (6) is satisfied. For $x, y \in Y$ nonexpansiveness of T yields the following:

$$\begin{aligned} H(Tx, Ty) &\leq \|x - y\| && (12) \\ \|x - y\| &\leq D(y, Tx) \\ H(Tx, Ty) &\leq D(y, Tx). && (13) \end{aligned}$$

Adding (12) and (13) yields

$$\begin{aligned} 2H(Tx, Ty) &\leq \|x - y\| + D(y, Tx) \\ \Longrightarrow D(y, Tx) - D(y, Ty) &\leq \frac{1}{2}\|x - y\| + \frac{1}{2}D(y, Tx) \\ \Longrightarrow D(y, Tx) &\leq \|x - y\| + 2D(y, Ty). && (14) \end{aligned}$$

Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated from the multivalued Krasnoselskii iteration scheme $x_{n+1} \in \lambda x_n + (1 - \lambda)Tx_n$ (for certain $\lambda \in (0, 1)$) for some $x_0 \in Y$ and $y \in \lambda x_0 + (1 - \lambda)Tx_0$ such that $x_1 = y$. It follows from Lemma 5 that for this initial guess $x_0 \in K$ the sequence associated with Krasnoselskii iteration $\{x_n\}_{n=1}^{\infty} = \{S_\lambda^n x_0\}_{n=1}^{\infty}$ converges to a fixed point of the nonexpansive mapping T since $Fix(T) \neq \emptyset$. Based on $Fix(T) \neq \emptyset$ we set $K_0 = Y \cap \{S_\lambda^n x_0\}_{n=1}^{\infty}$ and observe that K_0 is a nonempty bounded set. This follows since we can always find $n, m \in \mathbb{N}$ such that $\|S_\lambda^n x_0 - S_\lambda^m x_0\| \leq \|S_\lambda^m x_0 - S_\lambda^{n+1} x_0\|$ for any $x_0 \in K$. By Lemma 4 $S_\lambda^m x_0 \neq S_\lambda^{n+1} x_0$ if $x_0, S_\lambda x_0 \notin Fix(S_\lambda)$. Since $Fix(T) = Fix(S_\lambda)$ it follows that $T^m x_0 \neq T^{n+1} x_0$ if $x_0, Tx_0 \notin Fix(T)$.

In this case $D(y, S_\lambda y) = \inf_{z \in S_\lambda x_m}\{\|x_m - z\|\}$ since $x_k \in S_\lambda^k x_0$ while $\|x - y\| \leq D(y, Tx)$ takes the form $\|x_n - x_m\| \leq \inf_{z \in S_\lambda x_n} \|x_m - z\|$. Clearly, $\inf_{z \in S_\lambda x_n} \|x_m - z\| \leq \|x_n - x_m\|$. This follows from the fact that the converse condition $\|x_n - x_m\| \leq \inf_{z \in S_\lambda x_n} \|x_m - z\|$ implies that $n \geq m$ for n and m large enough contradicting convergence of the scheme. This means that $D(y, S_\lambda y) \leq \|x - y\|$ whenever $\|x - y\| \leq D(y, S_\lambda x)$ in K_0. This yields $D(y, Ty) + \lambda D(y, Ty) \leq \|x - y\|$ for all $\lambda \in [0, 1]$. This holds because K is metrically convex with respect to the norm of E. Therefore, since λ can be made as small as we please, it also holds that $D(y, Ty) \leq \|x - y\|$ in K_0, so (14) yields $D(y, Tx) \leq 3\|x - y\|$ in K_0. Let K_1 be the smallest open set in K containing K_0 and considering continuity of T we conclude that condition (6) is satisfied by multivalued nonexpansive mappings T for which $Fix(T) \neq \emptyset$.$\square$

Theorem 7 *Let K be a closed convex subset of a Banach space E and $T : K \longrightarrow K$ a multivalued nonexpansive operator. Suppose there exists an open subset $K_1 \subset K$ such that T satisfies the condition below:*

$$D(y, Tx) \quad \leq \quad M\|x - y\| \tag{15}$$

for some $M \geq 1$ for all $x, y \in K_1; x \neq y, x, y \notin Fix(T)$. Then T has a fixed point in K and the Krasnoselskii iteration scheme $x_{n+1} \in \lambda x_n + (1 - \lambda)T^n x_0, n \geq 0; x_0 \in K_1$ converges to a fixed point of T in K.

