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On Almost Contraction Principle and Property (XU)

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Abstract

We have established, in Banach spaces, an expansion of classes of examples concerning property (XU) (Monther A. Alfuraidana, Mostafa Bacharb, and, Mohamed A. Khamsi, *Almost monotone contractions on weighted graphs*, J. Nonlinear Sci. Appl. 9 (2016), 5189 - 5195.) and illustrate that it is invaluable in generalizations and extensions of quasicontractions and almost contractions. Our results opens new approaches for studies and investigations of weakly Picard operators.

1 Introduction

We investigate a property of an emerging class of operators which promises tremendous extensions of the fast developing theory of the class of almost contractions in Banach spaces. The foundation of theory of almost contractions consists of the definition of almost contraction, its existence results due to Berinde and a result on their continuity at fixed points due to Berinde and Pacura below together with their multivalued analogues:

Definition 1 *[2, 3, 7] Let (X, d) be a metric space, $\delta \in (0, 1)$ and $k \geq 0$, then a mapping $T : X \longrightarrow X$ is called $(\delta, k)-$weak contraction (or a weak contraction) if and only if*

$$d(Tx, Tx) \quad \leq \quad \delta d(x, y) + kd(y, Tx), \;\; for\ all\ x, y \;\in\; X. \tag{1}$$

Theorem 2 *[2, 3, 7] Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a $(\delta, k)-$weak contraction (i.e almost contraction). Then:*

(i) $Fix(T) = \{x \in X : Tx = x\} \neq \emptyset$.

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(ii) *For any* $x_0 \in X$ *the Picard iteration* $\{x_n\}$ *given by* $x_{n+1} = T^n x_0, n = 0, 1, 2, ...$ *converges to some* $x^* \in Fix(T)$.

(iii) *The following estimates*

$$\begin{aligned} d(x_n, x^*) &\leq \frac{\delta^n}{1-\delta} d(x_0, x_1), n = 0, 1, 2, , ... \\ d(x_n, x^*) &\leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), n = 0, 1, 2, ... \end{aligned}$$

hold, where δ *is the constant appearing in (1).*

(iv) *Under the additional condition that there exists* $\theta \in (0, 1)$ *and some* $k_1 \geq 0$ *such that*

$$d(Tx, Ty) \leq \theta d(x, y) + k_1 d(x, Tx), \text{ for all } x, y \in X \tag{2}$$

then the fixed point x^* *is unique and the Picard iteration converges at the rate* $d(x_n, x^*) \leq \theta d(x_{n-1}, x^*), n \in \mathbb{N}$.

Definition 3 *An operator* T *which satisfies (ii) above is called a Picard operator if the fixed point* x^* *is unique while* T *is called a weakly Picard operator if it satisfies (ii) without emphasis on uniqueness of fixed point* x^*.

Theorem 4 *[4, 7] Let* (X, d) *be a complete metric space and* $T : X \longrightarrow X$ *be an almost-contraction. The* T *is continuous at* p *for any* $p \in Fix(T)$.

Definition 5 *[12] By almost contraction principle is meant applications of any or all of Definition 1, Theorem 2 and Theorem 4.*

An unexpected application and extension of almost contraction principle obtained by Udo-utun et al [11] is the the following fixed point result for arbitrary Lipschitzian mappings among others:

Theorem 6 *[11] Let* K *be a closed convex subset of a Banach space* E *and* $T : K \longrightarrow K$ *an* $L-$*Lipschitzian operator. Suppose there exists an open subset* $K_1 \subset K$ *such that* T *satisfies the condition below:*

$$\|y - Tx\| \leq M\|x - y\| \text{ whenever } \|x - y\| \leq \|y - Tx\| \tag{3}$$

for some $M \geq 1$ *for all* $x, y \in K_1; x \neq y, x, y \notin Fix(T)$. *Then* T *has a fixed point in* K *and the Krasnoselskii iteration scheme* $x_{n+1} = \lambda x_n + (1 - \lambda)T^n x_0, n \geq 0; x_0 \in K_1$ *converges to a fixed point of* T *in* K.

Further, condition (3) generalizes contraction condition in Banach spaces.

In a reaction to condition (3) in [11] the authors of [1] defined property (XU), given below, and remarked that it applies only to identity mapping. It is the purpose of this article to emphasize the strength of significance of property (XU), since this was trivialized in [1], and to bring attention to important classes of examples where property (XU) are exhibited which can be very useful in various applications. In [1] property (XU) was defined (in connection with condition (3)) as follows:

Definition 7 *[1] Let (X, d) be a metric space. A map $T : X \longrightarrow X$ is said to satisfy the property (XU) in a nonempty subset $K \subset X$ if there exists $M \geq 1$ such that*

$$d(x, y) \leq d(x, Ty) \text{ implies } d(x, Ty) \leq Md(x, y) \tag{4}$$

for any $x \neq y$ in K.

Assuming (4), using the argument $d(y, Ty) \leq (M + 1)d(x, y)$ and without considerations of other obvious possibilities it was concluded in [1] that the possibility of $y = Ty$ implies (in error) that condition (XU) applies only to identity mapping. But if $y \neq Ty$ (which is the case reported in [10, 11]) the other possibility is that when $x, y \in K$ are made arbitrary close then $d(y, Ty)$ and $d(x, Ty)$ also become arbitrary close since for any mapping T the pair of inequalities in (5) below must hold simultaneously:

$$d(x, Ty) \leq d(y, Ty) + d(x, y) \text{ and } d(y, Ty) \leq d(x, Ty) + d(x, y). \tag{5}$$

In fact, the conclusion in [1] should be corrected accordingly in view of the classes of examples proved in the sequel.

Another important and very large class of weak contractions, of interest in the theory of weak contractions, is the class of quasicontractions introduced by Ciric [8].

Definition 8 *Let (X, d) be ametric space and $T : X \longrightarrow X$ a nonlinear mapping. Then T is called a quasicontraction if it satisfies*

$$d(Tx, Ty) \leq qM(x, y);\ 0 < q < 1, \tag{6}$$

where

$$M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

The classes of quasicontractions and almost contractions share in common the property of being extensions of the class of Zamfirescu operators [15] (see [9, 7]). But it is well known that the class of almost contractions includes so many weak contractive mappings as special cases except for the class of quasicontractions of which only a large subclass is in almost contractions. It is very important to comment that apart from generalizing the contraction condition, the condition (3) of Theorem 6 has been shown to include so many weak contractive conditions like the quasicontraction and almost contraction conditions as special cases [12]. In [10, 11] it was proved that a nonexpasive mapping has a fixed point if and only if (3) is satisfied. As important consequences of this condition we derive estimates of the (XU) constant, $M \geq 1$, for contraction mappings, almost contractions and for quasicontractions in the sequel.

2 Main Results

Proposition 9 *Let $T : K \longrightarrow K$ be a contraction mapping with contraction constant α, then T satisfies property (XU) in Banach spaces (i.e T satisfies (3)) if $M = 1 + \frac{\alpha}{1-\mu}$ for any fixed constant μ satisfying* $1 - \alpha \leq \mu < 1$.

PROOF
In view of symmetry considerations in (3), we may assume that $\|x - Ty\| \leq \|y - Tx\|$ for two points x and y in a small deleted neighborhood U of the fixed point p of T. Firstly, if $\|x - Tx\| < \|x - y\|$ then we have

$$\|y - Tx\| \leq \|x - y\| + \|x - Tx\| \leq 2\|x - y\|. \tag{7}$$

On the other hand, if $\|x - y\| \leq \|x - Tx\|$, we claim that, for a contraction $T : K \longrightarrow K$ with contraction constant α, the following holds

$$(1 - \mu)\|x - Tx\| \leq \|Tx - Ty\| \tag{8}$$

for all $\mu \in [1 - \alpha, 1)$ as shown immediately: Since K is convex and $\|x - y\| \leq \|x - Tx\|$, we can find $z \in K_1$ such that $\|x - Ty\| \leq \|x - z\|$ with $z = (1 - \mu)x + \mu Tx, \mu \in (0, 1)$. We obtain:

$$\begin{aligned}
\|x - Tx\| &\leq \|Tx - Ty\| + \|x - Ty\| \\
&\leq \|Tx - Ty\| + \|x - z\| = \|Tx - Ty\| + \|x - [(1 - \mu)x + \mu Tx]\| \\
&= \|Tx - Ty\| + \mu\|x - Tx\|.
\end{aligned}$$