Further, condition (15) generalizes Nadler's contraction condition (1) in Banach spaces.

PROOF

Let S_λ denote the multivalued avearged operator $S_\lambda = \lambda I + (1 - \lambda)T, \lambda \in [0, 1)$ associated with the multivalued nonexpansive operator T. Applying (9) we obtain

$$\begin{aligned} H(S_\lambda x, S_\lambda y) &\leq D(y, \lambda x + (1 - \lambda)Tx) + D(y, [\lambda y + (1 - \lambda)Ty]) \\ &= D(y, \lambda x + (1 - \lambda)Tx) + D((1 - \lambda)y + \lambda y, [\lambda y + (1 - \lambda)Ty]) \\ &\leq D(y, \lambda x + (1 - \lambda)Tx) + (1 - \lambda)D(y, Ty) \\ &\leq D(y, S_\lambda x) + (1 - \lambda)D((y, Tx) + (1 - \lambda)\|x - y\|. \end{aligned} \tag{16}$$

If $D(y, Tx) \leq \|x - y\|$ then (16) yields $H(S_\lambda x - S_\lambda y) \leq 2(1 - \lambda)\|x - y\| + \|y - S_\lambda x\|$. Clearly, if $\lambda \in \left(\frac{1}{2}, 1\right]$ then (16) yields:

$$H(S_\lambda x, S_\lambda y) \leq \delta\|x - y\| + D(y, S_\lambda x) \text{ where} \lambda \text{ and } \delta \text{ satisfy } \frac{\delta}{2} + \lambda < 1.$$

So in this case, when $D(y, Tx) \leq \|x - y\|$, S_λ is a $(\delta, k)-$weak contraction with $k = 1$ and δ as stipulated above.

On the other hand if (or when) $\|x - y\| \leq D(y, Tx)$ then by (15) equation (16) yields $(S_\lambda x, S_\lambda y) \leq (1 - \lambda)(M + 1)\|x - y\| + \|y - S_\lambda x\|$. So choosing $\lambda \in (0, 1)$ such that $(1 - \lambda) < \min\left\{\frac{1}{2}, \frac{1}{M+1}\right\}$ and $k = 1$ we conclude that S_λ is an almost contraction and by existence theorem of Berinde and Berinde [3] we conclude that S_λ has a fixed point in K. Therefore T has a fixed point in K.

To complete the proof we need to establish that condition (15) genralizes, in the Banach space context, the Nadler's contraction condition (1). The proof of this part follows as an application of Theorem 6 since the collection of all multivalued contractions is a proper subclass of the class of nonexpansive mappings T with $Fix(T) \neq \emptyset$. In other words, since nonexpasive condition of the later class imply (15) then all contractions satisfy condition (15) since they are special cases. $\square$

On merging Theorem 6 and Theorem 7 we obtain the following fixed-point-property characterization for nonexpansive mappings:

In conclusion we emphasize that:

1. It is important to note that Theorem 6 and Theorem 7 are respective formulations of necessary and sufficient aspects of Theorem 3.

2. Results established here are extremely non-triovial considering the fact that a very high percentage of results concerning existence of - or convergence of various schemes to fixed points for nonexpansive mappings often require specialized mappings defined on specialized Banach spaces and specialized iteration schemes.

3. The results in this article also apply without changes in metrically convex metric spaces.

4. Further studies include extensions of these reults to more general Lipschitzian mappings, continuous and discontinuous operators as well.

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