Next, $(1 - \mu)\|x - Tx\| \leq \|Tx - Ty\|$ yields $1 - \mu \leq \alpha < 1$ which in turn gives $\mu \in [1 - \alpha, 1)$.

Now, given a contraction T with contraction constant α, for any x and y in appropriate deleted neighjborhood U of p and for $\mu \in [1-\alpha, 1)$ we have:

$$\begin{aligned} (1-\mu)\|y-Tx\| &\leq (1-\mu)\|x-y\| + (1-\mu)\|x-Tx\| \\ &\leq (1-\mu)\|x-y\| + \|Tx-Ty\| \text{ (application of (8))} \\ \|y-Tx\| &\leq \left[1+\frac{\alpha}{1-\mu}\right]\|x-y\| \end{aligned} \tag{9}$$

Combination of (7) and (9) verifies that for a contraction T the condition (3) is satisfied in the open set $K_1 = U$ above if $M = 1 + \frac{\alpha}{1-\mu}$.

Theorem 10 *Let K be a closed convex subset of a real Banach space E and $T : K \longrightarrow K$ an almostcontraction (i.e $(\delta, k)-$weak contraction). Then T satisfies the property (XU) in Banach spaces (i.e T satisfies (3)) if $M = 1 + \frac{\delta}{1-\mu(L+1)}$ for $\mu \in \left[\frac{1-\delta}{L+1}, \frac{1}{L+1}\right)$.*

PROOF
Given a closed convex subset K of real Banach space E and $T : K \longrightarrow K$ a $(\delta, k)-$weak contraction, then by Theorem 2 $Fix(T) \neq \emptyset$. We shall use application of (8) to find $M \geq 1$ for validity of (3). Let U denote a deleted neighborhood of a fixed point of T and $x, y, z \in U$ be such that $\|x-y\| \leq \|y-Tx\|, x \neq y$, $\|x-y\| \leq \|x-Tx\|$ and $z = \mu x + (1-\mu Tx), \mu \in (0,1)$ is such that $\|x-Ty\| \leq \|x-z\|$. We assume that $\|x-Ty\| \leq \|y-Tx\|$ for symmetry considerations and proceed as in proof of Proposition 9:

$$\begin{aligned} \|x-Tx\| &\leq \|Tx-Ty\| + \|x-Ty\| \\ &\leq \|Tx-Ty\| + \|x-z\| = \|Tx-Ty\| + \|\mu x - [(1-\mu)x + \mu Tx]\| \\ &= \|Tx-Ty\| + \mu\|x-Tx\| \end{aligned}$$

In this case, $(1-\mu)\|x-Tx\| \leq \|Tx-Ty\|$ yields $(1-\mu)\|x-Tx\| \leq \delta\|x-y\| + \mu L\|x-Tx\|$ leading to

$$[1-\mu(L+1)]\|x-Tx\| \leq \delta\|x-y\| \tag{10}$$

and $[1-\mu(L+1)] \leq \delta$. It follows immediately that for $\mu \in \left[\frac{1-\delta}{L+1}, \frac{1}{L+1}\right)$ and we have:

$$\begin{aligned} [1-\mu(L+1)]\|y-Tx\| &\leq [1-\mu(L+1)]\|x-y\| + [1-\mu(L+1)]\|x-Tx\| \\ &\leq [1-\mu(L+1)]\|x-y\| + \delta\|x-y\| \text{ (by 10)} \\ \Longrightarrow \|y-Tx\| &\leq \left[1+\frac{\delta}{1-\mu(L+1)}\right]\|x-y\|. \end{aligned}$$

Hence almostcontractions satisfy property (XU) if $M = 1 + \frac{\delta}{1-\mu(L+1)}$ for $\mu \in \left[\frac{1-\delta}{L+1}, 1\right)$. This shows that for almostcontractions, estimations of M is not independent of the parameter L.

Our presentation would be incomplete if it is not shown that the quasicontractions of Ciric also satisfy the property (XU) (3). We recall that the class of quasicontractions is independent of the class of almostcontractions even though the latter includes a large portion of quasicontractions:

Theorem 11 *Let K be a closed convex subset of a real Banach space E and $T : K \longrightarrow K$ a quasicontraction, i.e T satisfies (6). Then T satisfies the property (XU) in Banach spaces (i.e T satisfies (3)) if $M = 1 + \frac{2q}{1-\mu-q}$.*

PROOF
Given a closed convex subset K of real Banach space E and $T : K \longrightarrow K$ a quasicontraction, then $Fix(T) \neq \emptyset$ since quasicontractions have unique fixed points. Let U denote a deleted neighborhood of the fixed point of T and as before for $x, y \in U, x \neq y$ we assume that $\|x - y\| \leq \|y - Tx\|$ and $\|x - y\| \leq \|x - Tx\|$ since otherwise there is nothing to prove. Also, we assume $\|x - Ty\| \leq \|y - Tx\|$ for symmetry considerations. We shall derive an estimate for $M \geq 1$ in condition (XU) for quasicontractive mappings. Since T is a quasicontraction, using $(1-\mu)\|x - Tx\| \leq \|Tx - Ty\|$ and the fact that $\|x - y\| \leq \|y - Tx\|$ and $\|x - y\| \leq \|x - Tx\|$, then for any constant $q \in (0, 1)$ and for $x, y \in U$ it follows that:

$$
\begin{aligned}
(1-\mu)\|y - Tx\| &\leq (1-\mu)\|x - y\| + \|Tx - Ty\|, \ \mu \in (0, 1-q) \\
&\leq (1-\mu)\|x - y\| + qM(x, y) \\
&\leq (1-\mu)\|x - y\| + q\max\{\|x - y\|, \|x - Tx\|, \|y - Tx\|, \|y - Ty\|\} \\
&\leq (1-\mu)\|x - y\| + q\max\{\|x - Tx\|, \|y - Tx\|, \|y - Ty\|\} \\
&\leq (1-\mu)\|x - y\| + q\max\{\|x - Tx\|, \|y - Tx\|\}. \qquad (11) \\
\Longrightarrow \|y - Tx\| &\leq \frac{1-\mu}{1-\mu-q}\|x - y\| \ (\text{using } M(x, y) = \|y - Tx\| \text{ in (11)}), \\
&= \left[1 + \frac{q}{1-\mu-q}\right]\|x - y\|. \qquad (12)
\end{aligned}
$$

But when when $M(x, y) = \|x - Tx\|$ in (11) it is straight forward to show that:

$$
\|y - Tx\| \leq \left[1 + \frac{2q}{1-\mu-q}\right]\|x - y\|, \ \mu \in (0, 1-q). \qquad (13)
$$

Without loss of generality we have assumed $\|y - Ty\| \leq \|x - Tx\|$ in the proofs above. Hence for quasicontractions, an estimate for the $(XU)-$ constant is $M = 1 + \frac{2q}{1-\mu-q}$, $\mu \in (0, 1-q)$.
□

3 Illustrative Examples

Below we illustrate particular cases from the classes of contractions, almost contractions and quasicontractions in view of the fact that our claims established above are given, respectively, in their most general situations. The estimates for the $(XU)-$constants M for contractions, almost contractions and quasicontractions as proved above are not the best as our method never targeted sharpness of the estimates. But the estimate for particular cases are shown, below, to be sharper than those for general examples above. This aspect would be pursued as further work. Our method below exploits symmetric property of the metric function which has intricate influence on validity of condition (XU):

Example 12 *(***Contractions***)*
Let $K \subset \mathbb{R}$ be given by $K = [0, 1]$ and $T : K \longrightarrow K$ be the operator $Tx = \frac{2}{3}$, then T is a Banach contraction with contraction constant $\frac{2}{3}$ with fixed point at zero. We shall show that, in this case, the $(XU)-$constant $M = \frac{8}{3} > 2$. Let $x, y \in K_1 = (0, 1)$ be such that $y > x$. Then $y = x + (y - x)$ and $|x - y| < |y - Tx|$ since

$$|y - Tx| = \frac{1}{3}x + (y - x). \tag{14}$$

But its symmetric counterpart gives

$$|x - Ty| = \frac{1}{3}|x - 2(y - x)| < |x - y| \tag{15}$$

since K_1 is a deleted neighborhood of the fixed point, zero, of a contraction mapping. Also, (15) yields $x - 2(y - x) < 3(y - x)$ giving $x < 5(y - x)$. On putting $x < 5(y - x)$ in (14) we obtain condition (XU) as $|y - Tx| = \frac{1}{3}x + (y - x) < \frac{5}{3}(y - x) + (y - x) = \frac{8}{3}(y - x)$. Therefore, T satisfies $|y - Tx| < \frac{8}{3}|x - y| = \left(2 + \frac{2}{3}\right)|x - y|$ for all $x, y \in K_1$.

Example 13 *[7](***Almost Contractions***)*
We now use a known example of almost contractions (see [7]) as a concrete illustration for almost contractions. In this case, $K \subset \mathbb{R}$ is as in Example 12 above but $T : K \longrightarrow K$ is defined as below

$$Tx = \begin{cases} \frac{2}{3}x; & if\ 0 \le x < \frac{1}{2} \\ \frac{2}{3}x + \frac{1}{3}; & if\ \frac{1}{2} \le x \le 1 \end{cases}$$

T has been illustrated in [7] as an almost contraction which, as a weakly Picard operator, is neither a contraction nor a quasicontraction with constants $\delta = \frac{2}{3}$, $L = 6$ and the fixed point set of T is $Fix(T) = \{0, 1\}$. We shall show that T satisfies condition (XU)) in two deleted neighborhoods K_1 and K_2 of its two fixed points points 0 and 1 respectively with $M_1 = 2 + \frac{2}{3}$ and $M_2 = 2 + \frac{2}{3}$.

We already know from Example 12 that in the deleted neighborhood $K_1 = (0, \frac{1}{2})$ of zero the $(XU)-$constant $M_1 = 2 + \frac{2}{3}$. So we are left to find $(XU)-$constant M_2 for the deleted neighborhood $K_2 = \left(\frac{1}{2}, 1\right)$ of the fixed point 1 as follows. Let $x, y \in K_2 = \left(\frac{1}{2}, 1\right)$ with $\frac{1}{2} < x < y < 1$. That $|x - y| < |x - Ty|$ follows from the fact that $x < y < Ty < 1$ and $Ty - y = \frac{1-y}{3} > 0$ for all $y \in (\frac{1}{2}, 1)$. Then, using $y = x + (y - x)$, we have

$$|x - Ty| = \frac{1}{3}(1 - x) + \frac{2}{3}(y - x) = \frac{1}{3}(1 - x) + \frac{2}{3}|x - y| \tag{16}$$

In this case $|y - Tx| < y - x = |x - y|$ since;

$$\begin{aligned} |y - Tx| &= y - x - \frac{1}{3}(1 - x). \\ \Longrightarrow \ 1 - x &< 6(y - x) = 6|x - y| \end{aligned} \tag{17}$$

We now put $1 - x < 6|x - y|$ from (17) into (16) to obtain $|x - Ty| < 2|x - y| + \frac{2}{3}|x - y|$. Therefore, for all $x, y \in K_2$ we have $|y - Tx| < \frac{8}{3}|x - y|$ or $|y - Tx| < (2 + \frac{2}{3})|x - y|$. So, $M_2 = \frac{8}{3} = 2 + \frac{2}{3}$.

Example 14 *[8](***Quasicontraction***)*
The following example was used in [8] by Ciric to demonstrate that Quasicontractions need not be generalized contractions. Ciric defined a metric space $(X, |.|)$ by $X = X_1 \cup X_2$, where X_1 and X_2 are given by:

$$X_1 = \left\{ \frac{m}{n} : m = 0, 1, 3, 9, \ \ldots; \ n = 1, 4, \ \ldots, 3k + 1, \ \ldots \right\}.$$

$$X_2 = \left\{ \frac{m}{n} : m = 1, 3, 9, 27 \ \ldots; \ n = 2, 5, \ \ldots, 3k + 2, \ \ldots \right\}.$$

In the example he defined a quasicontraction $T : X \longrightarrow X$ by

$$Tx = \begin{cases} \frac{3}{5}x; & if \ x \in X_1 \\ \frac{1}{8}x; & if \ x \in X_2 \end{cases}$$

and showed that T satisfies the quasicontraction condition with $q = \frac{3}{5}$, that is;

$$|Tx - Ty| \leq \frac{3}{5} \max\{|x - y|, |x - Tx|, |y - Ty|, |x - Ty|, |y - Tx|\}.$$

We shall show that T satisfies condition $(XU))$ in a deleted neighborhood $X_1 \backslash \{0\}$ of its fixed point zero with $M = 2 + \frac{2}{3}$. We have $|x - y| < |y - Tx|$ for $0 < x < y$ in K_1 as shown below:

$$|y - Tx| = \frac{2}{5}x + (y - x). \tag{18}$$

In this case $|x - Ty| < y - x = |x - y|$ *since;*

$$\begin{aligned} |x - Ty| &= \left|\frac{2}{5}x - \frac{3}{5}(y - x)\right| \text{ where } \frac{2}{5}x - \frac{3}{5}|x - y| < 0. \\ \implies \frac{2}{5}x &< \frac{8}{5}(y - x) = \frac{8}{5}|x - y| \end{aligned} \tag{19}$$

We now put $\frac{2}{5}x < \frac{8}{5}|x - y|$ *from (19) into (18) to obtain* $|y - Tx| < \frac{8}{5}|x - y| + |x - y|$. *Therefore, for all* $x, y \in K_1$ *we have* $|y - Tx| < \frac{13}{5}|x - y|$ *or* $|y - Tx| < (2 + \frac{3}{5})|x - y|$. *So,* $M = \frac{13}{5} = 2 + \frac{3}{5}$.

4 Conclusion

An important observation which reinforces the importance of Berinde and Pacura [4, 7] result on continuity at fixed points is the following:

$$\|Tx - Ty\| \leq \|y - Tx\| + \|y - Ty\| \leq (2M + 1)\|x - y\| \tag{20}$$

for all x and y in the subset K_1 of the convex set K in Theorem 6. This means that both classes of of quasicontractions and almost contractions are not only continuous at their fixed points but are also Lipschitz in certain small neighborhoods of their fixed points. An important direction is that of determining whether all weakly Picard operators are Lipschitz in some neighborhoods of their fixed points.

We need to comment on the significance of condition (XU) concerning unification of the classes of quasicontractions and almost contractions which is a very important problem initiated by Berinde [5, 6]. In [13] it was proposed that this unification need not be localized to weakly Picard operators suggesting that the largest class of weakly Picard operators may not unify the two classes and that a unifying class may require Krasnoselskii iterations for convergence to fixed points.

Consequences of property (5) can be explored beyond its usefulness in validating that an operator T which satisfy condition (XU) need not be identity map. We must appreciate the positive impact of the observations in [1] without which the strength of property (5) would still remain untapped. On the other hand, it is of much concern that an apparently valid analytic position such as reported in [1] can turnout so misleading. This is a reminder of our collective philosophical duty to use observations (examples or experiments) to reinforce results obtained via pure analysis since some analytic approaches may not account for all possibilities.

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